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The fundamental period of a G/SM/1 queue

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1. Introduction

We study the fundamental period of a PH-MR/SM/1 queue with a phase-type Markov renewal input and semi-Markovian services. The LST (Laplace-Stieltjes transform) of the fundamental period distribution has been obtained as a matrix-exponential form for a PH-MR/GI/1 queue with i.i.d. services [1, 2]. Lucantoni-Neuts recently generalized this result to a PH-MR/SM/1 queue[3], but its form does not seem to be suitable for computation. In this report, we give a different form for the LST of the fundamental period distribution. This form is more tractable for computation than Lucantoni-Neuts'.

2. Queueing Model and the Fundamental Period

We consider the G/SM/1 queueing model characterized as follows:

(a) The service times of successive customers form an irreducible semi-Markov process with m_1 possible states. Given that a service starts in state i, its service time follows distribution $H_i(x)$. Just after service completion, the state jumps from i to j with probability h_{ij}, where $H = (h_{ij})$ and the succeeding service time follows distribution $H_j(x)$. The service time distribution is given by $H(x) = (H_i(x)h_{ij})$. Suppose that when a system becomes empty in service state i $(i = 1, \cdots, m_1)$, the next service begins in the same service state i.

(b) The arrival process is a phase-type Markov renewal process (or a Markovian Arrival Process) with m_2 phase states and the interarrival time density is given by

$$f(x) = \alpha \exp(Tx)\, T^0, \tag{1}$$

where α, T and T^0 respectively have the $n \times m_2$, $m_2 \times m_2$ and $m_2 \times n$ matrix forms. Between successive arrivals, the fluctuation of m_2 states follows a Markov process with the infinitesimal generator T. We call these states "arrival phase states." The arrival process and the service times are mutually independent.

We assume that the service discipline is non-preemptive. We now consider the embedded Markov renewal process at departure epochs. Define τ_l to be the l-th departure epochs of customers from the system, with $\tau_0 = 0$, and (ξ_l, I_l, J_l) to be the number of customers in the system, the service state and the arrival phase state at τ_l+, immediately after τ_l. Then $(\xi_l, I_l, J_l, \tau_{l+1} - \tau_l)$ is a semi-Markov process in the state space $\{(i, j, k): i \geqq 0, 1 \leqq j \leqq m_1, 1 \leqq k \leqq m_2\}$. This process is positive recurrent when the traffic intensity $\rho = \lambda/\mu < 1$, where μ^{-1} and λ^{-1} are the mean service time and mean interarrival time. Now, we assume $\rho < 1$. We define level i to be the set of states $\{(i, j, k): 1 \leqq j \leqq m_1, 1 \leqq k \leqq m_2\}$, $i \geqq 0$. The states of level i are ordered lexicographically, that is, $(i, 1, 1), \cdots, (i, 1, m_2), (i, 2, 1), \cdots, (i, 2, m_2), \cdots, (i, m_1, 1), \cdots, (i, m_1, m_2)$. We call these m_1m_2 states "service-arrival states (of level i)."

Let us consider a service completion epoch τ_k at which the number of customers becomes level i+2 from level i+1. We now study the first passage time from level i+1 to level i $(i = 0, 1, \cdots)$. Let $G_{jj'}(x; i+1)$ be the probability that the first passage from state (i+1, j) to state (i, j') occurs no later than time x, and that (i, j') is the first state visited in level i, where $i \geq 0$, j and j' are lexicographically ordered service-arrival states, $1 \leq j, j' \leq m_1m_2$. Note that $G_{jj'}(x; i+1)$ is independent of $i \geq 0$. The Laplace-Stieltjes transform $G^*(s)$ of

$$G(x) = (G_{jj'}(x, i+1)) \ (1 \leq j, j' \leq m_1 m_2)$$

is given by [3]

$$G^*(s) = \int_0^\infty (dH(x) \otimes I_2) \exp\{(-sI + I_1 \otimes T + (I_1 \otimes T^0\alpha)G^*(s))\, x\}, \tag{2}$$

where I_i is $m_i \times m_i$ indentity matrix $(i = 1, 2)$.

This matrix-exponential form is the simple extension of the result for the i.i.d. service queue ([1], [2]). What is the difference between the cases in which H(x) has the matrix form (semi-Markovian service case) and H(x) has just a scalar form (i.i.d. service case)? In the latter, I_1 is scalar and we can simply substitute $sI - T - T^0\alpha G^*(s)$ into the variable w of the LST $H^*(w)$ of H(t). However, when H(x) has the matrix form, we have to directly compute the matrix-integration. Now, we will give the simpler form than Eq. (2).

Theorem The LST of the fundamental period length distribution is given by

$$G^*(s) = (G^*(s; j)h_{jj'})\ (1 \le j, j' \le m_1), \tag{3}$$

where

$$G^*(s; j) = (\bar{h}_j \otimes I_2) \int_0^\infty dH_j(x) \exp\{(-sI + I_1 \otimes T + (I_1 \otimes T^0\alpha)G^*(s))x\}\ (e_1 \otimes I_2). \tag{4}$$

for $\bar{h}_j = (h_{j1}, \cdots, h_{j,m_1})$.

Proof Let us assume that the service state at τ_k+ (just after τ_k) is j. Let G(x; j) ($m_2 \times m_2$ matrix) denote the fundamental period length distribution given that the service state at τ_k+ is j. Now, we shall consider that the service displine is LIFO (Last-In-First-Out) preemption during the considering fundamental period. Note that the fundamental period length remains unaltered for any service disciplines if they are work-conserving.

Denote X = Service time of the first customer whose service starts at τ_k+
and Y = Total service time of all customers in the fundamental period other than the first customer.

We define the conditional distribution given that X = x

$$m^{(j)}_{kk'}(y \mid x = x) = P\{Y \le y, K(T_f-) = k' \mid J(\tau_k+) = j, K(\tau_k+) = k, X = x\}.$$

and the transform

$$M^*(x, s; j) = \left(\int_0^\infty e^{-sy}\, m^{(j)}_{kk'}(x, d, y) \right)\ 1 \le k, k' \le m_2,$$

where T_f is the epoch at which the fundamental period ends, and J(t) and K(t) denote the service state and arrival phase state at t, respectively. Note that the service state at τ_k+ is equal to the service state at T_f- from the assumption of the LIFO preemptive discipline.

In order to derive $M^*(x, s; j)$, we consider the case in which there are n arrivals during the first custom service period (n = 0, 1, $\cdots$). When n = 0, its probability is given by exp(Tx) (= α_0). When n = i (> 0), service time of a customer who preempts the first customer service for the first time follows the distribution $H_{j'}(x)$ with probability $h_{jj'}$(j, j' = 1, 2, $\cdots$, m_1). And then , the new fundamental period whose length distribution is G(x) succeeds. When this new fundamental period ends, the service of the first customer is resumed. Therefore, the probability of this case is given by

$$\alpha_i = \int \cdots \int_{u_1+\cdots+u_{i+1}=x} \exp(Tu_1)\, T^0\alpha(\bar{h}_j \otimes I_2)G^*(s)$$
$$\times (I_1 \otimes \exp(Tu_2)T^0\alpha)G^*(s)) \cdots (I_1 \otimes \exp(Tu_i)T^0\alpha)\, G^*(s)\, (I_1 \otimes \exp(Tu_{n+1}))(e_1 \otimes I_2).$$

Then, we have

$$M^*(x, s; j) = \sum_{n=0}^{\infty} \alpha_n = (\bar{h}_j \otimes I_2) \exp\{(I_1 \otimes T + (I_1 \otimes T^0\alpha)G^*(s))x\}\ (e_1 \otimes I_2).$$

Since

$$G^*(s; j) = \int_0^\infty e^{-sx} M^*(x, s; j)\, dH_j(x),$$

we have (4). Since $\Pr\{J(T_f+) = j' \mid J(T_f-) = j\} = h_{jj'}$, we have (3).

References

[1] Machihara, F., *Proceeding of the 12th International Teletraffic Congress in Torino*, (1988) 5.4B 5.1-5.8. [2] Neuts, M. F., in *Probability, Statistics and Mathematics: Papers in Honor of Professor Samuel Karlin*, Academic Press, 1989. [3] Lucantoni, D. M. and Neuts, M. F., *J. Appl. Prob.*, **31** (1994), pp. 235-241.