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Research on Empty-Core Games

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The relation between the game theory and the combinatorial optimization theory has been extensively discussed. For example, according to the result of Shapley and Edmonds, any convex game is equivalent to a polymatroid. Here we are interested in another way. If the core of a game is empty, how is it related to combinatorial theory? How shall we characterize and research those games?

This paper denote a game by $G = (N, c)$, where N is the player set, and c is the characteristic function. Usually we use i to represent a general player in N. This paper assumes that c is a non-decreasing function. That is, for any $i \in S \subseteq N$, $c(S - \{i\}) \leq c(S)$. Furthermore, we suppose $c(\emptyset) = 0$. For fundamental terminology, such as imputation, core, convex game, τ-value see [2].

1 Base and Cut

Definition 1.1 *For a subset $S \subseteq N$, if $c(S) = c(N)$, we call S a* spanning set *of G (or N).*

Definition 1.2 *Given a subset $S \subseteq N$, if $c(T) < c(S)$ for any proper subset $T \subset S$, then S is called* independent.

Let $\mathcal{I}$ be the family of all independent sets.

Definition 1.3 *We call a player $i \in N$* null *player if for all $S \subseteq N - \{i\}$, $c(S \cup \{i\}) = c(S)$.*

Theorem 1.1 *For a convex game without null player, the only spanning set of N is N itself.*

Definition 1.4 *A subset $B \subseteq N$ is called a* base *of G (or N) if B is a* spanning set *of G and every proper subset of B is not a spanning set of G.*

Let $\mathcal{B}$ be the family of all bases.

Theorem 1.2 *If the core of G $core(G)$ is not empty, then for any imputation $x \in core(G)$, $x(i) = 0$ if there exits a base $B \subseteq N - \{i\}$.*

Definition 1.5 *For a set $S \subseteq N$, a subset $T \subset S$ is called a* cut *of S if $c(S - T) < c(S)$. T is called a* minimal cut *if any proper subset of T is not a cut. Especially, if T consists of a single player i, then i is called a* bridge *of S.*

Theorem 1.3 *Let D be the set of all bridges of N. If $c(G) > 0$ then $core(G) \neq \emptyset$ only if $D \neq \emptyset$. Furthermore, by then $x(D) = x(N)$ for any $x \in core(G)$.*

Definition 1.6 *Let $S \subseteq N$. For any set $T \subseteq S$, let $c_S(T) = c(T \cup (N - S))$. Then a new game (S, c_S) is called the* restricted game *of G on S.*

Theorem 1.4 *Let D be the set of all bridges of N and $D \neq \emptyset$. Then $core(G) \neq \emptyset$ if and only if $core(D, c_D) \neq \emptyset$.*

From the above definitions, we know that empty-core games share many properties with graphs. See Section 3.

2 Generalized Core and Generalized τ-Value

Definition 2.1 *An vector $x \in \mathcal{R}^n$ is called an* generalized imputation *if the following conditions are satisfied.*
1) (Generalized Individually Rationality)
$x_i \geq \min\{c(i), c(N) - c(N - \{i\})\}$;
2) (Efficiency) $\sum_{i \in N} x_i = c(N)$.

Note that if x is a imputation then $x(i) \geq c(i)$. Therefore imputation is a special case of generalized imputation.

Definition 2.2 *The set of all generalized imputation x which satisfy condition* 3) *is called* generalized core.
3) If there is a subset $S \subset N$ such that $x(S) < c(S)$ then $x(T) \leq c(T)$ holds for any subset $T \subset N$ with $|T| = |N| - 1$.

It is easy to check that generalized core contains core for any game. Furthermore, the following theorem shows that many empty-core games have nonempty generalized core.

Theorem 2.1 *If* $\sum_{i=1}^{|N|}(c(N) - c(N - \{i\})) \leq c(N)$, *then the generalized core is nonempty.*

Definition 2.3 *In game* (N, c), *two vectors* A *and* $B \in \mathcal{R}^{|N|}$ *are defined as follows: for all* $i \in N$,

$$A(i) = c(N) - c(N - \{i\}),$$

$$B(i) = \max_{\{S|i \in S \subseteq N\}} \left\{ c(S) - \sum_{k \in S - \{i\}} A(k) \right\}.$$

Recall that vectors A and B are the lower bound and upper bound in the definition of τ-value. We claim that the τ-value can be also applied to the case that $B \leq A$.

Definition 2.4 *If* $A \leq B$ *or* $B \leq A$ *then the* generalized τ-value *is defined as* $\tau = \lambda b + (1 - \lambda)B$, *when* $\lambda \in [0, 1]$ *is defined by the property* $\sum_i \tau(i) = c(N)$.

Theorem 2.2
If the generalized core is nonempty, then either $A \geq B$ *or* $B \geq A$.

3 Application

As an application, we investigate some games related to reliability theory. In reliability theory, how to measure the component importance is an important problem. Given a network system, represented by a connected graph $GRAPH(V, E)$, where V is the vertex set and E is the edge set where each edge may fail. This system works if and only if all vertices in V are connected by working edges. [3] defines a cooperative game by considering every edge of E as a player (i.e., let $V = E$). For any subset $S \subseteq N = E$ $c(S) = 1$ if S is a spanning set of E, otherwise $c(S) = 0$. In this way, almost all games have empty cores. Furthermore, the base (resp. cut, bridge) of a game (Section 1) corresponds to the base (resp. cut, bridge) of the related graph. On the other way, the famous Barlow-Proschan importance measure [1] for the graph is the Shapley value of the game.

There exists a problem from the view point of computation complexity.

Theorem 3.1 *Computing Barlow and Proschan measure is #P-complete, with respect to theinput length* $|N|$.

Therefore we now try to use the τ-value to measure the component importance. Since the games related to network system are usually with empty-core, we consider the generalized τ-value of Definition 2.4. Furthermore, it can be used to a weighted network in the following way.

We construct a game $G^\tau = (N, c^\tau)$ from a weighted network $GRAPH(V, N, w)$. For any subset $S \subseteq N$, $w(S) = \sum_{i \in S} w(i)$. Then define the characteristic function c^τ of game G^τ by

$$c^\tau(S) = \begin{cases} w(S) - w(B_S) & \text{if } S \text{ is a spanning set} \\ 0 & \text{otherwise} \end{cases}$$

where $w(B_S)$ is the weight of minimal spanning tree of S.

Definition 3.1 *The generalized* τ*-value for a game* (E, c^τ) *according to Definition 3.4 is called* τ*-measure for the components in a system.*

According to Theorem 2.1 and 2.2, the τ-measure can be applied to many network systems. Furthermore, the computation of τ-measure is in P. In fact, the computation of A is evidently easy. In order to compute B, let T_i be a spanning tree of $GRAPH(V, E)$ containing i with the least possible weight. T_i can be found by the greedy algorithm, starting from i and adding one edge each time with the least weight while keeping being a tree.

Theorem 3.2

$$B(i) = c(T_i) - \sum_{k \in T_i - \{i\}} A(k).$$

参考文献

[1] Barlow R. E. and Proschan F. (1975) Statistical Theory of Reliability and Life Testing. Holt, Rinehart and Winston Inc.

[2] Driessen T. S. H. (1988) Cooperative Games, Solutions and Applications. Kluwer Academic Publishers.

[3] Ramamurthy K. G. (1990) Coherent Structures and Simple Games. Kluwer Academic Publishers.