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Double Horn Functions

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1 Introduction

Boolean functions has been considerably investigated, as these functions play an important role in many fields such as artificial intelligence, database theory and operations research. In particular, class of *Horn* functions are one of the most important Boolean functions, because *satisfiability problem* of a Horn CNF (conjunctive normal form) (H-SAT) can be solved in polynomial time [5], whereas satisfiability problem (SAT) of a general CNF is NP-complete. Based on this result, Horn functions are widely used in practice as the production rules in expert systems. This is a motivation for recent increasing activities on Horn functions, e.g., [1, 6, 8].

In terms of sets $T(f)$ and $F(f)$, where $T(f)$ (resp., $F(f)$) denotes the set of true (resp., false) vectors of a Boolean function f, a Horn function has an elegant algebraic characterization: f is Horn if and only if $F(f)$ is closed under *intersection* of vectors (i.e., $v, w \in F(f)$ implies $v \bigwedge w \in F(f)$, where $\bigwedge$ denotes the componentwise AND operation). In this paper, we extend this semantical condition also to the set of true vectors, and define that a function f is *double Horn* if both $T(f)$ and $F(f)$ are closed under *intersection* of vectors. In other words, f is double Horn if and only if both f and $\bar{f}$ are Horn.

The other restrictions on Horn functions in the line of the above are the conditions such as (i) f and f^d are Horn, and (ii) f and f^* are Horn. Such functions are considered in [4, 7]. Moreover, [7] points out that condition (ii) characterizes the class of *submodular* functions, given by the inequation

$$f(x \wedge y) \vee f(x \vee y) \ \leq \ f(x) \vee f(y).$$

In this paper, we characterize a double Horn function in terms of disjunctive normal form (DNF), and on the computational side, consider the following problems, where pdBf stands for a partially defined Boolean function.

(Recognition): Given a DNF formula φ, does φ represent a double Horn function ?

(Extension): Given a pdBf (T, F), does it have a double horn extension ?

2 Preliminaries

Recall that a *Boolean function*, or a *function* in short, is a mapping $f : \{0,1\}^n \rightarrow \{0,1\}$. The sets $T(f) = \{v \mid f(v) = 1\}$ and $F(f) = \{v \mid f(v) = 0\}$ are the true vectors and false vectors of f, respectively. A pdBf is a pair (T, F) of subsets $T, F \subseteq \{0,1\}^n$. The pdBf is used to model a pair of a set of positive examples and a set of negative examples [2, 3]. Notice that (T, F) can be seen as a representation for all Boolean functions f such that $T \subseteq T(f)$ and $F \subseteq F(f)$; any such f is called an *extension* of (T, F). The main issue in the context of pdBf concerns existence and properties of extensions, subject to the condition that they are from a certain class of Boolean functions [2, 8].

Let $v \wedge w$ denote the *intersection* (i.e., the componentwise conjunction of vectors v and w, e.g., if $v = (1100)$ and $w = (1010)$, then $v \wedge w = (1000)$. A *Horn* function f has a well-known algebraic characterization, given by

$$f(v \wedge w) \leq f(v) \wedge f(w),$$

which is equivalent to $F(f) = Cl_{\wedge}(F(f))$, where $Cl_{\wedge}$ (S) is the closure of a set S of vectors under the intersection, also called the *intersection closure*. An equivalent definition of Horn functions can be given in terms of *disjunctive normal form* (DNF). A DNF

$\varphi = \bigvee_i t_i$ is called Horn if each t_i has at most one negative literal.

A function f is called *double Horn* if $T(f) = Cl_\wedge(T(f))$ and $F(f) = Cl_\wedge(F(f))$; the class of these functions is denoted by $\mathcal{C}_{DH}$. Note that f is double Horn if and only if f and $\overline{f}$ are Horn. For example,

$$f = \overline{x}_1 \vee x_2x_3\overline{x}_4 \vee x_2x_3x_5x_6\overline{x}_7$$

is double Horn, because

$$\begin{aligned}\overline{f} &= x_1(\overline{x}_2 \vee \overline{x}_3 \vee x_4)(\overline{x}_2 \vee \overline{x}_3 \vee \overline{x}_5 \vee \overline{x}_6 \vee x_7) \\ &= x_1\overline{x}_2 \vee x_1\overline{x}_3 \vee x_1x_4\overline{x}_5 \vee x_1x_4\overline{x}_6 \vee x_1x_4x_7.\end{aligned}$$

3 Double Horn functions

In this section, we consider a syntactical characterization of double functions, which will then be used for solving the recognition and extension problems efficiently.

Theorem 1 *Every $f \in \mathcal{C}_{DH}$ has a unique prime DNF, which has the form*

$$\varphi = \bigvee_{i=1}^{m} t_1t_2\cdots t_i\overline{x}_{\ell_i}, \tag{1}$$

where $m = 0$ means $\varphi = \perp$, and t_i and x_{ℓ_i}, $i = 1,2,\ldots,m$, are pairwise disjoint positive terms (in this case, x_{ℓ_i} are regarded as terms) and t_i for $i = 1,2,\ldots,m$ and x_{ℓ_m} are possibly empty. Conversely, every such formula φ represents an $f \in \mathcal{C}_{DH}$. □

From this theorem, we see that a double Horn function can be represented by a *linear read-once* formula [4].

Theorem 2 *Given a Horn DNF φ, checking if φ represents a double Horn function can be done in $O(|\varphi|)$ time.* □

However, in general, we have the following negative result.

Theorem 3 *Given a DNF φ, checking if φ represents a double Horn function is co-NP-complete.* □

Now, we turn to the problems about extensions.

Theorem 4 *Given a pdBf (T,F), where $T,F \subseteq \{0,1\}^n$, finding a double Horn extension can be done in $O(n(|T|+|F|))$ time.* □

Theorem 5 *Given a pdBf (T,F), there is a polynomial delay algorithm for enumerating all double Horn extensions of (T,F).* □

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