Double Horn Functions
Technische Universität Wien EITER Thomas
01001374 京都大学 茨木俊秀 (IBARAKI Toshihide)
02601514 京都大学 牧野和久 (MAKINO Kazuhisa)

1 Introduction

Boolean functions has been considerably investigated, as these functions play an important role in many fields such as artificial intelligence, database theory and operations research. In particular, class of Horn functions are one of the most important Boolean functions, because satisfiability problem of a Horn CNF (conjunctive normal form) (H-SAT) can be solved in polynomial time [5], whereas satisfiability problem (SAT) of a general CNF is NP-complete. Based on this result, Horn functions are widely used in practice as the production rules in expert systems. This is a motivation for recent increasing activities on Horn functions, e.g., [1, 6, 8].

In terms of sets \( T(f) \) and \( F(f) \), where \( T(f) \) (resp. \( F(f) \)) denotes the set of true (resp. false) vectors of a Boolean function \( f \), a Horn function has an elegant algebraic characterization: \( f \) is Horn if and only if \( F(f) \) is closed under intersection of vectors (i.e., \( v, w \in F(f) \) implies \( v \land w \in F(f) \), where \( \land \) denotes the componentwise AND operation). In this paper, we extend this semantical condition also to the set of true vectors, and define that a function \( f \) is double Horn if both \( T(f) \) and \( F(f) \) are closed under intersection of vectors. In other words, \( f \) is double Horn if and only if both \( f \) and \( f^* \) are Horn.

The other restrictions on Horn functions in the line of the above are the conditions such as (i) \( f \) and \( f^d \) are Horn, and (ii) \( f \) and \( f^* \) are Horn. Such functions are considered in [4, 7]. Moreover, [7] points out that condition (ii) characterizes the class of submodular functions, given by the inequation

\[
    f(x \land y) \lor f(x \lor y) \leq f(x) \lor f(y).
\]

In this paper, we characterize a double Horn function in terms of disjunctive normal form (DNF), and on the computational side, consider the following problems, where pdBf stands for a partially defined Boolean function.

(Recognition): Given a DNF formula \( \varphi \), does \( \varphi \) represent a double Horn function?

(Extension): Given a pdBf \((T, F)\), does it have a double horn extension?

2 Preliminaries

Recall that a Boolean function, or a function in short, is a mapping \( f : \{0, 1\}^n \to \{0, 1\} \). The sets \( T(f) = \{ v \mid f(v) = 1 \} \) and \( F(f) = \{ v \mid f(v) = 0 \} \) are the true vectors and false vectors of \( f \), respectively. A pdBf is a pair \((T, F)\) of subsets \( T, F \subseteq \{0, 1\}^n \). The pdBf is used to model a pair of a set of positive examples and a set of negative examples [2, 3]. Notice that \((T, F)\) can be seen as a representation for all Boolean functions \( f \) such that \( T \subseteq T(f) \) and \( F \subseteq F(f) \); any such \( f \) is called an extension of \((T, F)\). The main issue in the context of pdBf concerns existence and properties of extensions, subject to the condition that they are from a certain class of Boolean functions [2, 8].

Let \( v \land w \) denote the intersection (i.e., the componentwise conjunction of vectors \( v \) and \( w \), e.g., if \( v = (1100) \) and \( w = (1010) \), then \( v \land w = (1000) \). A Horn function \( f \) has a well-known algebraic characterization, given by

\[
    f(v \land w) \leq f(v) \land f(w),
\]

which is equivalent to \( F(f) = \text{Cl}_{\land}(F(f)) \), where \( \text{Cl}_{\land} (S) \) is the closure of a set \( S \) of vectors under the intersection, also called the intersection closure. An equivalent definition of Horn functions can be given in terms of disjunctive normal form (DNF). A DNF
$\varphi = \bigvee_i t_i$ is called Horn if each $t_i$ has at most one negative literal.

A function $f$ is called double Horn if $T(f) = C_{\leq}(T(f))$ and $F(f) = C_{\leq}(F(f))$; the class of these functions is denoted by $C_{DH}$. Note that $f$ is double Horn if and only if $f$ and $\overline{f}$ are Horn. For example,

$$f = \overline{x_1} \lor x_2 x_3 \overline{x_4} \lor x_2 x_3 x_5 \overline{x_7}$$

is double Horn, because

$$\overline{f} = x_1 \overline{x_2} \lor \overline{x_3} \lor x_4 (\overline{x_2} \lor \overline{x_3} \lor \overline{x_5} \lor \overline{x_6} \lor x_7) = x_1 \overline{x_2} \lor x_1 \overline{x_3} \lor x_1 x_4 \overline{x_5} \lor x_1 x_4 \overline{x_6} \lor x_1 x_4 x_7.$$

### 3 Double Horn functions

In this section, we consider a syntactical characterization of double functions, which will then be used for solving the recognition and extension problems efficiently.

**Theorem 1** Every $f \in C_{DH}$ has a unique prime DNF, which has the form

$$\varphi = \bigvee_{i=1}^m t_1 t_2 \cdots t_i \overline{x_i},$$

where $m = 0$ means $\varphi = \bot$, and $t_i$ and $x_i$, $i = 1, 2, \ldots, m$, are pairwise disjoint positive terms (in this case, $x_i$ are regarded as terms) and $t_i$ for $i = 1, 2, \ldots, m$ and $x_{i+1}$ are possibly empty. Conversely, every such formula $\varphi$ represents an $f \in C_{DH}$. □

From this theorem, we see that a double Horn function can be represented by a linear read-once formula [4].

**Theorem 2** Given a Horn DNF $\varphi$, checking if $\varphi$ represents a double Horn function can be done in $O(|\varphi|)$ time. □

However, in general, we have the following negative result.

**Theorem 3** Given a DNF $\varphi$, checking if $\varphi$ represents a double Horn function is co-NP-complete. □

Now, we turn to the problems about extensions.

**Theorem 4** Given a pdBF $(T, F)$, where $T, F \subseteq \{0, 1\}^n$, finding a double Horn extension can be done in $O(n(|T| + |F|))$ time. □

**Theorem 5** Given a pdBF $(T, F)$, there is a polynomial delay algorithm for enumerating all double Horn extensions of $(T, F)$. □

### References


