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A Simple Characterization of Returns to Scale in DEA

01302170 埼玉大学 刀根 薫 TONE Kaoru

Abstract

In this paper, we will present a simple method for deciding the local returns-to-scale characteristics of DMUs (Decision Making Units) in Data Envelopment Analysis. This method proceeds as follows: first, we solve the BCC (Banker-Charnes-Cooper) model and find the returns-to-scale of BCC-efficient DMUs and a reference set to each BCC-inefficient DMU. We can then decide the local returns-to-scale characteristics of each BCC-inefficient DMU by observing only the returns-to-scale characteristics of DMUs in their respective reference sets. No extra computation is required. We can apply this method to the output oriented model and to the additive model as well.

1 The BCC Model

The production possibility set of the BCC (Banker, Charnes and Cooper, 1984) model is described as:

$$P_B = \{(\boldsymbol{x}, \boldsymbol{y}) \mid \boldsymbol{x} \geq X\boldsymbol{\lambda},\ \boldsymbol{y} \leq Y\boldsymbol{\lambda},\ e\boldsymbol{\lambda} = 1,\ \boldsymbol{\lambda} \geq 0\}. \tag{1}$$

The BCC model evaluates the efficiency of each DMU_o $(\boldsymbol{x}_o, \boldsymbol{y}_o)$ $(o = 1, \ldots, n)$ by solving the following linear program:

$$\begin{array}{rrcl}
(BCC_o) & \min & & \theta_B \\
\text{subject to} & \theta_B \boldsymbol{x}_o - X\boldsymbol{\lambda} - \boldsymbol{s}_x & = & 0 \\
& Y\boldsymbol{\lambda} - \boldsymbol{s}_y & = & \boldsymbol{y}_o \\
& e\boldsymbol{\lambda} & = & 1 \\
& \boldsymbol{\lambda} \geq 0, & \boldsymbol{s}_x \geq 0, & \boldsymbol{s}_y \geq 0.
\end{array}$$

We express the dual program of (BCC_o) as:

$$\begin{array}{rrcl}
(DBCC_o) & \max \quad z & = & \boldsymbol{u}\boldsymbol{y}_o - u_0 \\
& \text{subject to} \quad \boldsymbol{v}\boldsymbol{x}_o & = & 1 \\
& -\boldsymbol{v}X + \boldsymbol{u}Y - u_0 e & \leq & 0 \\
& \boldsymbol{v} \geq 0, \ \boldsymbol{u} & \geq & 0,
\end{array}$$

where u_0 is free in sign.
As with the CCR case, we employ the two phase process for solving (BCC_o). Let an optimal solution in Phase II be $(\theta_B^*, \boldsymbol{\lambda}^*, \boldsymbol{s}_x^*, \boldsymbol{s}_y^*)$, based on which we define BCC-efficiency as follows:

Definition 1 (BCC-Efficiency)
A DMU_o is called BCC-efficient, *if it has $\theta_B^* = 1$, $\boldsymbol{s}_x^* = \boldsymbol{0}$ and $\boldsymbol{s}_y^* = \boldsymbol{0}$. Otherwise, it is called* BCC-inefficient.

If a DMU_o is BCC-efficient, then there exists an optimal solution $(\boldsymbol{v}^*, \boldsymbol{u}^*, u_0^*)$ with $\boldsymbol{v}^* > \boldsymbol{0}$ and $\boldsymbol{u}^* > \boldsymbol{0}$.

If a DMU_o is BCC-inefficient, the *reference set* E_o^B and the *BCC-projection* $(\boldsymbol{x}_e^B, \boldsymbol{y}_e^B)$ based on the reference set, are defined as:

$$E_o^B = \{j \mid \lambda_j^* > 0,\ j = 1, \ldots, n\} \tag{2}$$

$$\boldsymbol{x}_e^B = \theta_B^* \boldsymbol{x}_o - \boldsymbol{s}_x^* = \sum_{j \in E_o^B} \lambda_j^* \boldsymbol{x}_j \tag{3}$$

$$\boldsymbol{y}_e^B = \boldsymbol{y}_o + \boldsymbol{s}_y^* = \sum_{j \in E_o^B} \lambda_j^* \boldsymbol{y}_j. \tag{4}$$

We have:

Lemma 1 *Every DMU in the reference set E_o^B is BCC-efficient.*

Lemma 2 *The BCC projected activity $(\boldsymbol{x}_e^B, \boldsymbol{y}_e^B)$ is BCC-efficient.*

2 Returns to Scale of BCC-Efficient DMUs

Banker and Thrall (1992) demonstrated the following theorem on returns-to-scale of BCC efficient DMUs.

Theorem 1 (Returns-to-Scale)
Suppose DMU_o is BCC-efficient and let the sup *and* inf *of u_0 in the optimal solution for $(DBCC_o)$ be $\bar{u}_0$ and $\underline{u}_0$, respectively. Then, we have:*

1. *If $0 > \bar{u}_0$, then increasing returns-to-scale prevail in the DMU_o.*
2. *If $\bar{u}_0 \geq 0 \geq \underline{u}_0$, then constant returns-to-scale prevail in the DMU_o.*

3. *If $\underline{u}_0 > 0$, then decreasing returns-to-scale prevail in the DMU$_o$.*

We will denote *increasing, constant* and *decreasing* returns-to-scale by IRS, CRS and DRS, respectively.

Corollary 1 *DMU$_o$ is CCR-efficient if and only if it is BCC-efficient and displays CRS.*

Usually, we solve (BCC_o) by the simplex method of linear programming and obtain an optimal dual solution $(\boldsymbol{v}^*, \boldsymbol{u}^*, u_0^*)$ as the simplex multiplier of the Phase I optimal tableau. Thus, if $u_0^* > 0$, then we need to solve the lower bound $\underline{u}_0$, and if $u_0^* < 0$, then we need to solve the upper bound $\bar{u}_0$ for deciding the returns-to-scale characteristics of the DMU. If $u_0^* = 0$, the DMU shows CRS. The computation of $\underline{u}_0$ (or $\bar{u}_0$) is carried out in the primal side of LP.

3 Characterization of RTS

Theorem 2
If a DMU $(\boldsymbol{x}_o, \boldsymbol{y}_o)$ is BCC-inefficient, the reference set E_o^B to $(\boldsymbol{x}_o, \boldsymbol{y}_o)$ defined by (2), does not include both IRS and DRS DMUs.

Corollary 2 *Let a reference set to a BCC-inefficient DMU $(\boldsymbol{x}_o, \boldsymbol{y}_o)$ be E_o^B. Then, E_o^B consists of one of the following combinations of BCC-efficient DMUs.*

- (i) *All DMUs have IRS.*
- (ii) *Mixture of DMUs with IRS and CRS.*
- (iii) *All DMUs have CRS.*
- (iv) *Mixture of DMUs with CRS and DRS.*
- (v) *All DMUs show DRS.*

Theorem 3 (Characterization of RTS)
Let the BCC-projected activity of a BCC-inefficient DMU $(\boldsymbol{x}_o, \boldsymbol{y}_o)$ be $(\boldsymbol{x}_e^B, \boldsymbol{y}_e^B)$ and the reference set to $(\boldsymbol{x}_o, \boldsymbol{y}_o)$ be E_o^B. Then, $(\boldsymbol{x}_e^B, \boldsymbol{y}_e^B)$ belongs to

1. *IRS, if E_o^B consists of DMUs in categories (i) or (ii) of Corollary 2,*
2. *CRS, if E_o^B consists of DMUs in category (iii), and*
3. *DRS, if E_o^B consists of DMUs in categories (iv) or (v).*

4 Conclusion

In this paper we presented a simple alternative method for deciding the returns-to-scale characteristics of BCC (BCCO)-projected activities. This method is 'one' pass in the sense that BCC software equipped with a procedure for deciding returns-to-scale of BCC-efficient DMUs (*e.g.* Banker and Thrall (1992)), is sufficient for this purpose. No other software is needed. Since the number of BCC-efficient DMUs is considerably less than that of BCC-inefficient ones, this method will contribute to save computation time. Also, Theorems 2, 3 and Corollary 2 contribute to the development of theory and algorithms in DEA.

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