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An affine scaling algorithm for semidefinite programming

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1 Introduction

Let us consider the set of $n \times n$ symmetric matrices $\mathcal{S}(n)$. For $X \in \mathcal{S}(n)$, we denote by $X \succ (\succeq) 0$ that X is positive (semi-)definite.

Semidefinite programming problem is an optimization problem which minimizes a linear function over an intersection of the cone $\{X \in \mathcal{S}(n) \mid X \succeq 0\}$ and an affine space $\{X \in \mathcal{S}(n) \mid \mathrm{tr}(A_i X) = b_i, i = 1, \ldots, m\}$ where $A_i \in \mathcal{S}(n)$. The standard form semidefinite programming problem can be written as follows.

$$\left\{ \begin{array}{ll} \text{minimize} & \mathrm{tr}(CX) \\ \text{subject to} & \mathrm{tr}(A_i X) = b_i, i = 1, \ldots, m, \\ & x \succeq 0, \end{array} \right. \tag{1}$$

where $C \in \mathcal{S}(n)$.

The semidefinite programming can be viewed as an extension of linear programming which minimizes a linear function over an intersection of the cone of the positive orthant and an affine space. In fact, the above statement sounds more natural if we remind that the both cones are convex, closed, pointed, and self-dual.

The semidefinite programming is now considered to be an important class of optimization problems not only because this is a reasonable extension of linear programming, but also this problem arises in a wide variety of fields such as control theory or discrete optimization[1].

It is recently that interior point approach was turned out to be theoretically and practically efficient to semidefinite programming(e.g. [3, 4]). In this talk, an affine scaling algorithm for semidefinite programming is addressed.

The affine scaling algorithm for linear programming is proposed by Dikin[2], and well known for its simplicity and efficiency. In fact, the affine scaling algorithm is the first interior point algorithm which turned out to be practically as efficient as the simplex method.

Below we introduce an affine scaling algorithm for semidefinite programming and show some preliminary propositions without proof. In the talk, a more involved argument on the property of the sequence generated by the affine scaling algorithm will be shown.

2 The affine scaling algorithm

In linear programming, we have several ways to characterize the *affine scaling direction.* Below we introduce the affine scaling direction for a semidefinite programming problem by using *dual estimate,* which plays the same role as the shadow price does for the simplex method.

Given an interior (i.e., positive definite) feasible solution X, we define the dual estimate $S(X)$ as the solution of the following optimization problem:

$$\left\{ \begin{array}{ll} \text{minimize} & \|X^{1/2} S X^{1/2}\|^2 \\ \text{subject to} & S = C - \sum_i y_i A_i, \end{array} \right.$$

where $\|X\|^2 \triangleq \sum_{i,j} X_{ij}^2$ is called *Frobenius norm.* The solution can be explicitly written as

$$S = C - \sum_i \hat{y}_i A_i, \quad \hat{y} = G^{-1}p,$$

where $G_{ij} \triangleq \mathrm{tr}(A_i X A_j X)$ and $p_j \triangleq \mathrm{tr}(A_j XCX)$. In consideration of the symmetry, the affine scaling direction ΔX should be defined by

$$\Delta X \triangleq XS(X)X = X(C - \sum_i \hat{y}_i A_i)X.$$

Proposition 1 *We have*

$$\mathrm{tr}(C\Delta X) = \|X^{1/2}SX^{1/2}\|^2,$$

i.e., the affine scaling direction is a decent direction.

We define the iteration by using ΔX.

$$X_{k+1} = X_k - \alpha \frac{X_k S_k X_k}{\|X_k^{1/2} S_k X_k^{1/2}\|}.$$

The following proposition ensures that the iteration is well-defined for all k, if $\alpha < 1$.

Proposition 2 *If* $\alpha \leq (<)1$, *then* $X_{k+1} \succeq (\succ) 0$.

If $A_i, i = 1, \ldots, m$ and C are diagonal matrices, then putting the diagonal element of A_i to $a_i \in R^n$ and C to $c \in R^n$, we have a standard form linear programming problem:

$$\begin{cases} \text{minimize} & c^t x \\ \text{subject to} & a_i^t x = b_i, \quad i = 1, \ldots, m, \\ & x \geq 0. \end{cases} \tag{2}$$

Proposition 3 $X^* \triangleq \mathrm{diag}(x^*)$ *is an optimal solution of the original semidefinite programming problem.*

Proposition 4 *Let* Δx *be the affine scaling direction for* (2) *at an interior feasible solution* $x > 0$. *Then* $\mathrm{diag}(x)$ *is feasible for* (1), *and the affine scaling direction at* $\mathrm{diag}(X)$ *is* $\mathrm{diag}(\Delta x)$.

Corollary 5 *([5]) If all* A_i, C *and* X^0 *(initial point) are diagonal, then the affine scaling algorithm for semidefinite programming converges to an optimal solution if* $\alpha \leq 2/3$.

References

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