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Mean Queue Lengths of the Alternating Traffic with Starting Delays

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1. Model Description

We analyze an alternating traffic crossing a narrow one-lane bridge on a two-lane road. Once a car begins to cross the bridge in one direction, arriving cars from the other direction must wait, forming a queue, until all the arrivals in the direction finish crossing the bridge. The queues of the right- and the left-hand side traffic will be referred to as Q_1 and Q_2, respectively. Cars arriving at Q_i will be also referred to as type-i cars. Cars arrive at the queues according to independent Poisson processes. Denote by λ_i the arrival rate at Q_i. Let B_i be the durations of a period during which the bridge is continuously occupied by type-i cars. Once a period of one type ends, cars of the other type in queue, if any, initiate a period of the other type. Each car in queue needs the starting delay τ seconds, that is, if there are k cars waiting in Q_i when $B_j (j \neq i)$ ends, it takes $k\tau$ seconds for the kth car to start crossing the bridge. In this case, if the $(k+1)$st car arrives in $k\tau$ seconds from the starting point of B_i, then the $(k+1)$st car also needs the starting delay. Once the cars start crossing the bridge, they pass through the bridge after T seconds later, a constant, being independing of the number of cars in the bridge. The arriving type-i car which finds that Q_i is empty but there are still some type-i cars in the bridge, can start crossing the bridge immediately and takes only T seconds to pass through the bridge. B_i will be extended as long as the next type-i car arrives while type-i cars are crossing the bridge. Even if there is no car in Q_i when B_j ends, the signal controls the traffic so that type-i cars have a priority at least V_i seconds, that is, even if no type-i car arrives during V_i, type-j cars can not cross the bridge during the period. If the type-i cars arrive during V_i, B_i will be extended in the same manner.

The queueing model is a modified version of the simple traffic model by Greenberg, et al.[2], which incorporates neither the signal control nor starting delays. Note that introducing the starting delays makes the model realistic but the analysis of the model would be very difficult. We analyze the queueing model and obtain the mean queue length when B_i starts, for the special case of $V_i = V_j = T$.

2. Embedded Markov Chain

First of all, we formulate the Laplace transform of B_{i,k_i}, the durations of a period during which the bridge is continuously occupied by type-i cars, conditioned that there are k_i cars waiting in Q_i when B_i starts. First, we consider the case $k_i = k > 0$. Since we have to take the starting delays into account when Q_i is not empty, we divide the period $B_{i,k}$ into two parts, i.e., the durations of period during which type-i cars are waiting in Q_i (referred to as $B_{i,k}^W$) and during which Q_i is empty but there are still some type-i cars crossing the bridge (referred to as B_i^C), that is, $B_{i,k} = B_{i,k}^W + B_i^C$. We have the Laplace transform of $B_{i,k}^W$ and B_i^C from Chatani[1] and Greenberg, et al.[2], respectively:

$$f^*_{B_{i,k}^W}(s) = \sum_{n=0}^{\infty} \frac{[\lambda_i(k+n)\tau]^{n-1}\lambda_i k\tau}{n!} e^{-(\lambda_i+s)(k+n)\tau},$$

$$f^*_{B_i^C}(s) = \frac{\lambda_i + s}{s e^{(\lambda_i+s)T} + \lambda_i}.$$

Since $B_{i,k}^W$ and B_i^C are independent each other, we have the Laplace transform of $B_{i,k}$ as

$$f^*_{B_{i,k}}(s) = f^*_{B_{i,k}^W}(s) f^*_{B_i^C}(s) \tag{1}$$

Next, we consider the case $k_i = 0$ where there is no type-i car waiting in Q_i when B_j ends. We refer to the duration as $B_{i,0}$. In this case, the signal controls the traffic so that type-i cars have a priority at least V_i seconds, that is, $B_{i,0}$ continues at least V_i seconds. If no type-i car arrives during V_i with probability $e^{-\lambda_i V_i}$, the duration of $B_{i,0}$ results in V_i:

$$P\{B_{i,0} = V_i\} = e^{-\lambda_i V_i}.$$

On the other hand, if any type-i cars arrive during V_i, B_i will be extended. The distribution of the extended part is same as B_i^C. Here, $B_{i,0}^E$ denotes the duration from the starting instant of $B_{i,0}$ to the arriving instant of the first type-i car, i.e., the period during which the bridge is empty. The Laplace transform of $B_{i,0}^E$ is easily obtained by

$$f^*_{B_{i,0}^E}(s) = \frac{\lambda_i}{\lambda_i + s}[1 - e^{-(\lambda_i+s)V_i}].$$

Combining two cases, we have the Laplace transform of $B_{i,0}$ as

$$f^*_{B_{i,0}}(s) = e^{-(\lambda_i+s)V_i} + \frac{\lambda_i[1-e^{-(\lambda_i+s)V_i}]}{\lambda_i + se^{(\lambda_i+s)T}}. \quad (2)$$

Now, we construct an infinite state embedded Markov chain. Let us consider the instants when B_1 or B_2 ends as the embedded points. Let $i(i=1,2)$ be the indicator variable that shows which duration (B_1 or B_2) starts. k_i denote the number of cars waiting in Q_i when B_i starts. Then the state (i,k_i) has Markov property. Let p_{i,k_i} denotes the steady state probability of the state (i,k_i). Since B_1 and B_2 appear exactly alternately, the state transitions such as from (i,k_i) to (i,k_i') never occur. Hence q_{i,k_i,k_j} denotes the transition probability from state (i,k_i) to (j,k_j) $(i \neq j)$. Then we have the system of linear equations as follows:

$$p_{j,k_j} = \sum_{k_i=0}^{\infty} p_{i,k_i} q_{i,k_i,k_j} \quad (i,j=1,2, i\neq j). \quad (3)$$

q_{i,k_i,k_j} can be expressed by

$$\begin{aligned} q_{i,k_i,k_j} &= \int_0^\infty \frac{(\lambda_j t)^{k_j}}{k_j!} e^{-\lambda_j t} f_{B_{i,k_i}}(t)dt \\ &= \left. \frac{(-1)^{k_j}\lambda_j^{k_j}}{k_j!} \frac{d^{k_j}}{ds^{k_j}} f^*_{B_{i,k_i}}(s) \right|_{s=\lambda_j} \quad (4) \end{aligned}$$

where $f_{B_{i,k_i}}(t)$ is the stochastic density function of B_{i,k_i}. Then we are able to calculate the transition probabilities using the Laplace transform of $f^*_{B_{i,k_i}}(s)$ in (1) and (2). In particular,

$$q_{i,0,k_j} = e^{-\lambda_i V_i} \frac{(\lambda_j V_i)^{k_j}}{k_j!} e^{-\lambda_j V_i}$$
$$+ \left. \frac{(-1)^{k_j}\lambda_j^{k_j}}{k_j!} \frac{d^{k_j}}{ds^{k_j}} \frac{\lambda_i[1-e^{-(\lambda_i+s)V_i}]}{\lambda_i + se^{(\lambda_i+s)T}} \right|_{s=\lambda_j} \quad (5)$$

Throughout the paper, the stability condition given by $\lambda_j\tau + \lambda_i\tau - 1 < 0$. is assumed to hold.

3. Mean Queue Length

Here, we analyze the mean queue lengths at the embedded points. First, using (3) we have

$$\sum_{k_j=0}^{\infty} k_j p_{j,k_j} = \sum_{k_i=0}^{\infty}\sum_{k_j=0}^{\infty} k_j q_{i,k_i,k_j} p_{i,k_i}. \quad (6)$$

For $k_i = 0$, from (5) we have

$$\sum_{k_j=0}^{\infty} k_j q_{i,0,k_j} = \lambda_j \frac{e^{\lambda_i T}-1}{\lambda_i} - \lambda_j \frac{e^{\lambda_i(T-V_i)}-1}{\lambda_i}.$$

For $k_i > 0$, from (1) and (4) we get

$$\sum_{k_j=0}^{\infty} k_j q_{i,k_i,k_j} = \lambda_j \left[\frac{k_i \tau}{1-\lambda_i\tau} + \frac{e^{\lambda_i T}-1}{\lambda_i} \right].$$

Substituting them into (6) we get

$$\sum_{k_j=0}^{\infty} k_j p_{j,k_j} = \frac{\lambda_j\tau}{1-\lambda_i\tau}\sum_{k_i=0}^{\infty} k_i p_{i,k_i}$$
$$+ \frac{e^{\lambda_i T}-1}{2\lambda_i} - \lambda_j \frac{e^{\lambda_i(T-V_i)}-1}{\lambda_i} p_{i,0}. \quad (7)$$

Note that $\sum_{k_i=0}^{\infty} p_{i,k_i} = 1/2$, since B_1 and B_2 appear exactly alternately. (7) means that we can get the mean queue length at the embedded points if we obtained $p_{1,0}$ and $p_{2,0}$.

Here, we consider the special case $V_i = V_j = T$. Then (7) is equivalent to

$$\sum_{k_j=0}^{\infty} k_j p_{j,k_j} = \frac{\lambda_j\tau}{1-\lambda_i\tau}\sum_{k_i=0}^{\infty} k_i p_{i,k_i} + \frac{e^{\lambda_i T}-1}{2\lambda_i}. \quad (8)$$

By solving (8) the mean queue length at the embedded points, L_i^T, is obtained by

$$L_i^T = 2\sum_{k_i=0}^{\infty} k_i p_{i,k_i}$$
$$= \frac{(1-\lambda_i\tau)\{\lambda_j^2\tau(e^{\lambda_i T}-1) + \lambda_i(1-\lambda_j\tau)(e^{\lambda_j T}-1)\}}{(1-\lambda_i\tau-\lambda_j\tau)\lambda_j}. \quad (9)$$

When $V_i < T$ and $V_j < T$, (9) gives the upper bounds of $L_i^{V_i}$, and the bounds become tight as $V_i, V_j \to T$. The bounds also become tight as $\lambda_i\tau + \lambda_j\tau \to 1$, because $p_{i,0}$ in (7) approaches 0 as $\lambda_i\tau + \lambda_j\tau \to 1$. Similarly, when $V_i > T$ and $V_j > T$, (9) gives the lower bounds of $L_i^{V_i}$

The mean waiting time to start crossing the bridge is more useful information for the drivers than the performance measures obtained in this paper, but it is not straightforward.

References

[1] K.Chatani, *Modeling and analysis of the alternating traffic*, Master Thesis in Dept. of Mechanical Eng., Sophia University. 1993.

[2] B.S.Greenberg, R.C.Leachman, and R.W.Wolff, "Predictin dispatching delays on a low speed single track railroad", *Transportation Science*. vol.22, pp.31-38, 1988.