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# Geometric Decay of the Steady-State Distribution in Two-Stage Tandem Queues

01302440 東京工業大学 高橋 幸雄 TAKAHASHI Yukio
01605320 東京工業大学 牧本 直樹 MAKIMOTO Naoki
02501400 東京工業大学 *藤本 衡 FUJIMOTO Kou

## 1 Introduction

Tandem queues are basic models in the queueing theory and have been studied for a long time. However, the stationary state probabilities or even basic properties of them are scarcely known except for some simple cases with product form solutions. Then numerical computation and simulations are the main tools to evaluate these models.

Recently, the importance of the asymptotic analysis of queues is increasing since the asymptotic property can be applied to evaluate very small ($\sim 10^{-10}$) loss probabilities of the queueing networks with ATM multiplexer. Although there are many results on the asymptotic behavior of the steady-state distributions of queues (see e.g., [1] and the references therein), most of them are restricted to single queues.

In this paper, we show some results on the geometric decay of the steady-stage distribution in two-stage tandem queues via quasi-birth-death (QBD) processes with countable number of phases.

## 2 Model description

We consider a two-stage tandem queueing system $PH/PH/1 \to /PH/1$ with the ordinary FIFO queuing discipline. The $k$th stage ($k = 1, 2$) has a single server and a buffer of infinite capacity, so that no loss or blocking occurs. Interarrival times of customers are independent and identically distributed random variables subjecting to a phase-type distribution $(\boldsymbol{\alpha}, \boldsymbol{T})$. Service times at the $k$th stage are also i.i.d. variables subjecting to a phase-type distribution $(\boldsymbol{\beta}_k, \boldsymbol{S}_k)$. The interarrival times and the service times are assumed to be mutually independent. The state of the system is represented by a quintuple $(n_1, n_2; i_0, i_1, i_2)$, where $i_0$ is the phase of the arrival process, $i_k$ is the phase of the service process at the $k$th stage, and $n_k$ is the number of customers in the $k$th stage. Then the system behaves as a QBD process with countable number of phases if we appropriately rearrange the states.

The asymptotic analysis of $GI/PH/c$ is based on the theory of matrix-geometric form solutions of the QBD process [3]. This theory was extended to the QBD process with countable number of phases [2], [4]. In what follows, we first provide some asymptotic properties of the QBD process with countable number of phases and then show the geometric decay of the stationary distribution of the tandem queue.

## 3 QBD with countable number of phases

We consider a time-continuous Markov chain $\{X(t)\}$ on the state space $\mathcal{S} = \{(n, i) \mid n, i \in N\}$. We define the level $k$ of the state space by $\mathcal{L}_k = \{(k, i) \mid (k, i) \in \mathcal{S}\}$. Let $\boldsymbol{Q}$ be the infinitesimal generator of $\{X(t)\}$ with lexicographic order of states. We assume that $\{X(t)\}$ is ergodic, and that $\boldsymbol{Q}$ is of block-tridiagonal form:

$$\boldsymbol{Q} = \begin{pmatrix} \boldsymbol{B}_0 & \boldsymbol{A}_0 & & & \\ \boldsymbol{C}_1 & \boldsymbol{B} & \boldsymbol{A} & & \\ & \boldsymbol{C} & \boldsymbol{B} & \boldsymbol{A} & \\ & & \boldsymbol{C} & \boldsymbol{B} & \ddots \\ & & & \ddots & \ddots \end{pmatrix}. \quad (1)$$

In this case, there exists a stationary probability vector $\boldsymbol{\pi}$. Moreover, if we divide $\boldsymbol{\pi} = (\boldsymbol{\pi}_0\ \boldsymbol{\pi}_1\ \cdots)$ according to $\mathcal{L}_k$, then $\boldsymbol{\pi}_n$ satisfies

$$\boldsymbol{\pi}_n = \boldsymbol{\pi}_1 \boldsymbol{R}^{n-1}$$

where $\boldsymbol{R}$, called *rate matrix*, is the minimal nonnegative matrix which satisfies $\boldsymbol{A} + \boldsymbol{R}\boldsymbol{B} + \boldsymbol{R}^2\boldsymbol{C} = \boldsymbol{O}$.

**Theorem 3.1.** Suppose that there exist a positive constant $\eta (< 1)$ and positive vectors $\boldsymbol{x}_\eta$ and $\boldsymbol{y}_\eta$ such that

$$\boldsymbol{x}_\eta(\boldsymbol{A} + \eta\boldsymbol{B} + \eta^2\boldsymbol{C}) = \mathbf{o}, \quad (2)$$
$$(\boldsymbol{A} + \eta\boldsymbol{B} + \eta^2\boldsymbol{C})\boldsymbol{y}_\eta = \mathbf{o}, \quad (3)$$

$$x_\eta y_\eta < \infty, \quad x_\eta e < \infty. \tag{4}$$

Then, under some mild conditions,

1. $x_\eta R = \eta x_\eta$.
2. $y = (\eta^{-1}A - G)y_\eta$ is a non-zero, non-negative vector such that $Ry = \eta y$.
3. $x_\eta y < \infty$, and hence $R$ is $\eta$-positive.

**Theorem 3.2.** Under the assumptions of Theorem 3.1, $\pi$ has a geometric tail

$$\pi_n \sim C\eta^n x_\eta \quad (n \to \infty).$$

## 4 Geometric tail of the steady-state distribution in tandem queues

We rearrange the states of the Markov chain derived from the tandem queues as $(n_2, n_1, i_0, i_1, i_2)$ and define the level by $\mathcal{L}_k = \{(n_2, n_1, i_0, i_1, i_2) : n_2 = k\}$. Then the infinitesimal generator $Q$ of this chain has a block-tridiagonal form as in (1). If the traffic intensities $\rho_1$ and $\rho_2$ of the first and the second stages are less than 1, the chain is ergodic and hence the steady-state probability $\pi$ exists and has the matrix-geometric form $\pi_{n_2} = \pi_1 R^{n_2-1}$.

We show that there exist $\eta$, $x_\eta$ and $y_\eta$ which satisfy the conditions in Theorem 3.1 as follows. Define $\eta_1$ and $\eta_2$ by

$$\eta_2 = \{S_2^*(\omega_2)\}^{-1}, \quad \eta_1 = T^*(\omega_0),$$

where $T^*(s)$ and $S_k^*(s)$ $(k = 1, 2)$ are the Laplace-Stieltjes Transforms of the interarrival and the $k$th stage's service time distributions, and $(\omega_0, \omega_2)$ is the non-zero solution $(s_0, s_2)$ of the equation

$$\begin{cases} T^*(-s_2)S_2^*(s_2) &= 1, \\ T^*(s_0)S_1^*(-s_0 - s_2)S_2^*(s_2) &= 1. \end{cases}$$

We also define the column vector

$$y_{\eta_2} = \begin{pmatrix} v_0 \otimes v_2 \\ \eta_2^{-1} v_0 \otimes e_1 \otimes v_2 \\ \eta_2^{-2} v_0 \otimes e_1 \otimes v_2 \\ \vdots \end{pmatrix},$$

where $v_0 = -(-\omega_2 I_0 - T)^{-1} T e_0$ and $v_2 = -(\omega_2 I_2 - S_2)^{-1} S_2 e_2$. Then $y_{\eta_2}$ satisfies (3).

Let $D = diag(y_{\eta_2})$, and consider the matrix

$$K = D^{-1}(A + \eta_2 B + \eta_2^2 C)D.$$

Since $(A + \eta_2 B + \eta_2^2 C)$ has a block-tridiagonal form, it can be seen that $K$ is an infinitesimal generator of a QBD process which is ergodic iff $\eta_1 < \eta_2$. Then if we assume $\eta_1 < \eta_2$ there exists a positive vector $x$ such that $xK = \mathbf{o}$ and $xe = 1$, and $x_{\eta_2} = xD^{-1}$ satisfies (2) and (4). Therefore, the following theorem is proved.

**Theorem 4.1.** If $\eta_1 < \eta_2$, then the steady-state probability vector $\pi = (\pi_0 \ \pi_1 \ \cdots)$ has a geometric tail

$$\pi_{n_2} \sim C_1 \eta_2^{n_2} x_{\eta_2} \quad (n_2 \to \infty).$$

If we divide the vector $x_{\eta_2}$ into subvectors according to $n_1$ as $x_{\eta_2} = (x_{\eta_2}(0) \quad x_{\eta_2}(1) \quad \cdots)$, then $x_{\eta_2}$ has a matrix geometric form and a geometric tail

$$x_{\eta_2}(n_1) \sim C\eta_1^{n_1} \widehat{x} \quad (n_1 \to \infty),$$

where

$$\begin{aligned} \widehat{x} = \ & [\alpha(\omega_0 I_0 - T)^{-1} diag(v_0)^{-1}] \\ & \otimes \beta_1\{-(\omega_0 + \omega_2)I_1 - S_1\}^{-1} \\ & \otimes [\beta_2(\omega_2 I_2 - S_2)^{-1} diag(v_2)^{-1}]. \end{aligned}$$

## References


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