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On Generalizing Divisor Method for the Apportionment Problem

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1. Introduction

Given the set of N political constituencies as $S = \{1, 2, \ldots, N\}$, the population of political constituency $i \in S$ as p_i, the total population as P, and the total number of seats as K, the "ideal" number of seats allocated to the constituency i, *i.e.*, the "exact quota" q_i, is given by $q_i = \frac{p_i K}{P}, i \in S$ where $P = \sum_{i \in S} p_i$. Then the apportionment problem is to partition a given positive integer K into nonnegative integral parts $\{d_i \mid i \in S\}$ such that $\sum_{i \in S} d_i = K;\ \ d_i \geq 0,\ \ integer,\ i \in S$ and such that these parts are "as near as possible" proportional, respectively, to a set of nonnegative integers $\{p_1, p_2, \ldots, p_N\}$, *i.e.*, $\{q_1, q_2, \ldots, q_N\}$.

2. $GPDM$ and global optimization criterion

Let $v(d)$ be a monotone increasing function defined for all integers $d \geq 0$ and also satisfying $d \leq v(d) \leq d + 1$. Then we define the rounding process for the divisor λ by $[\frac{p_i}{\lambda}]_r = d_i \ \ i \in S$ where $v(d_i - 1) < \frac{p_i}{\lambda} \leq v(d_i)$ $i \in S$. Defining the rank function $r(p_i, d_i)$ as $r(p_i, d_i) = \frac{p_i}{v(d_i)},\ i \in S$, then we can write the above relation as

$$\max_{d_i \geq 0} r(p_i, d_i) \leq \min_{d_j > 0} r(p_j, d_j - 1)$$

Based upon different divisor functions we can define an infinite number of different divisor methods (see *e.g.*, [1,2,3,4]). There are five traditional divisor methods as well as the parametric divisor method (PDM). Using a parameter t_0 such that $0 \leq t_0 \leq 1$, the divisor function of the parametric divisor method (PDM) can be written as $v_{PD}(d, t_0) = d + t_0$. We generalise the parametric divisor method as follows, which we call general parametric divisor method ($GPDM$).

$$v_{GPD}(d_i; t_0, t_1, t_2) = d_i^{t_1}(d_i + t_0)^{t_2}$$

where $t_1 + t_2 = 1, 0 \leq t_0, t_1, t_2 \leq 1$

Then traditional divisor methods including PDM and $GPDM$, and their corresponding global optimization criteria can be summarized as in the table below.

Method	(t_0, t_1, t_2)	$\mathrm{Min}_d \mathrm{Max}_{j \in S}$
GDM	(1, 0, 1)	$\frac{p_j}{d_j+1}$
MFM	($\frac{1}{2}$, 0, 1)	$\frac{p_j}{d_j+1/2}$
EPM	($1, \frac{1}{2}, \frac{1}{2}$)	$\frac{p_j}{\sqrt{d_j(d_j+1)}}$
SDM	(0, 0, 1)	$\frac{p_j}{d_j}$
PDM	(*, 0, 1)	$\frac{p_j}{d_j+t_0}$
$GPDM$	(*, *, *)	$\frac{p_j}{d_j^{t_1}(d_j+t_0)^{t_2}}$

(* : arbitrary)

3. Local measure of inequity

We denote the measure of inequity between two constituencies i and j as $E(p_i, d_i; p_j, d_j)$. Then Huntington's rule says that we should transfer a seat from a more favored constituency i to a less favored constituency j when it brings a smaller measure of inequity. So the "desirable apportionment" is obtained when no switching of seats between constituencies can improve the measure of inequity between any such pair of constituencies. The attainment of this state is referred to as a stable assignment of seats.

Based upon the definition of the local measure of inequity in Oyama['91] $GPDM$'s local measure can be defined as follows.

$$E_{GPD}(p_i, d_i; p_j, d_j; t_0, t_1, t_2) =$$

$$\frac{1}{2}\log\frac{p_i}{p_j} - t_1 \log d_j + t_2 \log(d_i + t_0 - 1)$$

For the $GPDM$ we change Huntington's transfer rule such that we should transfer a seat from i to j when $\frac{(d_i+t_0-1)^{\frac{t_2}{2}}}{p_i} \geq \frac{(d_j+t)^{\frac{t_1}{2}}}{p_j}$ and the relation $E_{GPD}(p_i, d_i; p_j, d_j; t_0, t_1, t_2) > E_{GPD}(p_i, d_i - 1; p_j, d_j + 1; t_0, t_1, t_2)$ holds for all pairs i and j with i favored over j. Then we obtain the following theorem.

Theorem 1 *For the pair of constituencies i and j with populations p_i and p_j, apportionments d_i and d_j, respectively, the following holds.*

$$E_{GPD}(p_i, d_i; p_j, d_j; t) \leq$$
$$E_{GPD}(p_i, d_i - 1; p_j, d_j + 1; t) \qquad i, j \in S$$

if and only if

$$\frac{p_j}{d_j^{t_1}(d_j + t_0)^{t_2}} \leq \frac{p_i}{(d_i - 1)^{t_1}(d_i + t_0 - 1)^{t_2}}$$

4. Combined apportionment methods

We consider to combine two traditional divisor methods so that the linear combination of two global optimization criteria can be optimized. First we define the criteria function $c(p_i, d_i)$ such that the convergence criteria can be expressed as

$$\max_{d_i \geq 0} c(p_i, d_i) \leq \min_{d_j > 0} c(p_j, d_j - 1)$$

Then for the traditional divisor method the criterion function can be given by the corresponding rank function. The final allocation of seats among constituencies can be given by $\{d_i\}$'s such that the above convergence criterion can be satisfied. For example, we know that MFM and EPM minimise $\sum_i p_i(\frac{d_i}{p_i} - r)^2$ and $\sum_i d_i(\frac{p_i}{d_i} - s)^2$, where $r = \frac{K}{P}$ and $s = \frac{P}{K}$, respectively. Namely, MFM and EPM minimise $z_m = \sum_i \frac{d_i^2}{p_i}$ and $z_e = \sum_i \frac{p_i^2}{d_i}$, respectively. Defining the combined global optimal criterion as $z = c_m z_m + c_e z_e$, where we assume $c_m + c_e = 1, 0 \leq c_m, c_e \leq 1$. Then we obtain the following theorem.

Theorem 2 *Let the criterion function be*

$$c(p_i, d_i; c_m, c_e) = -c_m \frac{2d_j + 1}{d_j} + c_e \frac{p_j^2}{d_j(d_j + 1)}$$

Then the allocation of seats based upon the criterion function $c(p_i, d_i; c_m, c_e)$ is optimal with respect to the global optimal criterion $z = c_e z_e + c_e z_e$.

For other combinations of two traditional divisor methods, we can obtain similar results to the above theorem.

5. Summary

In conclusion, we believe that the method LFM, which of course satisfies the quota property, gives "a most reasonable and impartial" assignment of seats to the constituency although it does not satisfy the house monotone property. We try to find an appropriate "divisor" method giving the allocation mostly satisfying quota property. We also believe that both $GPDM$ and combined divisor methods can contribute to this target. Currently we are are trying to find an apportionment method whose allocation mostly "guarantees" quota property.

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