The Existence of Equilibrium in Symmetric Arbitration Games FOA and DOA

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1 Introduction

A classical existence theorem of a mixed-strategy Nash equilibrium requires the continuity of payoff functions and the compactness of strategy sets. Dasgupta and Maskin (1986) made a great improvement on the requirement of continuity. Recently, Méndez-Naya (1996) gives some existence results for a zero-sum game without the compactness of strategy sets but the continuity of payoff function is required there.

Arbitration games FOA and DOA originate from arbitration procedures final-offer arbitration (FOA) and double-offer arbitration (DOA), see Zeng et al. (1996). To study an arbitration game, it is usually supposed that the arbitrator has a notion z_a of fair settlement. However, two players have only incomplete information about z_a, therefore they estimate z_a by a probability distribution. Furthermore, two players are supposed to share the common estimate with distribution function $A(z_a)$. Hence the game becomes a constant-sum game. We call an arbitration game *symmetric* if $A(m+z_a)+A(m-z_a)=1$, where m is the mean. This paper only studies symmetric arbitration games, hence in the following we sometimes omit the word 'symmetric'.

In arbitration games FOA and DOA, neither the results of Dasgupta and Maskin (1986) nor the results of Méndez-Naya (1996) are directly applicable to show the existence of equilibrium. This paper first gives an existence result for such general games, and then applies the result to arbitration games FOA and DOA. It is shown that if the players' estimate about z_a is symmetric and whose support is bounded, then a (mixed-strategy) Nash equilibrium exists in both games FOA and DOA. Furthermore, in game FOA the offers in the equilibrium do not converge, but in game DOA the offers in the equilibrium converge. The results extend the main result of Zeng et al. (1996) from another viewpoint.

2 An Existence Result

A mixed-strategy is defined as a distribution function in Médez-Naya (1996). Although only a distribution function over interval such as $[a,b)$ is defined there, the definition can be easily extended to distribution functions over other kinds of intervals. Based on the concepts, a game can be defined as follows.

Definition 2.1 Let I_1, I_2 be two real intervals. We call system $\Gamma=(C(I_1),C(I_2),K)$ a *zero-sum game* over $I_1\times I_2$, where $C(I_1)$ is the first player's mixed-strategy set, $C(I_2)$ is the second player's mixed-strategy set and K is the expected payoff to the first player given by

$$K(\mu_1,\mu_2)=\int_{I_1}\int_{I_2}k(x,y)d\mu_1(x)d\mu_2(y)$$

for all $(\mu_1,\mu_2)\in C(I_1)\times C(I_2)$, where k is a function from $I_1\times I_2$ to $\mathcal{R}$.

We call game $\Gamma_{I_1^0,I_2^0}=(C(I_1^0),C(I_2^0),K)$ a *restricted game* of $\Gamma=(C(I_1),C(I_2),K)$ if intervals $I_1^0\subseteq I_1$ and $I_2^0\subseteq I_2$.

In game Γ, we define

$$\overline{V}=\inf_{\mu_2\in C(I_2)}\sup_{\mu_1\in C(I_1)}K(\mu_1,\mu_2)$$
$$\underline{V}=\sup_{\mu_1\in C(I_1)}\inf_{\mu_2\in C(I_2)}K(\mu_1,\mu_2).$$

Definition 2.2 We say that game Γ *has a value* V if $\overline{V}=\underline{V}=V$. The first (resp. second) player has an *optimal strategy* μ_1^* (*resp.* μ_2^*) if

$$\inf_{\mu_2\in C(I_2)}K(\mu_1^*,\mu_2)=V$$
$$(\text{resp.}\quad \sup_{\mu_1\in C(I_1)}K(\mu_1,\mu_2^*)=V).$$

We provide one existence result which will be applied to games FOA and DOA later.

Lemma 2.1 *Suppose that in zero-sum game* Γ,
1) *there exist closed intervals,* $I_1^0 \subseteq I_1$ *and* $I_2^0 \subseteq I_2$ *such that for all* $\mu_2 \in C(I_2)$, *there is an* $x_{\mu_2} \in I_1^0$ *such that* $K(x_{\mu_2}, \mu_2) \geq V$; *for all* $\mu_1 \in C(I_1)$, *there is a* $y_{\mu_1} \in I_2^0$ *such that* $K(\mu_1, y_{\mu_1}) \leq V$;
2) *restricted games* $\Gamma_{I_1^0, I_2}$ *and* Γ_{I_1, I_2^0} *have values, furthermore, the first player has an optimal strategy* μ_1^* *in* $\Gamma_{I_1^0, I_2}$ *and the second player has an optimal strategy* μ_2^* *in* Γ_{I_1, I_2^0},
then the game Γ *has value* V, *and both players have optimal strategies* μ_1^* *and* μ_2^* *respectively.*

3 Arbitration Game FOA

Without loss of generality, we suppose that the mean of the estimate is 0. The game is denoted as Γ^{FOA}. Suppose that there is a finite positive number z_a^m such that $A(z_a^m) = 1$ and $A(-z_a^m) = 0$, where $A(z_a)$ is the probability distribution function of z_a. In other words, the support of $A(\cdot)$ is bounded. Under FOA, disputants s and b give offers x_s and x_b respectively. If $x_s \leq x_b$, then the arbitration result is $(x_s + x_b)/2$; If $x_s > x_b$, the arbitrator compares the distance between two offers with his/her z_a. The disputant with closer offer wins and his/her offer becomes the arbitration result.

Theorem 1 *Game* Γ^{FOA} *has value 0, and there exists an atomless symmetric equilibrium* (μ^*, μ^*) *in game* Γ^{FOA}. *Furthermore, the support of* μ^* *is contained in* $[0, 5z_a^m]$.

The above theorem shows that, in the symmetric Nash equilibrium, the probability that two players make the same zero-offer is zero, therefore the probability that two offers converge is zero.

As a special case, Kilgour (1994) gives an expression for the optimal strategies when $A(z_a)$ is a discrete distribution which takes values 1 and -1 with probability 1/2. Theorem 1 indicates that the support of the mixed strategy should be containted in $[0, 5]$. The support of the mixed strategy of Kilgour is actually $[\sqrt{5} - 2, \sqrt{5} + 2] \subset [0, 5]$.

4 Arbitration Game DOA

Zeng et al. (1996) propose a new procedure double-offer arbitration (DOA) to improve FOA. Under DOA, disputants s and b give double offers (x_s, y_s) and $(-x_b, -y_b)$, respectively. The offers x_s and $-x_b$ are called *primary*, while y_s and $-y_b$ are called *secondary*. Based on these, the arbitrator, considering z_a as a fair settlement, determines the arbitration settlement as follows. If the primary offers converge ($x_s \leq -x_b$), we have a settlement $(x_s - x_b)/2$; if not, but the secondary offers converge ($x_s > -x_b$ and $y_s \leq -y_b$), the arbitrator provides a settlement $(y_s - y_b)/2$; if neither converges, then the arbitrator evaluates the offers by the following criterion functions for s and b,

$$C_s(x_s, y_s | z_a) = \alpha |y_s - x_s| + (1 - \alpha)(y_s - z_a)$$

$$C_b(x_b, y_b | z_a) = \alpha |y_b - x_b| + (1 - \alpha)(z_a + y_b),$$

where $\alpha \in (0, 1/2)$ is a constant to be determined and announced in advance by the arbitrator. The disputant with smaller criterion function value wins and his/her offer is chosen by the arbitrator.

We now construct an auxiliary game DOA^0, in which although two players' secondary offers converge at 0 (i.e., $y_s = y_b = 0$), the arbitrator chooses x_s or $-x_b$ as the final result according to the criterion functions. We can show that there is a symmetric equilibrium (λ^*, λ^*) in game DOA^0 in the above two cases, furthermore, the support of λ^* is contained in $[0, \infty)$. The main conclusion is as follows.

Theorem 2 *Game DOA has value* 0, *and both players have optimal strategies* $\mu^* = (\lambda^*, 0)$.

The secondary offers in the Nash equilibrium of Theorem 2 converge. Comparing with the conclusion of Section 3, we know that arbitration procedure DOA is better than FOA, in the sense of offers' convergence.

References

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