Application of the BD-based Diffusion Approximation to General Closed Queueing Networks

01105381 北海道大学 木村 俊一 KIMURA TOSHIKAZU

1 Introduction

Consider a closed queueing network in equilibrium with M service stations and N customers, in which we assume that

1. the routing probability p_{ij} that a customer leaving station i enters station j is independent of the state of the system ($i, j = 1, \ldots, M$);
2. customers are served under the first-come first-served discipline at all stations;
3. service times of customers at station i are iid with a general distribution and independent of the arrival process at station i ($i = 1, \ldots, M$);
4. each station has multiple identical services in parallel and a local buffer with unlimited capacity.

The system is further specified by the following notations: Let F_i be the service-time cdf with finite mean μ_i^{-1} and let c_i^2 be the squared coefficients of variation (*i.e.*, variance divided by the square of mean) of F_i ($i = 1, \ldots, M$). Let N_i denote the number of customers at station i and let $p_i(n) = P(N_i = n)$ ($i = 1, \ldots, M$, $n = 0, \ldots, N$). The problem we focus on here is to obtain all of the marginal distributions $\{p_i(n)\}$.

For the case of exponential service times, this problem is so classical that some efficient computational algorithms have been developed; see, *e.g.*, Bruell and Balbo [1]. For the general service-time case, however, the problem becomes quite difficult to solve. In addition, some studies indicated that the classical exponential network will *not* yield satisfactory results for typical performance measures in the non-exponential network. This is the reason why we need approximations.

Among numerous work to this direction, Marie [3] developed a very accurate approximation, which is often called the *exponentialization approach*. The idea of his approach is to transform the original network into an approximately equivalent exponential network, where each station has exponential service times with state-dependent rates. Yao and Buzacott [4] applied Marie's approximation to networks with limited local buffers and/or dynamic routing. Although our approximation also can be extended to such general networks, we restrict ourselves to the most basic case (*i.e.*, 1–4 above and single-server stations) for the ease of presentation.

2 The Exponentialization Approach

Let $\nu_i(n)$ be 'equivalent' service rate of customers at station i in the equivalent exponential network when $N_i = n$ ($i = 1, \ldots, M$, $n = 0, \ldots, N$), where $\nu_i(0) = 0$ for all i. Let $p_i^*(n)$ denote the marginal probability of having n customers at station i in the exponential network characterized by the set of service rates $\{\nu_i(n)\}$. Obviously, the success of the exponentialization approach strongly depends on how to approximate $\{\nu_i(n)\}$ for which $p_i(n) = p_i^*(n)$ ($i = 1, \ldots, M$, $n = 0, \ldots, N$). The essentials in the approaches of Marie [3] and Yao and Buzacott [4] can be summarized as the following algorithm:

Step 0 For $i = 1, \ldots, M$ and $n = 0, \ldots, N$, set $\nu_i(n) := \mu_i$.

Step 1 Solve the exponential network characterized by the set of service rates $\{\nu_i(n)\}$ and the routing matrix $P = (p_{ij})$ to obtain the marginal distribution $\{p_i^*(n)\}$.

For $i = 1, \ldots, M$, set

$$\lambda_i(n) := \begin{cases} 0, & n = N \\ \dfrac{p_i^*(n+1)}{p_i^*(n)} \nu_i(n+1), & 0 \le n \le N-1. \end{cases}$$

Step 2 For $i = 1, \ldots, M$, analyze station i as an isolated $M(n)/G/1/N$ queue having Poisson arrivals with state-dependent arrival rates $\{\lambda_i(n)\}$ and the service-time cdf F_i, obtaining its steady-state distribution $\{\pi_i(n); n = 0, \ldots, N\}$ as an approximation for $\{p_i(n)\}$.

Step 3 For a given error bound $\varepsilon > 0$, if $\max_{i,n} |p_i^*(n) - \pi_i(n)| < \varepsilon$, then $p_i(n) := \pi_i(n)$ for all i and n, and stop; otherwise set

$$\nu_i(n) := \begin{cases} 0, & n = 0 \\ \dfrac{\pi_i(n-1)}{\pi_i(n)} \lambda_i(n-1), & 1 \le n \le N, \end{cases}$$

for all i, and go to Step 1.

In each iteration of Step 2, Marie [3] and Yao and Buzacott [4] proposed to calculate $\{\pi_i(n)\}$ exactly by using the Coxian service-time distribution. Clearly, this increases the computational time significantly. In this paper, we will simplify the calculation in Step 2 by using the BD (birth-and-death)-based diffusion approximation for the $M(n)/G/1/N$ queue, which has recently been developed by Kimura [2]. The simplification makes the exponentialization approach be more tractable as a quick modeling tool for performance evaluation.

3 The BD-based Diffusion Approximation

Following Kimura [2], we provide an approximation for the distribution $\{\pi_i(n)\}$: For $n = 1, \ldots, N$, let

$$a_i(n) = \lambda_i(n) + \mu_i c_i^2,$$
$$a_i^*(n) = \lambda_i(n) + \mu_i,$$
$$\alpha_i(n) = \frac{a_i^*(n)}{a_i(n)},$$
$$\gamma_i(n) = \left(\frac{a_i^*(n)}{a_i^*(n-1)} \frac{\lambda_i(n-1)}{\mu_i} \right)^{\alpha_i(n)},$$
$$\xi_i(n) = \frac{1}{a_i(n)} \prod_{j=1}^{n} \gamma_i(j).$$

Then, the BD-based diffusion approximation for $\{\pi_i(n)\}$ is given by

$$\pi_i(n) = \begin{cases} \dfrac{\sum_{j=1}^{N} \{\mu_i - \lambda_i(j)\} \xi_i(j)}{\sum_{j=1}^{N} \{\mu_i + \lambda_i(0) - \lambda_i(j)\} \xi_i(j)}, & n = 0 \\ \dfrac{\lambda_i(0)\xi_i(n)}{\sum_{j=1}^{N} \{\mu_i + \lambda_i(0) - \lambda_i(j)\} \xi_i(j)}, & 1 \le n \le N. \end{cases}$$

Remark 1 Instead of the point discretization in Kimura [2], we have used a modified discretization method based on a rate conservation law. Check that our approximation satisfies the rate conservation condition

$$\sum_{n=0}^{N-1} \lambda_i(n)\pi_i(n) = (1 - \pi_i(0)) \mu_i, \quad 1 \le i \le M.$$

References

[1] Bruell, S.C. and Balbo, G., *Computational Algorithms for Closed Queueing Networks*, North-Holland, New York, 1980.

[2] Kimura, T., "Birth-and-Death-Based Diffusion Approximation for Queues," Proceedings of the Symposium on Performance Models for Information Communication Networks, Kyoto, pp. 289–300, 1996.

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[4] Yao, D.D. and Buzacott, J.A., "The Exponentialization Approach to Flexible Manufacturing System Models with General Processing Times," *European Journal of Operational Research*, **24** (1986) 410–416.