Application of the BD-based Diffusion Approximation to General Closed Queueing Networks

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1 Introduction

Consider a closed queuing network in equilibrium with $M$ service stations and $N$ customers, in which we assume that

1. the routing probability $p_{ij}$ that a customer leaving station $i$ enters station $j$ is independent of the state of the system ($i,j = 1,\ldots,M$);
2. customers are served under the first-come first-served discipline at all stations;
3. service times of customers at station $i$ are iid with a general distribution and independent of the arrival process at station $i$ ($i = 1,\ldots,M$);
4. each station has multiple identical services in parallel and a local buffer with unlimited capacity.

The system is further specified by the following notations: Let $F_i$ be the service-time cdf with finite mean $\mu_i^{-1}$ and let $c_i^2$ be the squared coefficients of variation (i.e., variance divided by the square of mean) of $F_i$ ($i = 1,\ldots,M$). Let $N_i$ denote the number of customers at station $i$ and let $p_i(n) = P(N_i = n)$ ($i = 1,\ldots,M$, $n = 0,\ldots,N$). The problem we focus on here is to obtain all of the marginal distributions $\{p_i(n)\}$.

For the case of exponential service times, this problem is so classical that some efficient computational algorithms have been developed; see, e.g., Bruell and Balbo [1]. For the general service-time case, however, the problem becomes quite difficult to solve. In addition, some studies indicated that the classical exponential network will not yield satisfactory results for typical performance measures in the non-exponential network. This is the reason why we need approximations.

Among numerous work to this direction, Marie [3] developed a very accurate approximation, which is often called the exponentialization approach. The idea of his approach is to transform the original network into an approximately equivalent exponential network, where each station has exponential service times with state-dependent rates. Yao and Buzacott [4] applied Marie’s approximation to networks with limited local buffers and/or dynamic routing. Although our approximation also can be extended to such general networks, we restrict ourselves to the most basic case (i.e., 1–4 above and single-server stations) for the ease of presentation.

2 The Exponentialization Approach

Let $\nu_i(n)$ be ‘equivalent’ service rate of customers at station $i$ in the equivalent exponential network when $N_i = n$ ($i = 1,\ldots,M$, $n = 0,\ldots,N$), where $\nu_i(0) = 0$ for all $i$. Let $p_i^*(n)$ denote the marginal probability of having $n$ customers at station $i$ in the exponential network characterized by the set of service rates $\{\nu_i(n)\}$. Obviously, the success of the exponentialization approach strongly depends on how to approximate $\{\nu_i(n)\}$ for which $p_i(n) = p_i^*(n)$ ($i = 1,\ldots,M$, $n = 0,\ldots,N$). The essentials in the approaches of Marie [3] and Yao and Buzacott [4] can be summarized as the following algorithm:

Step 0 For $i = 1,\ldots,M$ and $n = 0,\ldots,N$, set $\nu_i(n) := \mu_i$.

Step 1 Solve the exponential network characterized by the set of service rates $\{\nu_i(n)\}$ and the routing matrix $P = (p_{ij})$ to obtain...
the marginal distribution \( \{p_i^*(n)\} \).

For \( i = 1, \ldots, M \), set
\[
\lambda_i(n) := \begin{cases} 
0, & n = N \\
p_i^*(n+1) \nu_i(n+1), & 0 \leq n \leq N-1.
\end{cases}
\]

**Step 2** For \( i = 1, \ldots, M \), analyze station \( i \) as an isolated \( M(n)/G/1/N \) queue having Poisson arrivals with state-dependent arrival rates \( \{\lambda_i(n)\} \) and the service-time cdf \( F_i \), obtaining its steady-state distribution \( \{\pi_i(n); n = 0, \ldots, N\} \) as an approximation for \( \{p_i(n)\} \).

**Step 3** For a given error bound \( \varepsilon > 0 \), if \( \max_{i,n} |p_i^*(n) - \pi_i(n)| < \varepsilon \), then \( p_i(n) := \pi_i(n) \) for all \( i \) and \( n \), and stop; otherwise set
\[
\nu_i(n) := \begin{cases} 
0, & n = 0 \\
\pi_i(n-1) \lambda_i(n-1), & 1 \leq n \leq N,
\end{cases}
\]
for all \( i \), and go to Step 1.

In each iteration of Step 2, Marie [3] and Yao and Buzacott [4] proposed to calculate \( \{\pi_i(n)\} \) exactly by using the Coxian service-time distribution. Clearly, this increases the computational time significantly. In this paper, we will simplify the calculation in Step 2 by using the BD (birth-and-death)-based diffusion approximation for the \( M(n)/G/1/N \) queue, which has recently been developed by Kimura [2]. The simplification makes the exponentialization approach be more tractable as a quick modeling tool for performance evaluation.

3 The BD-based Diffusion Approximation

Following Kimura [2], we provide an approximation for the distribution \( \{\pi_i(n)\} \): For \( n = 1, \ldots, N \), let
\[
\gamma_i(n) = \left( \frac{a_i^*(n)}{a_i^*(n-1)} \mu_i \right)^{a_i(n)} \\
\xi_i(n) = \frac{1}{a_i(n)} \prod_{j=1}^n \gamma_i(j).
\]

Then, the BD-based diffusion approximation for \( \{\pi_i(n)\} \) is given by
\[
\pi_i(n) = \begin{cases} 
\frac{\sum_{j=1}^N \{\mu_i - \lambda_i(j)\} \xi_i(j)}{\sum_{j=1}^N \{\mu_i + \lambda_i(0) - \lambda_i(j)\} \xi_i(j)}, & n = 0 \\
\frac{\sum_{j=1}^N \{\mu_i + \lambda_i(0) - \lambda_i(j)\} \xi_i(j)}{\sum_{j=1}^N \{\mu_i + \lambda_i(0) - \lambda_i(j)\} \xi_i(j)}, & 1 \leq n \leq N.
\end{cases}
\]

**Remark 1** Instead of the point discretization in Kimura [2], we have used a modified discretization method based on a rate conservation law. Check that our approximation satisfies the rate conservation condition
\[
\sum_{n=0}^{N-1} \lambda_i(n) \pi_i(n) = (1 - \pi_i(0)) \mu_i, \quad 1 \leq i \leq M.
\]

**References**


