Augmenting edge-connectivity and vertex-connectivity simultaneously

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1 Introduction

Let $G = (V, E)$ stand for an undirected multigraph with a set of V of *vertices* and a set E of *edges*, where an edge with end vertices u and v is denoted by (u, v). For two disjoint subsets X, $Y \subset V$, we denote by $E_G(X, Y)$ the set of edges, one of whose end vertices is in X and the other is in Y, and by $c_G(X, Y)$ the number of edges in $E_G(X, Y)$. In particular, $E_G(u, v)$ implies the set of edges with end vertices u and v, and $c_G(u, v) = |E_G(u, v)|$. We denote $n = |V|$, $e = |E|$. For a vertex set $X \subset V$, $\{v \in V \mid (u, v) \in E$ for some $u \in X\}$ is called a neighbor set of X, denoted by $\Gamma_G(X)$. A *cut* is defined as a subset X of V with $\emptyset \neq X \neq V$, and the *size* of cut X is denoted by $c_G(X, V - X)$, which may also be written as $c_G(X)$. If $X = \{x\}$, $c_G(x)$ denotes the degree of vertex x. A cut with the minimum size is called a (*global*) *minimum cut*, and its size, denoted by $\lambda(G)$, is called the *edge-connectivity* of G. The local edge-connectivity $\lambda_G(x, y)$ for two vertices x, $y \in V$ is defined to be the minimum size of a cut in G that separates x and y. It is known that there are $\lambda_G(x, y)$ edge-disjoint paths in G for every $x, y \in V$ [1]. A *separator* is defined as a subset S of V with $\emptyset \neq S \neq V$ such that the removal of S in G leaves at least two components. If $G \neq K_n$, a separator with the minimum size is called a (*global*) *minimum separator*, and its size, denoted by $\kappa(G)$, is called the *vertex-connectivity* of G. If $G = K_n$, define $\kappa(G) = n - 1$. The local vertex-connectivity $\kappa_G(x, y)$ for two vertices x, $y \in V$ is defined to be the number of vertex-disjoint paths between x and y, and $\kappa(G) = \min\{\kappa_G(x, y) | x, y \in V\}$ holds. For a separator S, β_S denotes the number of components in $G - S$. Let $\beta(G) := \max\{\beta_S : S$ is a minimum separator in G $\}$.

Given (i) a multigraph $G = (V, E)$, (ii) the demand function $r_1(x, y) \in Z^+$ (Z^+: the set of nonnegative integers) for each $x, y \in V$, (iii) the demand function $r_2(x, y) \in Z^+$ for each x, $y \in V$, the *edge and vertex-connectivities augmentation problem*, denoted by $EVAP$ asks to augment G by adding the smallest number of new edges to G so that the resulting graph G' satisfies $\lambda_{G'}(x, y) \geq r_1(x, y)$ for each $x, y \in V$ and $\kappa_{G'}(x, y) \geq r_2(x, y)$ for each $x, y \in V$. When the demand functions r_1 and r_2 satisfy $r_1(x, y) = k \in Z^+$ for each $x, y \in V$ and $r_2(x, y) = l \in Z^+$ for each $x, y \in V$ respectively, this problem is denoted by $EVAD(k, l)$. Without loss of generality, $k \geq l$ is assumed. When $r_2(x, y) = 0$ holds for each $x, y \in V$, this problem implies the *edge-connectivity augmentation problem.* When $r_1(x, y) = 0$ holds for each $x, y \in V$, this problem implies the *vertex-connectivity augmentation problem.*

2 Definitions

A *partition* $X_1, \cdots, X_t$ of vertex set V means a family of disjoint nonempty subsets of V whose union is V, and a *subpartition* of V means a partition of a subset of V.

2.1 Edge-Splitting

We introduce a tool that is helpful to solve the edge-connectivity augmentation problem, called *edge-splitting.*

Given a multigraph $G = (V, E)$, a designated vertex $s \in V$, vertices $u, v \in \Gamma_G(s)$, and a nonnegative integer $\delta \leq \min\{c_G(s, u), c_G(s, v)\}$, we construct graph $G' = (V, E')$ from G by deleting δ edges from $E_G(s, u)$ and $E_G(s, v)$, respectively, and adding new δ edges to $E_G(u, v)$. That is to say, G' satisfies $c_{G'}(s, u) := c_G(s, u) - \delta$, $c_{G'}(s, v) := c_G(s, v) - \delta$, $c_{G'}(u, v) := c_G(u, v) + \delta$, and $c_{G'}(x, y) := c_G(x, y)$ for all other pairs $x, y \in V$. A splitting is *complete* if G' does not have any neighbor of s.

Theorem 2.1 [2] *Let $G = (V, E)$ be a multigraph with a designated vertex $s \in V$ with $c_G(s) \neq 1, 3$ and $\lambda_G(x, y) \geq 2$ for every pair $x, y \in V - s$. Then for each edge $(s, u) \in E$ there is edge $(s, v) \in E$ such that $\lambda_{G'}(x, y) \geq k$ for for every pair $x, y \in$*

$V - s$ where G' is the resulting graph obtained by splitting (s,u) and (s,v). □

This says that if $c_G(s)$ is even, there is always a sequence of pairs of edges, whose splitting together give rise to a complete feasible splitting at s.

3 The EVAP(k,2)

In this section, we show that the EVAP(k,2) with an arbitrary k can be solved in polynomial time when the input graph is connected. In case of the input graph is not connected, the EVAP(k,2) can be solved in the same manner, but for simplicity, we assume that the input graph is connected. The EVAP(k, 2) when the input graph is connected is the following:

Input: A connected multigraph $G = (V, E)$
and an integer $k \geq 2$,
Output: $G' = (V, E \cup F)$ where F is the smallest number of edge set with
$\lambda(G') \geq k$ and
$\kappa(G') \geq 2$. □

To solve EVAP(k,2), it is necessary to add at least $k - c_G(X)$ edges to $E_G(X, V - X)$ for each $\emptyset \neq X \subset V$, to add at least $2 - |\Gamma_G(X)|$ edges to $E_G(X, V - X)$ for each $\emptyset \neq X, V - X - \Gamma_G(X) \subset V$, and to add at least $b_S - 1$ edges to connect components of $G - S$ for each separator S. Hence a lower bound of the number that must be added to G to solve this problem is presented as follows:

The Lower Bound $\gamma(G)$:
$\gamma(G) = \max\{\lceil \alpha(G)/2 \rceil, \beta(G) - 1\}$ where

$$\alpha(G) = \max\{\sum_{i=1}^{p}(k - c_G(X_i)) + \sum_{i=p+1}^{q}(2 - |\Gamma_G(X_i)|)\},$$

$\{X_1, \cdots, X_q\}$ is a subpartition of V with $\emptyset \neq V - X_i - \Gamma_G(X_i)$, $i = p+1, \cdots, q$ □

We show that the following main theorem holds.

Theorem 3.1 *Given a connected graph G and an integer $k \geq 2$, G can be made k-edge-connected and biconnected by adding $\gamma(G)$ new edges.* □

Before proving this theorem, we will show some lemmas.

Lemma 3.1 *Let $G = (V, E)$ be a multigraph. We can augment G by adding a new vertex s and $\alpha(G)$ new edges between s and V so that the resulting graph $G_1 = (V \cup \{s\}, E \cup F_1)$ satisfies $\lambda_{G_1}(x,y) \geq k$ and $\kappa_{G_1}(x,y) \geq 2$ for each $x, y \in V$, where F_1 denotes added the set of new edges.* □

Lemma 3.2 *Let $G = (V, E)$ and $G_2 = (V \cup \{s\}, E \cup F)$, where $F = E_{G_2}(s, V)$, and assume that (i) $\lambda_{G_2}(x,y) \geq k$ and (ii) $\kappa_{G_2}(x,y) \geq 2$ hold for every $x, y \in V$. If $c_{G_2}(s)$ is even and $\lceil \alpha(G)/2 \rceil \geq \beta(G) - 1$ holds, then there is a complete splitting at s without violating (i) and (ii).* □

The Proof of Theorem 3.1
Case-(1) $\lceil \alpha(G)/2 \rceil \geq \beta(G) - 1$
From Lemma 3.1, we can obtain $G_1 = (V \cup \{s\}, E \cup F_1)$ with $\lambda_{G_1}(x,y) \geq k$ and $\kappa_{G_1}(x,y) \geq 2$ for each $x, y \in V$ and $|F_1| = \alpha(G)$. From Lemma 3.2, we can obtain $G^* = (V, E \cup F^*)$ with $\lambda(G^*) \geq k$ and $\kappa(G^*) \geq 2$ by a complete splitting at s where edge set F^* is obtained by the splitting.
Case-(2) $\lceil \alpha(G)/2 \rceil < \beta(G) - 1$
Similarly to Case-(1), we first obtain $G_1 = (V \cup \{s\}, E \cup F_1)$ with $\lambda_{G_1}(x,y) \geq k$ and $\kappa_{G_1}(x,y) \geq 2$ for each $x, y \in V$ and $|F_1| = \alpha(G)$. From the proof of Lemma 3.2 (details are omitted), we can show that there is a complete splitting at s so that $\lambda(G^* \cup F') \geq k$ and $\kappa(G^* \cup F') \geq 2$ hold where $G^* = (V, E \cup F^*)$ is the resulting graph obtained by a complete splitting at s and F' is a new edge set with $|F'| = \beta(G) - 1 - \lceil \alpha(G)/2 \rceil$.

From above, we can show that EVAP(k,2) can be solved in polynomial time. □

Remarks Even if the input graph is not connected, the EVAP(k,2) can be solved in polynomial time similarly to the above procedure. Moreover, even if the demand function $r_1(x,y)$ for edge-connectivity is given individually for each pair $x, y \in V$ instead of $r_1(x,y) = k$ for every pair $x, y \in V$, this edge and vertex-connectivities augmentation problem can be solved in polynomial time, similarly to the above procedure. □

References

[1] L. R. Ford and D. R. Fulkerson, *Flows in Networks*, Princeton University Press, Princeton, N. J., 1962.

[2] W. Mader, *A reduction method for edge-connectivity in graphs*, Ann. Discrete Math., Vol.3, 1978, pp. 145-164.