Polynomial-Time Convergence of Predictor-Corrector Infeasible-Interior-Point Algorithms for Monotone SDLCP: Generalization and Inexact Approach

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1 Introduction

Monotone Semidefinite Linear Complementarity Problems (SDLCPs) give fundamental mathematical frameworks including various problems, such as Linear Programming Problems, Quadratic Convex Programming Problems, Semidefinite Programming Problems and monotone Linear Complementarity Problems.

Recently, many authors have discussed the generalization of path-following interior-point algorithms for LP and monotone LCP to the context of SDP and monotone SDLCP. Several search directions have been proposed for the SDP and monotone SDLCP, such as the AHO [1], the HRVW-KSH-M [2, 3, 4] and the NT search directions [8, 9]. To unify the above directions, different families of search directions have been proposed, such as the KSH [3], the MZ [7, 11], the Tseng [10] and the MT [6] families of search directions.

We propose the predictor-corrector infeasible-interior-point algorithms using more general choice of search directions (which contains all proposed search directions such as in the KSH, the MZ, the Tseng and the MT families) and show their polynomial-time convergence.

2 Admissible Search Directions

Let $\mathcal{S}$ denote the set of $n \times n$ real symmetric matrices, and $\mathcal{S}_+$ (and $\mathcal{S}_{++}$) the set of positive semidefinite (positive definite, respectively) matrices. We denote the standard inner product $\mathbf{Tr}\ \boldsymbol{XY}$ of $\boldsymbol{X}, \boldsymbol{Y}$ in $\mathcal{S}$ by $\boldsymbol{X} \bullet \boldsymbol{Y}$. Let $\mathcal{F}$ be an $\frac{n(n+1)}{2}$-dimensional affine subspace of $\mathcal{S} \times \mathcal{S}$ such that $(\boldsymbol{X} - \boldsymbol{X}') \bullet (\boldsymbol{Y} - \boldsymbol{Y}') \geq 0$ if $(\boldsymbol{X}, \boldsymbol{Y}), (\boldsymbol{X}', \boldsymbol{Y}') \in \mathcal{F}$, *i.e.*, the set $\mathcal{F}$ be a maximal monotone affine subspace of $\mathcal{S} \times \mathcal{S}$. We treat of Monotone Semidefinite Linear Complementarity Problem (SDLCP):

$$\text{Find } (\boldsymbol{X}, \boldsymbol{Y}) \in \mathcal{S}_+ \times \mathcal{S}_+, \quad \text{such that } (\boldsymbol{X}, \boldsymbol{Y}) \in \mathcal{F}, \boldsymbol{X} \bullet \boldsymbol{Y} = 0. \tag{1}$$

Let $(\boldsymbol{X}^0, \boldsymbol{Y}^0) \in \mathcal{S}_{++} \times \mathcal{S}_{++}$ be an initial point, $\mu_0 \equiv \boldsymbol{X}^0 \bullet \boldsymbol{Y}^0 / n > 0$ and $(\boldsymbol{X}', \boldsymbol{Y}') \in \mathcal{F}$. Let

$$\begin{aligned} \mathcal{F}(\theta) &= \mathcal{F} + \theta\left((\boldsymbol{X}^0 - \boldsymbol{X}', \boldsymbol{Y}^0 - \boldsymbol{X}')\right), \\ \mathcal{N}(\gamma, \theta) &= \left\{(\boldsymbol{X}, \boldsymbol{Y}) \in \mathcal{F}(\theta) \cap (\mathcal{S}_+ \times \mathcal{S}_+) : \left\|\boldsymbol{X}^{\frac{1}{2}}\boldsymbol{Y}\boldsymbol{X}^{\frac{1}{2}} - \theta\mu_0\boldsymbol{I}\right\|_F \leq \gamma\theta\mu_0\right\}. \end{aligned}$$

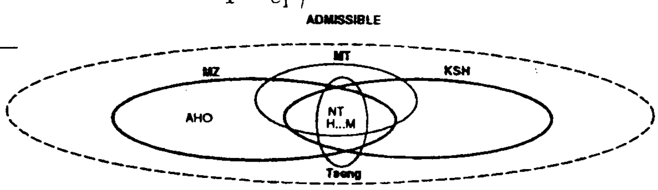
Let $(\boldsymbol{X}, \boldsymbol{Y}) \in \mathcal{N}(\gamma, \theta)$. Define $\psi = \begin{cases} 0 & \text{if } (\boldsymbol{X}^0, \boldsymbol{Y}^0) \text{ is feasible,} \\ 1 & \text{if } (\boldsymbol{X}^0, \boldsymbol{Y}^0) \text{ is infeasible.} \end{cases}$ Let $0 \leq \kappa < 1$ and $0 \leq \lambda$ be fixed. Let $\gamma \in [0, 1)$, $\theta \in (0, 1]$, $\beta \in [0, 1]$. We say that $(d\boldsymbol{X}, d\boldsymbol{Y})$ is an "admissible search direction" at $(\boldsymbol{X}, \boldsymbol{Y}) \in \mathcal{N}(\gamma, \theta)$, if

$$\left.\begin{aligned} &(\boldsymbol{X} + d\boldsymbol{X}, \boldsymbol{Y} + d\boldsymbol{Y}) \in \mathcal{F} + \beta\psi\theta\left((\boldsymbol{X}^0, \boldsymbol{Y}^0) - (\boldsymbol{X}'.\boldsymbol{Y}')\right), \\ &\left\|\frac{1}{2}\left[\boldsymbol{X}^{-\frac{1}{2}}(\boldsymbol{X}d\boldsymbol{Y} + d\boldsymbol{X}\boldsymbol{Y})\boldsymbol{X}^{\frac{1}{2}} + \boldsymbol{X}^{\frac{1}{2}}(d\boldsymbol{Y}\boldsymbol{X} + \boldsymbol{Y}d\boldsymbol{X})\boldsymbol{X}^{-\frac{1}{2}}\right] - \beta\theta\mu_0\boldsymbol{I} + \boldsymbol{X}^{\frac{1}{2}}\boldsymbol{Y}\boldsymbol{X}^{\frac{1}{2}}\right\|_F \\ &\qquad \leq \left[\kappa(\gamma + (1-\beta)\sqrt{n}) + \lambda(1-\beta)\sqrt{n}\right]\theta\mu_0. \end{aligned}\right\} \tag{2}$$

The admissible search direction has the following remarkable properties:

- If $\kappa = \lambda = 0$, an admissible search direction is unique and equal to the HRVW-KSH-M search direction. Hence for $\kappa > 0$ (and $\lambda \geq 0$), an admissible search direction can be seen as an inexact search direction of the HRVW-KSH-M search direction.
- The admissible search direction is a generalization of all proposed search directions such as the KSH, the MZ, the Tseng and the MT families of search directions with $\kappa = \frac{c_2\gamma}{1 - c_1\gamma}$, where c_1, c_2 are constants:

	γ	c_1	c_2	
KSH	≤ 1	1	$1 + 2\sqrt{2}$	[6]
MZ	$\leq 1/(1 + 2\sqrt{2})$	1	$2\sqrt{2}$	[7]
Tseng	≤ 1	1	2	[10]
MT	$\leq 1/\sqrt{2}$	$\sqrt{2}$	$2\sqrt{2}$	[5]



By taking a small γ, we can choose a parameter $\kappa < 1$. since the parameter κ can be chosen as almost linear as γ,

For the sake of simplicity, we assume that the solution of the SDLCP (1) exists.

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3 Predictor-Corrector Algorithms

In this section, we propose the predictor-corrector infeasible-interior-point algorithm, and show the polynomial-time convergence property of both algorithms.

Algorithm 3.1. [Predictor-Corrector Infeasible-Interior-Point Algorithm]

Step 0 Choose an accuracy parameter $\epsilon > 0$, a neighborhood parameter $\gamma \in (0,1)$ and an approximate parameter $\kappa \in [0,1)$ satisfying $\tau \equiv \kappa + \frac{(\kappa+1)^2\gamma}{2(1-\gamma)}\sqrt{\frac{1+\gamma}{1-\gamma}} < 1$. Let an initial point $(\boldsymbol{X}^0, \boldsymbol{Y}^0) \in \mathcal{N}(\tau\gamma, 1)$ and $\theta_0 = 1$. Set $k = 0$.

Step 1 (Predictor Step): Compute an admissible search direction $(d\boldsymbol{X}^k, d\boldsymbol{Y}^k)$ with $\beta = 0$ and $(\boldsymbol{X}, \boldsymbol{Y}) = (\boldsymbol{X}^k, \boldsymbol{Y}^k)$. Let $\alpha_k \equiv \max\left\{\alpha \in [0,1] : (\boldsymbol{X}^k(\alpha'), \boldsymbol{Y}^k(\alpha')) \in \mathcal{N}((1-\alpha')\theta_k, \gamma), \forall \alpha' \in [0,\alpha]\right\}$;

Let $(\hat{\boldsymbol{X}}^k, \hat{\boldsymbol{Y}}^k) \equiv (\boldsymbol{X}^k, \boldsymbol{Y}^k) + \alpha_k(d\boldsymbol{X}^k, d\boldsymbol{Y}^k)$ and $\theta_{k+1} = (1-\alpha_k)\theta_k$. If $\theta_{k+1} \leq \epsilon$ then stop.

Step 2 (Corrector Step): Compute an admissible search direction $(\widehat{d\boldsymbol{X}}^k, \widehat{d\boldsymbol{Y}}^k)$ with $\beta = 1$ and $(\boldsymbol{X}, \boldsymbol{Y}) = (\hat{\boldsymbol{X}}^k, \hat{\boldsymbol{Y}}^k)$; Set $(\boldsymbol{X}^{k+1}, \boldsymbol{Y}^{k+1}) \equiv (\hat{\boldsymbol{X}}^k, \hat{\boldsymbol{Y}}^k) + (\widehat{d\boldsymbol{X}}^k, \widehat{d\boldsymbol{Y}}^k)$. Replace k by $k+1$. Go to Step 1. ∎

Theorem 3.2. *Let $(\boldsymbol{X}^0, \boldsymbol{Y}^0) \in \mathcal{N}(\tau\gamma, 1)$, $\kappa \in [0,1)$ and $\lambda \geq 0$ satisfying the condition in Step 0. Then, for every $k \geq 0$, $(\boldsymbol{X}^k, \boldsymbol{Y}^k) \in \mathcal{N}(\tau\gamma, \theta_k)$ and $(\hat{\boldsymbol{X}}^k, \hat{\boldsymbol{Y}}^k) \in \mathcal{N}(\gamma, \theta_k)$ and*

$$\theta_{k+1} \leq (1-\bar{\alpha})^k\theta_0 (= (1-\bar{\alpha})^k), \quad \hat{\boldsymbol{X}}^k \bullet \hat{\boldsymbol{Y}}^k \leq (1+\gamma)(1-\bar{\alpha})^k\mu_0 n,$$

where $\bar{\alpha} = 1/O(n)$. Algorithm 3.1 terminates in at most $O\left(n \log \frac{1}{\epsilon}\right)$ iterations. ∎

Remark 3.3. If the initial point $(\boldsymbol{X}^0, \boldsymbol{Y}^0)$ is strictly feasible, Algorithm 3.1 terminates in at most $O\left(\sqrt{n}\log\frac{1}{\epsilon}\right)$ iterations by taking $\psi = 0$ and $\alpha = \delta/\sqrt{n}$ with sufficiently small positive δ. ∎

Superlinear Convergence of Predictor-Corrector Algorithm

We consider the superlinear convergence of Algorithm 3.1. We assume that the strictly complementary solution $(\boldsymbol{X}^*, \boldsymbol{Y}^*)$ of the SDLCP exists. Then, by taking the neighborhood parameter γ as $o(1)$ and the approximate parameters κ and λ as $o(\gamma)$, we can conclude the superlinear convergence using the admissible search directions. Note that we may need more than one iteration in the corrector step to take the neighborhood parameter γ as in order $o(1)$.

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