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Optimal Stopping in a Continuous-Time Dynamic Fuzzy System.

01702986 北九州大学 吉田祐治 YOSHIDA Yuji

In the discrete-time case, we introduced fuzzy rewards in a fuzzy decision process (see Kurano et al. [2]), and we also discussed an optimal stopping problem with fuzzy rewards (see Yoshida [3]). This talk deals with an optimal stopping problem with fuzzy rewards in a continuous-time dynamic fuzzy system introduced by Yoshida [4].

Let E be a convex compact subset of some Banach space, and let ρ denote the Hausdorff metric on the set of all closed subsets of E. We deal with fuzzy sets on E whose membership functions $\tilde{s} : E \mapsto [0,1]$ are upper semi-continuous and satisfy the normality condition: $\sup_{x\in E}\tilde{s}(x) = 1$. Then $\mathcal{F}_c(E)$ denotes the set of all convex fuzzy sets. For a fuzzy set $\tilde{s} \in \mathcal{F}_c(E)$, its α-cut is defined by $\tilde{s}_\alpha := \{x \in E \mid \tilde{s}(x) \geq \alpha\}$ $(\alpha \in (0,1])$ and $\tilde{s}_0 := \mathrm{cl}\{x \in E \mid \tilde{s}(x) > 0\}$, where cl means the closure of a set. Let $\tilde{g}$ be an upper semi-continuous convex fuzzy relation $\tilde{g} : E \times E \mapsto [0,1]$ satisfying a certain normality condition in [4], and let $\{\tilde{q}^t\}_{t\geq 0}$ be the family of fuzzy relations on $E \times E$ which is given by Yoshida [4]. Then, $\{\tilde{q}^t\}_{t\geq 0}$ has the following properties (i) and (ii):

(i) $\tilde{q}^{t+r}(x,y) = \sup_{z\in E}\{\tilde{q}^t(x,z) \wedge \tilde{q}^r(z,y)\}, \quad x,y \in E,\ t,r \geq 0.$

(ii) $\tilde{q}^0(x,y) = \begin{cases} 1 & \text{if } x = y \\ 0 & \text{if } x \neq y, \end{cases} \quad x,y \in E.$

For an initial fuzzy state $\tilde{s}_0 = \tilde{s} \in \mathcal{F}_c(E)$, we define a family of fuzzy states $\{\tilde{s}_t\}_{t\geq 0}$ by

$$\tilde{s}_t(y) := \sup_{x\in E}\{\tilde{s}(x) \wedge \tilde{q}^t(x,y)\}, \quad y \in E \quad \text{for } t \geq 0. \tag{1}$$

Assumption C. It holds that

$$\lim_{t\downarrow 0}\sup_{x\in E}\sup_{\alpha\in(0,1]}\rho\left(\tilde{q}^t_\alpha(x), \{x\}\right) = 0,$$

where $\tilde{q}^t_\alpha(x) := \{y \in E \mid \tilde{q}^t(x,y) \geq \alpha\},\ x \in E,\ t \geq 0,\ \alpha \in (0,1]$.

We suppose Assumption C throughout this paper. First, we define fuzzy rewards by fuzzy numbers. Next, we estimate fuzzy rewards by a fuzzy expectation and formulate an optimal stopping problem. Let $\mathbf{R}$ be the set of all real numbers. In this paper, upper-semicontinuous convex fuzzy sets on $\mathbf{R}$ with the normality condition and a compact support are called fuzzy numbers. $\mathcal{R}$ denotes the set of all fuzzy numbers, and $\mathcal{I}$ denotes the set of all bounded closed sub-intervals of $\mathbf{R}$.

We define a partial order $\succeq$ on $\mathcal{R}$: Let $\tilde{a}, \tilde{b} \in \mathcal{R}$. $\tilde{a} \succeq \tilde{b}$ means that

$$\tilde{a}^-_\alpha \geq \tilde{b}^-_\alpha \quad \text{and} \quad \tilde{a}^+_\alpha \geq \tilde{b}^+_\alpha \quad \text{for all } \alpha \in [0,1].$$

Then $(\mathcal{R}, \succeq)$ becomes a lattice, and $\succeq$ is called the fuzzy max order. We give fuzzy rewards by the following maps from fuzzy states to fuzzy numbers. We denote by $\mathcal{F}(E : \mathcal{R})$ the family of all maps $\tilde{f} : \mathcal{F}_c(E) \mapsto \mathcal{R}$, which are called fuzzy-number-valued functions on $\mathcal{F}_c(E)$.

Now, we formulate an optimal stopping problem. Throughout this paper, we fix a constant $\lambda > 0$, and we let $\tilde{r}, \tilde{c} \in \mathcal{F}(E : \mathcal{R})$ be bounded and satisfy Lipschitz conditions. Discounted fuzzy rewards with stopping times $\tau \in [0, \infty)$ are given by

$$\tilde{u}(\tilde{s}, \tau) := \int_0^\tau e^{-\lambda t} \tilde{r}(\tilde{s}_t)\, dt + e^{-\lambda\tau} \tilde{c}(\tilde{s}_\tau) \tag{2}$$

for initial fuzzy states $\tilde{s} \in \mathcal{F}_c(E)$, where $\{\tilde{s}_t\}_{t\geq 0}$ is defined by (1) and the integral is defind by one-dimensional Aumann integral at each $\alpha \in [0,1]$ (see [1]), and for $\tau = \infty$ we also define

$$\tilde{u}(\tilde{s}, \infty) := \int_0^\infty e^{-\lambda t} \tilde{r}(\tilde{s}_t)\, dt \tag{3}$$

for initial fuzzy states $\tilde{s} \in \mathcal{F}_c(E)$. Then, $\tilde{u}(\cdot, \tau) \in \mathcal{F}(E : \mathcal{R})$ for $\tau \in [0, \infty]$. Put a fuzzy goal by a fuzzy set $\varphi : \mathbf{R} \mapsto [0,1]$ which is a continuous and nondecreasing function with $\lim_{z\to-\infty} \varphi(z) = 0$ and $\lim_{z\to\infty} \varphi(z) = 1$. Then we note that $\varphi_\alpha = [\varphi_\alpha^-, \infty)$ for $\alpha \in (0,1)$. In this paper, we discuss the following optimal stopping problem.

Problem 1. Maximize

$$\tilde{E}(\tilde{u}(\tilde{s}, \tau)) := ⨍_{\mathbf{R}} \tilde{u}(\tilde{s}, \tau)(z)\, \mathrm{d}\tilde{P}(z) = \sup_{z\in\mathbf{R}} \{\tilde{u}(\tilde{s}, \tau)(z) \wedge \varphi(z)\} \tag{4}$$

over all $\tau \geq 0$, where $\tilde{P}$ is the possibility measure generated by the density φ and $⨍ d\tilde{P}$ denotes Sugeno integral.

In Problem 1, the fuzzy expectation implies the degree of satisfaction of discounted fuzzy rewards, and the fuzzy goal $\varphi(z)$ means a kind of utility function for fuzzy payoffs z in (4). We define a fuzzy reward $\tilde{v}(\tilde{s})$ by

$$\tilde{v}(\tilde{s}) := \bigvee_{\tau\geq 0} \tilde{u}(\tilde{s}, \tau) = \bigvee_{\tau\geq 0} \left\{ \int_0^\tau e^{-\lambda t} \tilde{r}(\tilde{s}_t)\, dt + e^{-\lambda\tau} \tilde{c}(\tilde{s}_\tau) \right\} \tag{5}$$

for $\tilde{s} \in \mathcal{F}_c(E)$, where $\vee$ means the supremum with respect to the fuzzy max order $\succeq$.

Theorem 1. *We define a stopping time*

$$\tau^* := \inf\{\tau \geq 0 \mid \tilde{v}(\tilde{s}_\tau)^+_{\alpha^*} = \tilde{c}(\tilde{s}_\tau)^+_{\alpha^*}\},$$

where $\inf \emptyset := +\infty$ *and* $\alpha^* := \sup\{\alpha \in [0,1] | \varphi_\alpha^- \leq \tilde{v}(\tilde{s})^+_\alpha\}$. *Then, the following (i) and (ii) hold:*

(i) *If* $\tau^* < \infty$, *then* τ^* *is an optimal stopping time.*

(ii) *If* $\tau^* = \infty$, *then* $\tilde{E}(\tilde{v}(\tilde{s})) = \tilde{E}(\tilde{u}(\tilde{s}, \infty))$.

References

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