# A Slacks-Based Measure of Efficiency in DEA

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## 1 Introduction

One of the drawbacks of $\theta^*$ (the scaler measure of efficiency) in the CCR-type models is the lack of consideration to the slacks, i.e. the input surplus $s_x^*$ and the output shortage $s_y^*$. If two DMUs $A$ and $B$ have the same $\theta^*$ , $A$ has larger slacks than $B$ and the slacks are factors to be accounted for in evaluationg efficiency, then it is reasonable to judge that $A$ is inferior to $B$. Attempts to reflect this slacks-based factor to $\theta^*$ will be well deserving of special attention.

On the other hand, the Additive model deals directly with input surplus and output shortage. Although this model can descriminate efficient and inefficient DMUs by the existence of slacks, it has no means to gauge the depth of inefficiency similar to the $\theta^*$ in the CCR-type models.

In this paper, we will introduce a scalar measure that unifies both factors and demonstrate its compatibility with other measures and its potential applicability to actual problems.

## 2 A Slacks-based Measure

In designing such a scalar measure, the following properties are considered as important:

1. (P1) Unit-invariant
2. (P2) Monotone in slacks

In an effort to estimate the efficiency of a DMU $(\boldsymbol{x}_o,\ \boldsymbol{y}_o)$, we formulate the following fractional program in $\lambda$, $s_x$ and $s_y$.

$$
\begin{aligned}
\text{[SBM]} \qquad \min \rho &= \frac{1-\frac{1}{m}\sum_{i=1}^{m} s_{xi}/x_{io}}{1+\frac{1}{s}\sum_{i=1}^{s} s_{yi}/y_{io}} \quad (1)\\
\text{subject to } \boldsymbol{x}_o &= X\lambda + s_x\\
\boldsymbol{y}_o &= Y\lambda - s_y\\
\lambda &\geq \mathbb{0},\ s_x \geq \mathbb{0},\ s_y \geq \mathbb{0}.
\end{aligned}
$$

In this model, we assume that $X \geq O$. If $x_{io} = 0$, then we delete the term$s_{xi}/x_{io}$ in the objective function. If $y_{io} \leq 0$, then we replace it by a very small positive number so that the term $s_{yi}/y_{io}$ plays a role of penalty. We can impose a constraint on $\lambda$ such as $e\lambda = 1$, corresponding to the basic Additive model, if the model belongs to variable returns-to-scale. The above formulation has the same production possiblity set as the CCR model (constant returns-to-scale).

From the conditions $X \geq$O and $\lambda \geq \mathbb{0}$, it holds

$$\boldsymbol{x}_o \geq s_x. \qquad (2)$$

It can be verified that the objective function value $\rho$ satisfies the properties (P1) (unit-invariant) and (P2) (monotone). Furthermore, from (2), it holds

$$0 < \rho \leq 1. \qquad (3)$$

## 3 Interpretation of SBM as Product of Input and Output Inefficiencies

The formula for $\rho$ in (1) can be transformed into

$$\rho = (\frac{1}{m}\sum_{i=1}^{m}\frac{x_{io}-s_{xi}}{x_{io}})(\frac{1}{s}\sum_{i=1}^{s}\frac{y_{io}+s_{yi}}{y_{io}})^{-1}.$$

The ratio $(x_{io} - s_{xi})/x_{io}$ evaluates the relative reduction rate of input $i$ and therefore the first term corresponds to the mean reduction rate of inputs or *input inefficiency.* Similarly, in the second term, the ratio $(y_{io}+s_{yi})/y_{io}$ evaluates the relative expansion rate of output $i$ and $(1/s)\sum(y_{io} + s_{yi})/y_{io}$ is the mean expansion rate of outputs. Its inverse, the second term, measures *output inefficiency.* Thus, SBM $\rho$ can be interpreted as the product of input and output inefficiencies. Further, we have the theorem:

**Theorem 1** *If DMU A dominates DMU B, i.e., $\boldsymbol{x}_A \leq \boldsymbol{x}_B$ and $\boldsymbol{y}_A \geq \boldsymbol{y}_B$, then it holds that $\rho_A^* \geq \rho_B^*$.*

## 4 Algorithm for Solving SBM

[SBM] can be transformed into the program below by introducing a positive scalar variable $t$. (See

Charnes and Cooper (1962).)

$$
\begin{aligned}
\text{[SBMt]} \qquad \min \tau &= t - \frac{1}{m}\sum_{i=1}^{m} ts_{xi}/x_{io} \qquad (4)\\
\text{subject to } 1 &= t + \frac{1}{s}\sum_{i=1}^{s} ts_{yi}/y_{io}\\
\boldsymbol{x}_o &= X\boldsymbol{\lambda} + \boldsymbol{s}_x\\
\boldsymbol{y}_o &= Y\boldsymbol{\lambda} - \boldsymbol{s}_y\\
\boldsymbol{\lambda} &\geq \mathbf{0},\ \boldsymbol{s}_x \geq \mathbf{0},\ \boldsymbol{s}_y \geq \mathbf{0},\ t > 0.
\end{aligned}
$$

Now, let us define

$$\boldsymbol{S}_x = t\boldsymbol{s}_x, \boldsymbol{S}_y = t\boldsymbol{s}_y, \text{and } \boldsymbol{\Lambda} = t\boldsymbol{\lambda}.$$

Then, [SBMt] becomes to the following linear program in $t$, $\boldsymbol{S}_x$, $\boldsymbol{S}_y$, and $\boldsymbol{\Lambda}$:

$$
\begin{aligned}
\text{[LP]} \qquad \min \tau &= t - \frac{1}{m}\sum_{i=1}^{m} S_{xi}/x_{io} \qquad (5)\\
\text{subject to } 1 &= t + \frac{1}{s}\sum_{i=1}^{s} S_{yi}/y_{io}\\
t\boldsymbol{x}_o &= X\boldsymbol{\Lambda} + \boldsymbol{S}_x\\
t\boldsymbol{y}_o &= Y\boldsymbol{\Lambda} - \boldsymbol{S}_y\\
\boldsymbol{\Lambda} &\geq \mathbf{0},\ \boldsymbol{S}_x \geq \mathbf{0},\ \boldsymbol{S}_y \geq \mathbf{0},\ t > 0.
\end{aligned}
$$

Let an optimal solution of [LP] be

$$(\tau^*, t^*, \boldsymbol{\Lambda}^*, \boldsymbol{S}_x^*, \boldsymbol{S}_y^*).$$

Then, we have an optimal solution of [SBM] as defined by,

$$\rho^* = \tau^*,\ \boldsymbol{\lambda}^* = \boldsymbol{\Lambda}^*/t^*,\ \boldsymbol{s}_x^* = \boldsymbol{S}_x^*/t^*,\ \boldsymbol{s}_y^* = \boldsymbol{S}_y^*/t^*. \qquad (6)$$

Based on this optimal solution, we decide a DMU as *SBM-efficient* as follows:

**Definition 1 (SBM-efficient)** *A DMU* $(\boldsymbol{x}_o, \boldsymbol{y}_o)$ *is SBM-efficient, if* $\rho^* = 1$.

For an SBM inefficient DMU $(\boldsymbol{x}_o, \boldsymbol{y}_o)$, we have the expression:

$$
\begin{aligned}
\boldsymbol{x}_o &= X\boldsymbol{\lambda}^* + \boldsymbol{s}_x^*\\
\boldsymbol{y}_o &= Y\boldsymbol{\lambda}^* - \boldsymbol{s}_y^*.
\end{aligned}
$$

The DMU $(\boldsymbol{x}_o, \boldsymbol{y}_o)$ can be improved and becomes efficient by deleting the input surplus and augmenting the output shortage as follows:

$$
\begin{aligned}
\boldsymbol{x}_o &\leftarrow \boldsymbol{x}_o - \boldsymbol{s}_x^* \qquad (7)\\
\boldsymbol{y}_o &\leftarrow \boldsymbol{y}_o + \boldsymbol{s}_y^*. \qquad (8)
\end{aligned}
$$

# 5 SBM and the CCR Measure

**Theorem 2** *The optimal SBM* $\rho^*$ *is not greater than the optimal CCR* $\theta^*$.

The relationship between CCR-efficiency and SBM-efficiency is shown by the following theorem.

**Theorem 3** *A DMU* $(\boldsymbol{x}_o, \boldsymbol{y}_o)$ *is CCR-efficient, if and only if it is SBM-efficient.*

# 6 Numerical Example

Table 1 exhibits the data of eight DMUs with two inputs $(x_1, x_2)$ and single output $(y = 1)$, along with CCR, SBM scores, slacks and reference set. Although DMUs $F$ and $G$ have full CCR-score $(\theta^* = 1)$, they have slacks against $C$ and turned out to show sharp drops in the SBM scores $\rho_F^* = 0.9$ and $\rho_G^* = 0.83333$. Also, the SBM scores of inefficient DMUs $A$, $B$ and $H$ dropped slightly from the CCR scores.

Table 1: Results of Example

| | Data | | | CCR | SBM | | | |
|---|---|---|---|---|---|---|---|---|
| DMU | $x_1$ | $x_2$ | $y$ | $\theta^*$ | $\rho^*$ | Rf | $s_{x1}^*$ | $s_{x2}^*$ |
| $A$ | 4 | 3 | 1 | 0.85 | 0.83 | $D$ | 0 | 1 |
| $B$ | 7 | 3 | 1 | 0.63 | 0.61 | $D$ | 3 | 1 |
| $C$ | 8 | 1 | 1 | 1.00 | 1.00 | $C$ | 0 | 0 |
| $D$ | 4 | 2 | 1 | 1.00 | 1.00 | $D$ | 0 | 0 |
| $E$ | 2 | 4 | 1 | 1.00 | 1.00 | $E$ | 0 | 0 |
| $F$ | 10 | 1 | 1 | 1.00 | 0.90 | $C$ | 2 | 0 |
| $G$ | 12 | 1 | 1 | 1.00 | 0.83 | $C$ | 4 | 0 |
| $H$ | 10 | 1.5 | 1 | 0.75 | 0.73 | $C$ | 2 | 0.5 |

# References


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[3] Thrall, "Goal vectors for DEA efficiency and inefficiency", Rice University, 1997.

[4] Tone, "An $\varepsilon$-free DEA and a new measure of efficiency", JORSJ, 36, 167-174, 1993.