Approximate Algorithms For Fair Division

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1 Introduction

How to fairly divide a cake among n players? There are many criteria of fairness. This paper only considers envy-freeness. A presuppose is that players may have different measures on the cake. A cake division is said to be envy-free if each player receives a piece he/she would not swap for that received by any other player.

Contrary to dividing a delicious cake, sometimes people have to divide chores or dirty work, in which every player hopes a small piece. We call this kind of problems as chores division. Many, but not all, results of cake division hold similarly for chores division.

We are interested in designing an algorithm, consisting of a set of rules and strategies for players. An envy-free algorithm outputs an envy-free division if n players divide the whole set according to the rules step by step, and use the strategies suggested in the algorithm. Brams and Taylor [1] give a finite discrete algorithm (for the definitions of various algorithms, see [2]) for envy-free division. A shortcoming is that, although the number of cuts necessary to obtain an envy-free division by that algorithm is finite, it could be made arbitrarily large by a suitable choice of the measures corresponding to the players' preferences.

An ϵ-approximate envy-free division is one that every player thinks his/her piece is not smaller than the largest piece by ϵ. Section 7.2 of [2] provides an idea for a very practical ϵ-approximate envy-free cake division algorithm, by use of a moving-knife. However, the idea does not apply to chores division problem.

This paper first evaluates two algorithms of Brams and Taylor for ϵ-approximate envy-free division, then provides a new moving-knife algorithm to approximate solve multi-fair division, which unifies both cake division and chores division. Finally, we define envy-free division with unequal ratios, and use the studied algorithms to approximately solve such division problems.

As notations, we denote $\mathcal{C}$ as the whole set (something like a cake or chores). Player i uses measure m_i on $\mathcal{C}$. All m_i are supposed to be additive, non-atomic probability measure defined on some common $\sigma-$algebra of subsets of $\mathcal{C}$. Without loss of generality, we let $n \geq 2$ and $\epsilon \in (0,1)$.

2 Two known algorithms

Brams and Taylor ([1]) provide a discrete algorithm for an exactly envy-free division. This algorithm can be adapted to an ϵ-approximate envy-free algorithm, which needs at most $2^{n-2}K_1 = O(n2^{2n-3})$ cuts, where $K_1 = n\left\lceil \frac{-\ln\epsilon}{\ln\left(1+\frac{1}{2^{n-2}}\right)}\right\rceil$. Section 7.2 of [2] provides another ϵ-approximate envy-free algorithm, which uses a moving knife. The process asks each player initially to call cut whenever he/she thinks the piece the moving-knife will yield is of size $1/n$, and thereafter to call cut whenever he/she thinks the new piece is larger by ϵ than the one he/she presently holds. Ties are broken at random. The number of cuts can be bounded by $\lceil 2K_2/\epsilon\rceil nK_2 = O(n^3)$, where $K_2 = \left\lceil \frac{\ln\frac{\epsilon}{2}}{\ln\left(1-\frac{1}{n}\right)}\right\rceil$.

3 A new algorithm

Given a set $\mathcal{C}$, integers n and l $(1 \leq l \leq n-1)$. Call n pieces $A_1, A_2, \cdots, A_n$ of $\mathcal{C}$ a *multi-fair division* if each $x \in \mathcal{C}$ appears in l of the n pieces, and by assigning piece A_i to player i we have $m_i(A_i) \geq m_i(A_j)$, for all $i, j = 1, 2, \cdots, n$. When $l = 1$ a multi-fair division problem degenerates into the ordinary envy-free cake division, in which each player gets all the piece he/she was assigned; When $l = n-1$ this problem turns to be the ordinary envy-free chores division, in which each player obtains the complement of the piece assigned to the player. Similarly, we say the pieces A_i $(i = 1, 2, \cdots, n)$ form an ϵ-approximate multi-fair division if $m_i(A_i) \geq m_i(A_j) - \epsilon$, for all $i, j = 1, 2, \cdots, n$.

For ϵ-approximate multi-fair division, we have the following algorithm. Let $K_3 = n \left\lceil \frac{\ln \frac{\epsilon}{2}}{\ln(1-\frac{1}{n})} \right\rceil$.

Algorithm for multi-fair division:

Step 0 Let $k = 1$, $A_i = \emptyset$ for $i = 1, 2, \cdots, n$, let $\mathcal{R} = \mathcal{C}$.

Step 1 For $i = 1, 2, \cdots, n$, $j = 1, 2, \cdots, l$, let $k_{i_j} = 1$.

Step 2 Let player n cut $\mathcal{R}$ into n pieces $A'_1, A'_2, \cdots, A'_n$.

Strategy: Cut $\mathcal{R}$ into n equal pieces.

Step 3 Let each player $i = 1, 2, \cdots, n$ label $i_1, i_2, \cdots, i_l$ on l different pieces.

Strategy: Each player labels l largest pieces.

Step 4 If each piece has l labels from l players, let $\mathcal{R}'$ be the total trimmings, go to Step 7. Otherwise go to Step 5.

Step 5 Choose a piece $\mathcal{B}$ which is labeled by at least $l + 1$ players. Let a knife move from its right edge to its left edge. Let the left side of the knife be $\mathcal{B}_1$ and the right side $\mathcal{B}_2$. Move the knife until a player s who is now labeling s_j on piece $\mathcal{B}$ with $k_{s_j} \leq \frac{2K_3}{(l+1)\epsilon}$ calls "cut". By then go to Step 6.

Strategy: Each player s calls cut if $m_s(\mathcal{B}_1) = (1 - \epsilon(l + 1)/(2K_3))m_s(A''_s)$, where A''_s is a piece which s measures largest among those which s did not label.

Step 6 Let $k_{s_j} = k_{s_j} + 1$. Cut piece $\mathcal{B}$ into $\mathcal{B}_1$ and $\mathcal{B}_2$. Let the player i who called cut move label s_j from $\mathcal{B}$ to a new piece, which is without any label of player s. Leave $\mathcal{B}_2$ aside as trimmings, replace $\mathcal{B}$ by $\mathcal{B}_1$. Go to Step 4.

Strategy: Player s now labels s_j on A''_s.

Step 7 Let $k = k + 1$, $\mathcal{R} = \mathcal{R}'$, $A_i = A_i \cup \left(\cup_j \{A'_j \mid \text{player } i \text{ has a label on } A'_j\}\right)$. If $k \leq K_3$, rename player n if necessary so that it represents the same player at most K_3/n times. (A_i and m_i should be renamed correspondently.) Go to Step 1. If $k > K_3$, give $\mathcal{R}'$ (randomly) to player 1, (i.e., let $A_1 = A_1 \cup \mathcal{R}'$,) go to Step 8.

Step 8 Output pieces $A_1, A_2, \cdots, A_n$, where A_i is assigned to the l players who labeled it. End.

Theorem 1 *For any given $\epsilon > 0$, Algorithm 5 obtains an ϵ-approximate envy-free division with at most $\left\lceil \frac{2K_3}{\epsilon(l+1)} \right\rceil n \min\{l, n-l\} K_3$ cuts.*

Corollary 3. 1 *For cake division problems, $l = 1$. The bound becomes $\lceil K_3/\epsilon \rceil n K_3 = O(n^5)$. For chores division problem, $l = n - 1$. The bound becomes $\lceil 2K_3/\epsilon \rceil K_3 = O(n^4)$.*

Therefore we know that, for cake division our algorithm is worse than that of Brams and Taylor[2], but ours is very good to solve chores division problem.

4 Approximate envy-Free division in unequal ratios

An envy-free division $A_1, A_2, \cdots, A_n$ with ratios $\langle \alpha_1, \alpha_2, \cdots, \alpha_n \rangle$ is a division in which player i receives A_i and $m_i(A_i)/\alpha_i \geq m_i(A_j)/\alpha_j$, for $i = 1, 2, \cdots, n$. Given $\epsilon > 0$, the division is called ϵ-approximate envy-free if $m_i(A_i)/\alpha_i \geq m_i(A_j)/\alpha_j - \epsilon$.

Theorem 2 *Given any $\epsilon > 0$ and entitlement sequence $\langle \alpha_1, \alpha_2, \cdots, \alpha_n \rangle$. Let $\alpha_{\min} = \min_i \alpha_i$, $\alpha_{\max} = \max_i \alpha_i$, $N = \left\lceil \frac{\alpha_{\max}}{\alpha^3_{\min}} \frac{5}{\epsilon} \right\rceil$, $\epsilon' = \frac{\epsilon}{2N} \alpha_{\min}$, $K_4 = \left\lceil \frac{\ln \frac{\epsilon'}{2}}{\ln(1-\frac{1}{N})} \right\rceil$. There is an algorithm to obtain an ϵ-approximate envy-free division with the given entitlement sequence, in which the number of necessary cuts is bounded by $\lceil 2K_4/\epsilon' \rceil N K_4$.*

For chores division problem, let $K_5 = N \left\lceil \frac{\ln \frac{\epsilon'}{2}}{\ln(1-\frac{1}{N})} \right\rceil$, where N and ϵ' are defined in Theorem 2. The number of necessary cuts can be bounded by: $\lceil 2K_5/\epsilon' \rceil K_5$.

References

[1] Brams, S. J. and Taylor, A. D., *An envy-free cake division protocol*, American Mathematical Monthly **102** (1995), 9-18.

[2] Brams, S. J. and Taylor, A. D., Fair Division: From Cake-Cutting to Dispute Resolution, Cambridge University Press, 1996.