A Finite Algorithm for Nonlinear General Integer Allocation Problems

01991000 東京工業大学 戴 陽 DAI YANG

1 Introduction

In this paper we consider the following general integer allocation problem:

Problem (P, M^0)

$$\begin{aligned}
\text{minimize} \quad & \sum_{i=1}^{n} f_i(x_i) \\
\text{subject to} \quad & \sum_{i=1}^{n} g_i(x_i) \le b, \\
& l_i \le x_i \le u_i, \quad i = 1, \ldots, n, \\
& x_i \quad \text{integer}, \quad i = 1, \ldots, n,
\end{aligned}$$

where $x_i \in R$, $i = 1, \ldots, n$; $f_i(x_i)$, $i = 1, \ldots, n$, are concave functions on R; $g_i(x_i)$, $i = 1, \ldots, n$, are convex functions on R; b is a constant; and l_i and u_i $(l_i \le u_i)$, $i = 1, \ldots, n$, are integers. We denote the rectangular constraint (1) by $M^0 = \{x \in R^n | l_i \le x_i \le u_i, i = 1, \ldots, n\}$. The problem is difficult to solve since even those with a linear constraint is shown to be NP-hard (see, e.g., [Sa74]). Although plenty of algorithms for solving general continuous concave minimization problems have been proposed (see [Ben95, HT90]), only a few deal with their integral counterparts. In Benson et al. [BE90, BEH90] general branch and bound methods are proposed for problems with polyhedral regions and integer constraints.

Here we develop a branch and bound algorithm for solving Problem (P, M) since none of the above mentioned algorithms can be used directly. Our algorithm takes advantage of the fact that Problem (P, M) has only one constraint. The algorithm solves a continuous subproblem of a more simple form, i.e., the objective function is linear, which is produced by taking a convex envelop of the original concave function over a subset of feasible region. The subproblem is solved by manipulation of the Kuhn-Tucker conditions.

To improve the efficiency of our algorithm some heuristic methods are incorporated to obtain feasible solutions whose objective function values may improve the upper bound. We report the computational results of the algorithm.

2 The Algorithm

In this paper we use convex envelope as an underestimating function of the original objective function.

Definition 1 *Let $M \subset R^n$ be convex and compact, and let $f : M \to R$ be lower semi-continuous on M. A function $\psi : M \to R$ is called the convex envelope of f on M if it satisfies:*

(a) ψ is convex on M,
(b) $\psi(x) \le f(x)$ for all $x \in M$,
(c) there is no function $\varphi : M \to R$ satisfying (a),(b), and $\psi(x) \le \varphi(x)$ for some $x \in M$.

By Theorem IV.8 of [HT90] if $f(x) = \sum_{i=1}^{n} f_i(x_i)$, then the convex envelope of $f(x)$ taken over a rectangle $\{x \mid r_i \le x_i \le s_i, i = 1, \ldots, n\}$ is equal to the sum of the convex envelopes of the function $f_i(x_i)$ taken over the line segments $r_i \le x_i \le s_i, i = 1, \ldots, n$. The convex envelope of a convex function $f_i(x_i)$ taken over a line segment $r_i \le x_i \le s_i$ is simply the affine function that agrees with f_i at the end points of this segment, i.e., the function $\psi_i(x_i) = f_i(r_i) + \frac{f_i(s_i)-f_i(r_i)}{s_i-r_i}(x_i - r_i), \quad r_i \le x_i \le s_i$.

At each node of the branch and bound tree we solve a continuous convex underestimating problem of Problem (P, M^t) over $M^t = \{x \mid l_i^t \le x_i \le u_i^t, \, i = 1, \ldots, n\}$, which is given as follows.

Problem (CUP, M^t)

$$\begin{aligned} \text{minimize} \quad & \sum_{i=1}^{n} \psi_i(x_i) \\ \text{subject to} \quad & \sum_{i=1}^{n} g_i(x_i) \le b, \\ & l_i^t \le x_i \le u_i^t, \quad i = 1, \ldots, n. \end{aligned}$$

The algorithm maintains a set W of rectangles which may contain the optimal solution. It starts with solving the problem (CUP, M^0). If it finds an integer optimal solution then the problem is solved. Otherwise the objective value provides a lower bound LB^0 for the optimal value of Problem (P, M^0). We denote the optimal solution of the relaxed problem as x^0 and $UB^0 = +\infty$. In the subsequence step t of the algorithm, we have a rectangle chosen from the previous step, the lower bound LB^t, the upper bound UB^t, the incumbent solution x^c and the optimal solution x^t associated with the rectangle. We subdivide the rectangle into two smaller rectangles $M^{(t,i)}$, $i = 1, 2$, each with integral extreme points. The new rectangle $M^{(t,1)}$ is generated by taking $u_j^t = \lfloor x_j^t \rfloor$ for some component x_j^t with non-integer value while keeping the other unchanged. Similarly, the rectangle $M^{(t,2)}$ is generated by taking $l_j^t = \lfloor x_j^t \rfloor + 1$. Then the two rectangles are added to the set W and the problems $(CUP, M^{(t,i)})$ are solved. If the problem is infeasible then it is deleted from the set W. Otherwise the optimal value of the problem is used to update the current lower bound to the new lower bound LB^{t+1}. Moreover, if the optimal solution x^t is integral, then the upper bound UB^t is updated by $UB^{t+1} = \min\{UB^t, \sum_{i=1}^{n} f_i(x_i^t)\}$ and the rectangle is deleted. If $UB^{t+1} = \sum_{i=1}^{n} f_i(x_i^t)$ then set $x^c = x^t$. When the upper bound is updated, rectangles whose correspounding problems having optimal values greater than or equal to UB^{t+1} are erased from the set W. If the set W is empty then we conclude that the point x^c is the optimal solution. Otherwise remove the rectangle associated with the lower bound LB^{t+1} from the set W and pass it to the step $t+1$.

A discussion similar to that of Benson et al. [BEH90] shows that the branch and bound algorithm terminates after a finite number of steps with an optimal solution x^c for Problem (P, M^0).

References

[Ben95] H. P. Benson, Concave minimization: theory, applications and algorithms, in R. Horst and P. M. Pardalos eds., *Handbook of Global Optimization* Kluwer Academic Publishers, Dordrecht, 1995.

[BE90] H. P. Benson and S. D. Erenguc, An algorithm for concave integer minimization over a polyhedron, *Naval Research Logistics* 37 (1990) 515-525.

[BEH90] H. P. Benson, S. D. Erenguc and R. Horst, A note on adapting methods for continuous global optimization to the discrete case, *Annals of Operations Research* 25 (1990) 243-252.

[BS95] K. M. Bretthauer and B. Shetty, The nonlinear resource allocation problem, *Operations Research* 43 (1995) 670-683.

[HT90] R. Horst and H. Tuy, *Global Optimization: Deterministic Approaches*, Spinger-Verlag, Berlin, 1990.

[Sa74] S. Sahni, Computationally related problems, *SIAM Journal on Computing* 3 (1974) 262-379.