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Limit Theorems for the Departure Process of a MAP/M/c Queue and an Application to Two–stage Composite Tandem Queues

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Abstract The departure process of a MAP/M/c queue is first investigated and the asymptotic logarithmic Laplace–Stieltjes transform (ALLST) of the process is identified in terms of the MAP and the exponential service rate. The ALLSTs of the superposed departure process and the randomly splitted process are also obtained. We apply these limit theorems to the asymptotic analysis of the tail behavior of two–stage composite tandem queues and show exponential decay of the stationary distributions.

1. Introduction

It is well known that in fairly general single–stage queues, the stationary waiting time distribution decays exponentially fast. Consider a single server queue with a stationary sequence of interarrival times $\{\tau_n\}$. If the asymptotic logarithmic Laplace–Stieltjes transform (ALLST) $\phi(s) = \lim_{n\to\infty} n^{-1} \log \mathrm{E}\left(\exp(-s\sum_{i=1}^{n}\tau_i)\right)$ exists and the service time distribution satisfies certain conditions, then

$$\lim_{w\to\infty} w^{-1} \log \mathrm{P}(W > w) = -\delta \tag{1}$$

and the decay parameter δ is determined by $\phi(s)$ and the LST of the service time distribution [2].

In tandem queues or queueing networks, an arrival process to one queue is formed by departure processes of the other queues. Thus, to apply the above asymptotic result to queueing networks, we need to investigate the departure process. Chang [1] considered a discrete time single server queue with constant service rate. Assuming that external arrivals satisfy sample path large deviation principle, he derived a relation between the ALLSTs of the arrival and departure process. O'Connell [4] obtained a similar result for a single server queue with stochastic service times. Though these results are potent tool to analyze queueing networks, it seems difficult to extend them to multiple server queues since the analysis is essentially based on the Lindley equation.

In this paper, we consider a multiple server queue with Markovian arrival process (MAP) and exponential service time distribution. The departure process of a MAP/M/c queue is first investigated and the ALLST of the process is identified in terms of the MAP and the exponential service rate. The ALLSTs of a superposed departure process and a randomly splitted departure process are also derived. We apply these limit theorems to the asymptotic analysis of the tail behavior of two–stage composite tandem queues and show exponential decay of the stationary distributions. Our analysis is based on a matrix algebraic approach and is very different from those in [1, 4].

2. The Departure Process of a MAP/M/c Queue

We consider the following stable MAP/M/1 queue. The background Markov chain of the MAP has an irreducible representation $(\mathbb{A}_0, \mathbb{A}_1)$ where $\mathbb{A}_1$ ($\mathbb{A}_0$, respectively) governs transitions with (without, respectively) an arrival. The service rate is μ_k when there are k customers in the system. We assume that $0 < \mu_k \le \mu$ for $k = 1, \cdots, c-1$ and $\mu_k = \mu$ for $k = c, c+1, \cdots$. Note that a MAP/M/c is a special case with $\mu_k = k\mu/c$ for $k = 1, \cdots, c-1$. Let D_n denote the nth departure epoch with the convention $D_0 \equiv 0$. The queueing process observed just after the departure epoch forms a discrete time Markov chain $\{(\hat{N}_n, \hat{I}_n);\ n \ge 0\}$ which is ergodic by the stability condition. The stationary distribution of $\{(\hat{N}_n, \hat{I}_n)\}$ is denoted by $\pi = (\pi_k)$ with $\pi_k = (\pi_{(k,i)})$, $\pi_{(k,i)} = \lim_{n\to\infty} \mathrm{P}\left((\hat{N}_n, \hat{I}_n) = (k, i)\right)$.

We denote by $u_{(k,i)(\ell,j)}(s)$ the Laplace–Stieltjes transform (LST) of the interdeparture time conditioned on the number of customers and states of the MAP, that is

$$u_{(k,i)(\ell,j)}(s) = \int_0^\infty e^{-st}\, \mathrm{dP}\left(D_1 \le t,\ (\hat{N}_1, \hat{I}_1) = (\ell, j) \,\middle|\, (\hat{N}_0, \hat{I}_0) = (k, i)\right)$$

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and let $U(s) = \left(u_{(k,i)(\ell,j)}(s)\right)$. To make the departure process stationary, we assume that $(\hat{N}_0, \hat{I}_0)$ is distributed according to the stationary distribution $\boldsymbol{\pi}$. Then, the logarithmic LST of nth departure epoch will be given as

$$n^{-1} \log \mathrm{E}\left(\exp(-sD_n) \,\Big|\, (\hat{N}_0, \hat{I}_0) \stackrel{d}{=} \boldsymbol{\pi}\right) = n^{-1} \log \boldsymbol{\pi} \boldsymbol{U}^n(s) \mathbf{1}^\top.$$

Let $f(s)$ be the Perron–Frobenius eigenvalue of $\boldsymbol{A}_1(s\boldsymbol{I} - \boldsymbol{A}_0)^{-1}$ and define a bivariate function

$$g(s,x) = f(s+x)\frac{\mu}{\mu - x}.$$

Also let x_0 be the unique positive solution of $g(0,x) = 1$ and define

$$\mathcal{E}(s) = \begin{cases} \{\, x \mid -s \leq x \leq 0 \,\}, & 0 \leq s \leq x_0, \\ \{\, x \mid -s \leq x < x_0 - s \,\}, & x_0 < s. \end{cases}$$

Then, $\phi(s)$ is given by the next theorem.

Theorem 1 $\quad \phi(s) = \inf_{x \in \mathcal{E}(s)} \log g(s,x)$.

It is observed from this theorem that μ_k $(k = 1, \cdots, c-1)$ does not affect $\phi(s)$ provided $\mu_k \leq \mu$.

3. Superposition and Random Splitting

Suppose there are K_1 queues $\mathrm{MAP}_i/\mathrm{M}_i/\mathrm{c}_i$ $(i = 1, \cdots, K_1)$ operating independently of each other. The ALLST $\phi_i(s)$ of $\mathrm{MAP}_i/\mathrm{M}_i/\mathrm{c}_i$ can be obtained from Theorem 1. Let $N_i(t)$ be the number of departures from $\mathrm{MAP}_i/\mathrm{M}_i/\mathrm{c}_i$ during $[0,t)$. From the inversion theorem [3], $\psi_i(s) = \lim_{t \to \infty} t^{-1} \log \mathrm{E}\left(\exp(-sN(t))\right)$ exists and is given by $-\phi_i^{(-1)}(-s)$ where $\phi_i^{(-1)}(s)$ is the inverse function of $\phi_i(s)$. Let $\hat{\phi}(s)$ be the ALLST of the superposed process consisting of K independent departure processes.

Theorem 2 $\quad \hat{\phi}(s) = \psi^{(-1)}(s)$ where $\psi(s) = \sum_{i=1}^{K} \psi_i(s)$. In particular, if $\phi_i(s) = \phi_1(s)$ for all $i = 1, \cdots, K$ then $\hat{\phi}(s) = \phi_1(s/K)$.

Next, we consider the ALLST of a process obtained by splitting the original process by certain routing discipline. Suppose there are K_2 possible routes. Let $\{N_k; k = 1, 2, \cdots\}$ be a random sequence of customers who select (say) route 1. We assume that $\{N_{k+1} - N_k\}$ forms an i.i.d. sequence of random variables whose z–transform is denoted by $G(z) = \sum_{n=1}^{\infty} z^n \mathrm{P}\left(N_{k+1} - N_k = n\right)$. The ALLST $\tilde{\phi}(s)$ of the splitted departure process is expressed in terms of the ALLST $\phi(s)$ of the original process.

Theorem 3 $\quad \tilde{\phi}(s) = \log G(\exp(\phi(s)))$.

4. Application to Two–stage Composite Tandem Queues

We consider a two–stage tandem queue where the first stage consists of K_1 independent MAP/M/c queues while there are K_2 single server queues having general service time distributions in the second stage. We write jth queue in the ℓth stage as Q_j^ℓ $(\ell = 1, 2;\ j = 1, \cdots, K_\ell)$. A customer who completes the service in the first stage moves to one of the queues in the second stage according to a Markovian routing discipline. Let $p_{ij}^{(k)}$ be a probability that those who depart from Q_k^1 will be routed to Q_j^2 given that the previous customer went to Q_i^2. Under this setting, $N_{k+1} - N_k$ in Section 3 follows a discrete phase–type distribution and its z–transform can be easily calculated from $\boldsymbol{P}^{(k)} = (p_{ij}^{(k)})$. Note that the arrival process into Q_j^2 is a superposition of departure processes of Q_k^1 $(k = 1, \cdots, K_1)$ that are routed to Q_j^2. Therefore, the ALLST $\phi_j(s)$ of the arrival process into Q_j^2 can be obtained by combining Theorems 1, 2 and 3. Assume now that $\phi_j(s) + \log h_j(-s) = 0$ has a positive solution $s = \delta_j$ where $h_j(s)$ is the LST of the service time distribution of Q_j^2. Then the distribution of the stationary waiting time W_j in Q_j^2 decays exponentially as

$$\lim_{w \to \infty} w^{-1} \log \mathrm{P}\left(W_j > w\right) = -\delta_j$$

as indicated in (1).

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