1999年度日本オペレーションズ・リサーチ学会
春季研究発表会

THE OPTIMAL PLANNED REPLACEMENT MODEL WITH MINIMAL AND PERFECT REPAIR

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1. Introduction

In this paper we investigate a replacement model with two types of repairs. Repairs are classified into minimal and perfect repair. An operating unit is completely replaced whenever it reaches age $\tau(\tau > 0)$ at a cost c_1(planned replacement). If it fails at age $t < \tau$, it is either restores by a entire unit with probability $p(t)$ at a cost c_2(perfect repair), or it undergoes minimal repair with probability $\overline{p}(t) = 1 - p(t)$ at a cost c_3. After a planned replacement, the procedure is repeated. The aim of this paper is to find a condition to exist the optimal planned replacement period τ^* which minimizes the total long-run expected cost per unit time.

2. Problem Formulation

Let X_k be the time interval failure time between $(k-1)$st. If F is the failure time distribution of the unit, then the failure time distribution following minimal repair for a unit which fails at time t is given by

$$P\left[X_k \le x \middle| \sum_{i=1}^{k-1} X_i = t\right] = \frac{F(t+x) - F(t)}{\overline{F}(t)}, \quad x > t.$$

Let t be the age of a unit and $\lambda(t)$ be the failure rate(or the hazard rate) function belonging to $F(t)$. Let $G(t)$ be the distribution function of random variable Y = total time among perfect repairs. Formally, $Y_1, Y_2, \ldots, Y_{n-1}$ be a sequence of nonnegative independent random variables with a common distribution $G(t)$. Then

$$Y = \sum_{i=1}^{n-1} Y_i.$$

Also let $G(t) = P(Y \le t)$.

The survival distribution of the time between successive perfect repairs is given by

$$\overline{G}(t) = \exp\left[-\int_0^t p(x)\lambda(x)\,dx\right], \tag{1}$$

where $\overline{G}(t) = 1 - G(t)$ and $G(0) = 0$. Hence $r(t) = p(t)\lambda(t)$ denotes the failure rate function belonging to $G(t)$.

Let $\{\hat{N}(t),\ t \ge 0\}$ be the number of perfect repairs that occur during $[0,\ t]$.

Let $\{N(t),\ t \ge 0\}$ be the number of minimal repairs that occur during $[0,\ t]$.

Let $\{N^*(\tau),\ \tau \ge 0\}$ be the number of minimal repairs whenever it reaches age τ from a final perfect repair.

Therefore, we constitute the total time W and the total cost C until a unit completely replaced whenever it reaches age τ is given by

$$W = \sum_{i=1}^{n-1} Y_i + \tau, \tag{2}$$

if $Y_1 < \tau,\ Y_2 < \tau,\ \ldots,\ Y_{n-1} < \tau,\ Y_n \ge \tau.$
And

$$C = c_1 + c_2\hat{N}(W) + c_3\big[N(W) + N^*(\tau)\big], \tag{3}$$

if $Y_1 < \tau,\ Y_2 < \tau,\ \ldots,\ Y_{n-1} < \tau,\ Y_n \ge \tau.$

3. Analysis of Optimal Model

In this section, we shall explain two part of expected total time $E[W]$ and expected total cost $E[C]$.

Lemma 3.1. *$E[W]$ denotes the expected total time of the unit until planned replacement. Thus it follow that*

$$E[W] = \frac{\int_0^\tau \overline{G}(t)\,dt}{\overline{G}(\tau)}.$$

Lemma 3.2. *Let $N^*(\tau)$ denotes the number of minimal repairs in $[0,\ \min(Y_n,\ \tau)]$, for all $\tau > 0$, where Y_n denotes the random length of the final perfect repair before planned replacement.*

$$E\big[N^*(\tau)\big] = \Lambda(\tau) - R(\tau).$$

Lemma 3.3. *Let $N(W)$ denotes the random number of minimal repairs during $[0,\ W]$.*

$$E\big[N(W)\big] = \frac{1}{\overline{G}(\tau)}\left[\int_0^\tau \Lambda(t)\,dG(t) + \big[1 + R(\tau)\big]\overline{G}(\tau) - 1\right].$$

Lemma 3.4. *$\hat{N}(W)$ be the random number of perfect repairs between $[0,\ W]$. If only perfect repair actions are taken, the renewal process is well known to be a process $\{\hat{N}(W),\ W \ge 0\}$.*

$$E\big[\hat{N}(W)\big] = \frac{G(\tau)}{\overline{G}(\tau)}.$$

Theorem 3.1. *The total long-run expected cost per unit time can be obtained by using the theory of renewal process and is equal to:*

$$K(\tau) = \frac{c_1 + c_2E\big[\hat{N}(W)\big] + c_3E\big[N(W) + N^*(\tau)\big]}{E[W]}. \tag{4}$$

Theorem 3.2. *Let $F(t)$ have failure rate function $\lambda(t)$ and suppose that the functions $r(t)$ is continuous. Then if λ and r are monotonically increasing function and monotonically decreasing function, respectively. And that*

$$\lim_{x\to+\infty}\lambda(x)\int_0^x \overline{G}(t)\,dt > \frac{c_1}{c_3},$$

there exists at least one finite positive period τ^* which minimizes the total long-run expected cost per unit time $K(\tau)$.

Proof. We can differentiate K with respect to τ. Noting that $d\overline{G}(\tau)/d\tau = -p(\tau)\lambda(\tau)\overline{G}(\tau)$ and $dR(\tau)/d\tau = r(\tau) = p(\tau)\lambda(\tau)$. From (4), it follows that

$$(5)\qquad \frac{dK(\tau)}{d\tau} = \frac{\overline{G}(\tau)}{\left[\int_0^\tau \overline{G}(t)dt\right]^2}\Theta(\tau) = 0,$$

where

$$\Theta(\tau) = c_3\left[\int_0^\tau [\lambda(\tau)-\Lambda(t)r(t)]\overline{G}(t)\,dt - \overline{G}(\tau)\Lambda(\tau)\right]$$

$$(6)\quad -[c_1+c_3-c_2]\left[\int_0^\tau [r(\tau)-r(t)]\overline{G}(t)\,dt\right] - c_1.$$

We assumed that the cost of planned replacement is higher than the cost of perfect repair and minimal repair. Differentiating the right-hand side in (6), we have

$$c_3\left[\lambda'(\tau)\int_0^\tau \overline{G}(t)\,dt\right] - [c_1+c_3-c_2]\left[r'(\tau)\int_0^\tau \overline{G}(t)\,dt\right],$$

which is nonnegative, if we assume $\lambda'(\tau)\geq 0$ and $r'(\tau)\leq 0$.

In (6) is a continuous increasing function of τ which is negative $(-c_1)$ at $\tau\to 0$ and $c_1 > c_2 \geq c_3$. And tends to $+\infty$ if $\lim_{\tau\to+\infty}\lambda(t) = \lambda(+\infty) = +\infty$ as $\tau\to+\infty$. Hence there always exists a unique solution $\tau=\tau^*$, $0<\tau^*<+\infty$ of (5). Since $dK(\tau)/d\tau$ has the same pattern $(-\infty, 0, +\infty)$, it follows that $K(\tau)$ has a minimizes at τ^*. Under the strictly increasing assumption, $K(\tau)$ is strictly increasing, so τ^* is unique. If τ^* is the solution, then from (4) and (6) it is easy to get

$$(7)\qquad K(\tau^*) = c_3\lambda(\tau^*) - (c_1+c_3-c_2)r(\tau^*).$$

Remark 3.1. *If $c_1+c_3<c_2$, the optimal planned replacement age is $\tau^*=+\infty$.*

4. Example

In this section, we consider the particular case where $p(t)\equiv p$, $0<p<1$. In this case, we assume that the probability of a perfect repair does not depend on age. We have $r(t)=p\lambda(t)$, $G(t) = 1-\left[\overline{F}(t)\right]^p$, and

$$(8) K_p(\tau) = \frac{c_1\overline{G}(\tau) + \left[c_2 + \left(\frac{1-p}{p}\right)c_3\right]G(\tau)}{\int_0^\tau \overline{G}(t)dt}.$$

The p.d.f. of the Weibull distribution with shape parameter α and scale parameter β.

From (8), it follows that

$$(9)\qquad \frac{dK_p(\tau)}{d\tau} = \frac{\overline{G}(\tau)}{\left[\int_0^\tau \overline{G}(t)dt\right]^2}\theta(\tau) = 0,$$

where

$$\theta(\tau) = \left[c_2 + \left(\frac{1-p}{p}\right)c_3 - c_1\right]\left[\lambda(\tau)\int_0^\tau \overline{G}(t)\,dt + \frac{1}{p}\overline{G}(\tau)\right] - \frac{1}{p}\left[c_2+\left(\frac{1-p}{p}\right)c_3\right].$$

Consider the case in which $c_2 + \left[(1-p)/p\right]c_3 \leq c_1$, $\theta(\tau)<0$ for $\tau>0$, then we have the optimal planned replacement period is $\tau=+\infty$. Supposing that the $c_2+\left[(1-p)/p\right]c_3 > c_1$, and that

$$\lim_{x\to+\infty}\lambda(x)\int_0^x \overline{G}(t)\,dt > \frac{1}{p}\left[\frac{c_2+\left(\frac{1-p}{p}\right)c_3}{c_2+\left(\frac{1-p}{p}\right)c_3-c_1}\right].$$

By (9), If $\alpha>1$, a finite optimal planned replacement period τ^* exists, since $\lambda(+\infty)=+\infty$. θ is increasing since $\lambda(\tau)$ and easy computation shows that

$$\theta'(\tau) = \lambda'(\tau)\int_0^\tau \overline{G}(t)\,dt \geq 0.$$

Also note that $\theta(0) = -c_1/p < 0$. Thus if the $\theta(\tau)=0$ has a solution τ^*, then $K_p'(\tau)$ is negative on $(0,\ \tau^*)$ and positive on $(\tau^*,\ +\infty)$.

$$\lambda(\tau^*)\int_0^{\tau^*}\overline{G}(t)\,dt - G(\tau^*) = \frac{pc_1}{p(c_2-c_1)+(1-p)c_3}.$$

Equation (9) reduces to the equation

$$\left(\frac{\alpha}{\beta}\right)\left(\frac{t}{\beta}\right)^{\alpha-1}\int_0^\tau \exp\left[-p\left(\frac{t}{\beta}\right)^\alpha\right]dt + \frac{1}{p}\exp\left[-p\left(\frac{t}{\beta}\right)^\alpha\right] = \frac{1}{p}\left[\frac{c_2+\left(\frac{1-p}{p}\right)c_3}{c_2+\left(\frac{1-p}{p}\right)c_3-c_1}\right],$$

which is easily solved by numerical methods. If $\alpha=1$, the equation above has no solution.

References

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