A Correlated Markov Chain Model and Its Application to Risk Management

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1 The model

We consider a set of Markov chains $\{X_t^k, t = 0,1,2,\cdots\}$, $k = 1,2,\cdots,n$, each defined on the state space $\mathcal{S} = \{1,2,\cdots,K,K+1\}$, where $1 < K < \infty$. The Markov chain $\{X_t^k\}$ may represent the dynamics of a credit rating of firm k. In this case, state 1 represents the highest credit class, state 2 the second highest, $\cdots$, state K the lowest credit class, and state $K+1$ designates default. While state 1 is a retaining boundary, it is usually assumed for simplicity that the default state $K+1$ is absorbing.

In our model, the dynamics of $\{X_t^k\}$ is assumed to follow

$$X_{t+1}^k = \begin{cases} \xi\Big(X_t^k + Z_{t+1}^k + (-1)^{\delta_k} B_{t+1}^k Y_{t+1}\Big), & X_t^k \neq K+1, \\ X_t^k, & X_t^k = K+1, \end{cases} \tag{1}$$

where δ_k is either 0 or 1, B_t^k are independent, identically distributed (IID) Bernoulli random variables with parameter α_k, i.e.

$$P\{B_t^k = 1\} = 1 - P\{B_t^k = 0\} = \alpha_k \tag{2}$$

for some $0 \le \alpha_k \le 1$, Y_t are integer-valued IID random variables with mean 0, and Z_{t+1}^k denotes the increment dependent on X_t^k, i.e.

$$P\{Z_{t+1}^k = j - i | X_t^k = i\} = q_{ij}^k \tag{3}$$

for $i, j \in \mathcal{S}$. The function $\xi(x)$ determines the boundary conditions so that

$$\xi(x) = \begin{cases} 1, & x \le 1, \\ x, & 1 < x \le K, \\ K+1, & x \ge K+1. \end{cases} \tag{4}$$

The kth set of random variables $\{B_t^k\}$ and $\{Z_t^k\}$ is independent of $\{Y_t\}$ and the other processes. Hence, the Markov chains $\{X_t^k\}$ are correlated only through the process $\{Y_t\}$ which is the common factor to all the Markov chains. Note that, when $B_t^k = 0$ with probability 1 for all t and k, the processes $\{X_t^k\}$ constitute independent, time-homogeneous Markov chains each with one-step transition probabilities q_{ij}^k.

The model (1) can be seen as follows. Suppose that $X_t^k \neq K+1$ and that $X_{t+1}^k \neq 1, K+1$. Then, from (1) and (4), the increment of the Markov chain $\{X_t^k\}$ is given by

$$X_{t+1}^k - X_t^k = (-1)^{\delta_k} B_{t+1}^k Y_{t+1} + Z_{t+1}^k. \tag{5}$$

Hence, the common random variable Y_{t+1} is considered to represent a *systematic risk*, while Z_{t+1}^k is a *specific risk* of firm k. The factor $(-1)^{\delta_k} B_{t+1}^k$ corresponds to the "β" in the CAPM. It also concerns with the correlation coefficient between the Markov chains. To see this, define

$$c_{k\ell}^n = \mathrm{Cov}\Big(X_{t+1}^k - X_t^k,\ X_{t+1}^\ell - X_t^\ell\Big),$$

where Cov denotes the covariance operator. For simplicity, assume that (5) holds true. It then follows from (1) and the imposed independent assumptions that

$$c_{k\ell}^n = (-1)^{\delta_k + \delta_\ell} \alpha_k \alpha_\ell V[Y_{t+1}],$$

where $V[Y_t]$ is the variance of Y_t. Also, since $E[Y_{t+1}] = 0$ by our assumption, we obtain

$$V[X_{t+1}^k - X_t^k] = V[Z_{t+1}^k] + \alpha_k V[Y_{t+1}].$$

It follows that the correlation coefficient between the increments of the Markov chains is given by

$$\ell_{k\ell} = \lambda_k \lambda_\ell, \qquad k, \ell = 1, 2, \cdots, n, \tag{6}$$

where

$$\lambda_k = (-1)^{\delta_k} \sqrt{\alpha_k},$$

provided that $V[Z^k_{t+1}]$ are negligible.

In the following, we assume that the correlation coefficient between the increments of the Markov chains is given by (6).

2 Observed probabilities

For the Markov chain $\{X^k_t\}$, let

$$p^k_{ij} = P\{X^k_{t+1} = j | X^k_t = i\}, \qquad i, j \in \mathcal{S},$$

and suppose that

$$P\{Y_t = i\} = r_i, \quad i = 0, \pm 1, \pm 2, \cdots. \tag{7}$$

We note that the distribution (7) must satisfy the condition $E[Y_t] = 0$. Also, the transition probabilities p^k_{ij} determine a marginal conditional distribution in the sense that they do not determine the *joint* distribution of $(X^1_t, X^2_t, \cdots, X^n_t)$.

Suppose $X^k_t \neq K+1$ and suppose first that $X^k_{t+1} \neq 1, K+1$. Then, from (1) and the law of total probability, we obtain

$$p^k_{ij} = (1-\alpha_k) q^k_{ij} + \alpha_k \sum_{m \in \mathcal{S}} q^k_{im} r_{j-m} \tag{8}$$

where $j \neq 1, K+1$. Next, for $j = 1$, we have

$$p^k_{i1} = (1-\alpha_k) q^k_{i1} + \alpha_k \sum_{m \in \mathcal{S}} q^k_{im} R_{1-m}, \tag{9}$$

whereas for $j = K+1$

$$p^k_{i,K+1} = (1-\alpha_k) q^k_{i,K+1} + \alpha_k \sum_{m \in \mathcal{S}} q^k_{im} \overline{R}_{K+1-m}, \tag{10}$$

where $R_i = \sum_{k \leq i} r_k$ and $\overline{R}_i = \sum_{k \geq i} r_k$.

Denote the transition matrix of $\{X^k_t\}$ by $\mathbf{P}_k = (p^k_{ij})$. Also, let $\mathbf{Q}_k = (q^k_{ij})$ and define

$$\mathbf{R} = \begin{pmatrix} R_0 & r_1 & r_2 & \cdots & \overline{R}_K \\ R_{-1} & r_0 & r_1 & \cdots & \overline{R}_{K-1} \\ R_{-2} & r_{-1} & r_0 & \cdots & \overline{R}_{K-2} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ R_{-K} & r_{-K+1} & r_{-K+2} & \cdots & \overline{R}_0 \end{pmatrix}. \tag{11}$$

It follows from (8) – (10) that

$$\mathbf{P}_k = \mathbf{Q}_k \Big((1-\alpha_k)\mathbf{I} + \alpha_k \mathbf{R} \Big), \tag{12}$$

where $\mathbf{I}$ denotes the identity matrix of order $K+1$. It should be noted that the transition matrix $\mathbf{P}_k$ is observable in the market. The α_k can also be obtained from the market data through (6). Assuming that the systematic risk (r_i) can be determined by some means, we need to recover the data $\mathbf{Q}_k$ from (12). That is, if the matrix

$$\tilde{\mathbf{R}} = (1-\alpha_k)\mathbf{I} + \alpha_k \mathbf{R} \tag{13}$$

is convertible, the unknown matrix can be calculated by

$$\mathbf{Q}_k = \mathbf{P}_k \tilde{\mathbf{R}}^{-1}.$$

Especially, in the simplest case that

$$P\{Y_t = 1\} = P\{Y_t = -1\} = r, \quad P\{Y_t = 0\} = 1 - 2r \tag{14}$$

for some $0 < r \leq 1/2$, the matrix in (11) becomes the tri-diagonal matrix

$$\mathbf{R} = \begin{pmatrix} 1-r & r & 0 & \cdots & 0 \\ r & 1-2r & r & \cdots & 0 \\ 0 & r & 1-2r & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1-r \end{pmatrix},$$

whereas the matrix in (13) is given by

$$\tilde{\mathbf{R}} = \begin{pmatrix} 1-r\alpha_k & r\alpha_k & 0 & \cdots & 0 \\ r\alpha_k & 1-2r\alpha_k & r\alpha_k & \cdots & 0 \\ 0 & r\alpha_k & 1-2r\alpha_k & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1-r\alpha_k \end{pmatrix},$$

which is convertible.