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# Credit Events and the Valuation of Credit Derivatives of Basket Type

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## 1 Introduction

Suppose that an agent faces a set of credit events to be considered. A credit event is typically default of a corporate bond or that of the counterparty itself. If a credit event occurs, the agent will not be able to collect the whole amount of promised money at the maturity and so, it may be willing to use a financial instrument, called a *default swap*, that protects the agent partially from the unpredictable credit event.

One of the most common default swaps traded in practice is a credit derivative with the first-to-default feature. In order to explain this derivative with maturity $T$, suppose there are $n$ credit events and let $\tau_j$ denote the occurrence time of credit event $j$, $j = 1, 2, \cdots, n$. Of course, $\tau_j$ are random variables and we shall call them *default times*. Let $I_F$ denote the random variable that represents the identity of the first default before the maturity $T$. If no default occurs until the maturity, we denote $I_F = 0$. That is,

$$I_F = \begin{cases} 0, & \text{if } \min_j \tau_j > T, \\ j, & \text{if } \tau_j = \min_k \tau_k \leq T. \end{cases}$$

Suppose further that the payment $S_j$, when the event $\{I_F = j\}$ occurs, is positive constant and the claimholder will receive $S_j$ at the maturity $T$ of the contract. Then, the payoff from this derivative at the maturity is given by

$$X_F = \sum_{j=0}^{n} S_j 1_{\{I_F = j\}}, \tag{1}$$

where $1_A$ denotes the indicator function. Note that $S_0$ is the payoff when no default occurs until the maturity $T$, and typically $S_0 = 0$. A similar credit derivative is considered in Duffie (1998) by analyzing the first-to-default time in terms of stochastic intensity processes.

Another default swap traded frequently in practice is a credit derivative that protects claimholders from the first *two* defaults. In order to explain this derivative with maturity $T$, we need to consider the following two cases:

- There is the exactly one default before the maturity;
- There are more than two defaults before the maturity.

Let $I_D^1$ denote the random variable that represents the identity of default if only one default occurs before the maturity $T$. If no default occurs until the maturity, we denote $I_D^1 = 0$. Also, let $I_D^2$ be the ordered pair of identities $j$ and $k$ if $j$ defaults first and then $k$ defaults second before the maturity $T$. That is,

$$I_D^1 = \begin{cases} 0, & \text{if } \min_j \tau_j > T, \\ j, & \text{if } \tau_j \leq T \text{ and } \tau_k > T \text{ for all } k \neq j, \end{cases}$$

and $I_D^2 = (j, k)$ if $\tau_j = \min_i \tau_i \leq T$ and $\tau_k = \min_{i \neq j} \tau_i \leq T$. Note that, for the event $\{I_F = j\}$, there may be more defaults after $j$ while $\{I_D^1 = j\}$ means that there is no more default before the maturity. Since these events are mutually exclusive, the payoff of this derivative at the maturity $T$ is given by

$$X_D = \sum_{j=0}^{n} S_j 1_{\{I_D^1 = j\}} + \sum_{j=1}^{n} \sum_{k \neq 0, j} (S_j + S_k) 1_{\{I_D^2 = (j,k)\}}.$$

A default swap that protects claimholders from a more general type of defaults can be defined in the same manner.

In this paper, we consider default swaps that protect claimholders from the first default (called type F) and the first two defaults (called type

D) and provide their pricing formulas under the risk-neutral valuation framework. To this end, we intend to formulate directly the joint survival probability

$$S(t_0, t_1, \cdots, t_n) = P\{\tau_0 > t_0, \tau_1 > t_1, \cdots, \tau_n > t_n\}$$

rather than to formulate the first-to-default and the second-to-default times. This is the significant difference of this paper from others such as Duffie (1998). The default time $\tau_0$ is needed to represent the discount factor of the *default-free* interest rates.

Throughout this paper, we fix the probability space $(\Omega, \mathcal{F}, P)$ and denote the expectation operator by $E$. The probability measure $P$ is the risk-neutral measure, since we are interested in pricing of financial instruments. The canonical filtration generated by the underlying stochastic structure is denoted by $\{\mathcal{F}_t\}$, which defines the information available at each time. The conditional probability measure given $\mathcal{F}_t$ is denoted by $P_t$ and the associated conditional expectation operator is $E_t$.

## 2 A Model for Default

In what follows, we consider $n$ credit events and their default times are denoted by $\tau_j$, $j = 1, 2, \cdots, n$. Also, we shall denote the killing time associated with the default-free spot rate by $\tau_0$. The corresponding default processes are $h_j(t)$, $j = 0, 1, \cdots, n$. In order to price credit derivatives of basket type, it is essential to obtain the joint distribution of the default times $\tau_j$.

Suppose that $h_j(t)$ are stochastic processes satisfying the "integrability" condition and that they are *not* independent. However, we assume that, given $\mathcal{F}_T$ where $T \geq \max_j t_j$, the default times $\tau_j$ are *conditionally independent*. The conditional independence means that, given the realization $\mathcal{F}_T$, we have

$$P_T\{\tau_0 > t_0, \cdots, \tau_n > t_n\} = \exp\left\{-\sum_{j=0}^{n} H_j(t_j)\right\}.$$

By the law of total probability, it follows that

$$P\{\tau_0 > t_0, \tau_1 > t_1, \cdots, \tau_n > t_n\} = E\left[\exp\left\{-\sum_{j=0}^{n} H_j(t_j)\right\}\right].$$

It should be noted that the conditional independence does not imply the ordinary independence and vice versa.

## 3 Valuation of Default Swaps

In this section, we consider the pricing formula of the type F default swap (the type D swaps will be discussed at the presentation). Note that we will not specify the default processes explicitly here, but use the joint survival probability for this purpose. In the following, it is assumed that the payments $S_j$ are nonnegative constant and claimholders will receive them at the maturity $T$ of the contract according to the occurrence of credit events. From (1), the current price of the default swap of type F is given by

$$\pi_F = \sum_{j=0}^{n} S_j E\left[e^{-H_0(T)} 1_{\{I_F = j\}}\right]$$

under the risk-neutral valuation framework.

**Proposition 1** *Given the default processes $h_j(t)$, suppose that the default times $\tau_j$ are conditionally independent. Then the price of the type F default swap is given by*

$$\sum_{j=1}^{n} S_j \int_0^T E\left[h_j(t) e^{-H_0(T) - \sum_{k=1}^{n} H_k(t)}\right] dt + S_0 E\left[\exp\left\{-\sum_{k=0}^{n} H_k(T)\right\}\right].$$


## References

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