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Two-queue and two-server model with a threshold control service policy

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1. The Model

Threshold-based service policies have been applied by many authors to queueing systems as policies to control service rate, number of servers or vacation, and proved to be optimal to some queueing systems. In this paper we consider a threshold-based service policy for a queueing system with two queues: Q_1 and Q_2, two homogeneous servers and infinite buffer capacities. The arrival processes in Q_i is a Poisson with rate λ_i, and the service time distribution at Q_i is exponential with the parameter μ_i. The two control levels $v(\geq 0)$ and $N(\geq v)$ are set up in Q_2. The service policy is as follows.

(1). At each epoch of service completion in Q_1, (i) when the two servers all serve in Q_1, if the number of customers in Q_2 exceeds the threshold v, the server switches its service to Q_2, otherwise it continues to serve the customers in Q_1; (ii) when the two servers server respectively in Q_1 and Q_2, if the number of customers in Q_2 exceeds the threshold N, the server switches its service to Q_2, otherwise it continues to serve the customers in Q_1. (2). At each epoch of service completion in Q_2, (i) when the two servers all serve in Q_2, if the number of customers in Q_2 drops below the threshold N, the server switches its service to Q_1, otherwise it continues to serve the customers in Q_2; (ii) when the two servers server respectively in Q_1 and Q_2, if the number of customers in Q_2 drops the threshold v, the server switches its service to Q_1, otherwise it continues to serve the customers in Q_2. (3). The server does not idle if there are customers present at either queue. The service is first-come-first-served within each queue and non-preemptive.

We derive the generating functions of the joint stationary queue-length distribution, and calculate the mean queue lengths and the mean waiting times.

2. Generating function equations

Let $\rho_i \equiv \lambda_i/\mu_i, i = 1,2$. The system is ergodic iff $\rho \equiv \rho_1 + \rho_2 < 2$. Let $Q_i(t)$ be the number of customers waiting for service in queue Q_i at time t, and $I_i(t)$ the number of servers serving in queue Q_i at time t. Then $\{(I_1(t), I_2(t), Q_1(t), Q_2(t))\}_{t\geq 0}$ is an irreducible continuous-time Markov chain. We denote its equilibrium probabilities by $\{p_{i,j,n,m},\ 0 \leq i+j \leq 2; n,m \geq 0\}$, that is,

$$p_{i,j,n,m} = \lim_{t\to\infty} P((I_1(t), I_2(t), Q_1(t), Q_2(t)) = (i,j,n,m)). \tag{2.1}$$

For $|z| \leq 1, |w| \leq 1$, define respectively two-dimensional generating functions and one-dimensional generating functions as follows

$$\Psi_i(z,w) = \sum_{n=0}^{\infty}\sum_{m=0}^{\infty} p_{i,n,m} z^n w^m, \quad \psi_{im}(z) = \sum_{n=0}^{\infty} p_{i,n,m} z^n, \qquad i = 0,1,2;\ 0 \leq m \leq N. \tag{2.2}$$

Then we derive the following generating function equations,

$$\alpha_0(z,w)\Psi_0(z,w) + \mu_1\Psi_1(z,w) = 2\mu_2 \sum_{m=0}^{N}(\psi_{0m}(z) - \psi_{0m}(0))w^m + \mu_1\sum_{m=0}^{N}(\psi_{1m}(z) - \psi_{1m}(0))w^m + 2\mu_2 p_{0,0,0} + \mu_1 p_{1,0,0} - \lambda_2 w p_{0,1} \tag{2.3}$$

$$\alpha_1(z,w)z\Psi_1(z,w) + 2\mu_1 z\Psi_2(z,w) = -2\mu_2 w\sum_{m=0}^{N}(\psi_{0m}(z) - \psi_{0m}(0))w^m - \mu_1 w\sum_{m=0}^{N}(\psi_{1m}(z) - \psi_{1m}(0))w^m + \mu_2 z\sum_{m=0}^{v}(\psi_{1m}(z) - \psi_{1m}(0))w^m + 2\mu_1 z\sum_{m=0}^{v}(\psi_{2m}(z) - \psi_{2m}(0))w^m + \mu_2 z p_{1,0,0} + 2\mu_1 z p_{2,0,0} + \lambda_1 z w p_{0,1} + \lambda_2 z w p_{1,0} \tag{2.4}$$

$$\alpha_2(z,w)z\Psi_2(z,w) = \mu_2\sum_{m=0}^{v}(\psi_{1m}(z) - \psi_{1m}(0))w^m + 2\mu_1\sum_{m=0}^{v}(\psi_{2m}(z) - \psi_{2m}(0))w^m + \lambda_1 z p_{1,0}, \tag{2.5}$$

where $\alpha_0(z,w) = 2\mu_2(1-w) - w(\lambda_1(1-z) + \lambda_2(1-w))$,

$\alpha_1(z,w) = \mu_2(1-w) - w(\lambda_1(1-z) + \lambda_2(1-w) + \mu_1)$, $\quad \alpha_2(z,w) = \lambda_1(1-z) + \lambda_2(1-w) + 2\mu_1$.

Furthermore, we have the following equations for one-dimensional generating functions,

$$\beta_0(z)\psi_{00}(z) = 2\mu_2 p_{0,0,1} + \mu_1 p_{1,0,1} + \lambda_2 p_{0,1},$$

$$\beta_0(z)\psi_{0m}(z) = \lambda_2\psi_{0(m-1)}(z) + 2\mu_2 p_{0,0,m+1} + \mu_1 p_{1,0,m+1}, \quad 1 \leq m \leq N-1, \tag{2.6}$$

$$\beta_1(z)\psi_{10}(z) = 2\mu_2\psi_{00}(z) - 2\mu_2 p_{0,0,0} - \mu_1 p_{1,0,0} + \mu_2 z p_{1,0,1} + 2\mu_1 z p_{2,0,1} + \lambda_1 z p_{0,1} + \lambda_2 z p_{1,0} + \mu_2 z(p_{1,0,1}\delta_{\{v>0\}} + \psi_{11}(z)\delta_{\{v=0\}}) + 2\mu_1 z(p_{2,0,1}\delta_{\{v>0\}} + \psi_{21}(z)\delta_{\{v=0\}}),$$

$$\beta_1(z)\psi_{1m}(z) = 2\mu_2\psi_{0m}(z)+\lambda_2 z\psi_{1(m-1)}(z)+\mu_2 z\psi_{1(m+1)}(z)\delta_{\{m+1>v\}}+2\mu_1 z\psi_{2(m+1)}(z)\delta_{\{m+1>v\}}-2\mu_2 p_{0,0,m}$$
$$+\mu_2 z p_{1,0,m+1}\delta_{\{m+1\leq v\}} - \mu_1 p_{1,0,m} + 2\mu_1 z p_{2,0,m+1}\delta_{\{m+1\leq v\}}, \quad 1\leq m\leq N-1, \tag{2.7}$$
$$\beta_2(z)\psi_{20}(z) = \mu_2\psi_{10}(z) - \mu_2 p_{1,0,0} - 2\mu_1 p_{2,0,0} + \lambda_1 z p_{1,0},$$
$$\beta_2(z)\psi_{2m}(z) = \mu_2\psi_{1m}(z)+\lambda_2 z\psi_{2(m-1)}(z)-\mu_2 p_{1,0,m}-2\mu_1 p_{2,0,m}, \quad 1\leq m\leq v, \tag{2.8}$$
$$\beta_3(z)\psi_{2m}(z) = \lambda_2\psi_{2(m-1)}(z), \quad v<m\leq N, \tag{2.9}$$
where $\beta_0(z) = \lambda_1(1-z)+\lambda_2+2\mu_2$, $\beta_1(z) = z(\lambda_1(1-z)+\lambda_2+\mu_2)-\mu_1(1-z)$,
$\beta_2(z) = z(\lambda_1(1-z)+\lambda_2)-2\mu_1(1-z)$, $\beta_3(z) = \lambda_1(1-z)+\lambda_2+2\mu_1$.

3. Determination of the function equations

In this section, we determine the two-dimensional generating functions $\Psi_i(z,w), i=0,1,2$.

Theorem 1. If $\rho<2$, then for every fixed $|z|\leq 1$,

(i) $\alpha_i(z,w)$ has exactly two zeros $y_i(z), w_i(z)$ for $i=0,1$, and $\alpha_2(z,w)$ has exactly one zero $y_2(z)$.

$$y_i(z), w_i(z) = \frac{\lambda_1(1-z)+\lambda_2+i\mu_1+(2-i)\mu_2}{2\lambda_2} + \frac{\sqrt{(\lambda_1(1-z)+\lambda_2+i\mu_1+(2-i)\mu_2)^2-4(2-i)\lambda_2\mu_2}}{2\lambda_2}, \quad i=0,1, \tag{3.1}$$
$$y_2(z) = \frac{\lambda_1(1-z)+\lambda_2+2\mu_1}{\lambda_2} \tag{3.2}$$

(ii) The zeros $y_i(z), i=0,1,2$ are in $|w|>1$, and $w_i(z), i=0,1$ are in $|w|\leq 1$. Furthermore, $w_0(z)\neq w_1(z)$.

(iii) $w_i(z)$ is analytic in $|z|<1$ and continuous in $|z|\leq 1$ for $i=0,1$.

Using the zeros $w_i(z), i=0,1$ and the equations (2.6)–(2.9), we obtain the matrix equation
$$\mathcal{M}(z)(\mathbf{\Pi}(z)-\mathbf{\Pi}(0)) = \mathcal{E}(z)\mathbf{P}+\mathcal{K}(z)\hat{\mathcal{P}}, \tag{3.3}$$
where $\mathbf{\Pi}(z) = (\mathbf{\Pi}_1^\tau(z), \mathbf{\Pi}_2^\tau(z), \psi_{0N}(z))^\tau$, $\mathbf{P} = (\mathcal{P}_0^\tau, \mathcal{P}_1^\tau, \mathcal{P}_2^\tau)^\tau$, $\hat{\mathcal{P}} = (p_{0,0}, p_{0,1}, p_{1,0})^\tau$,
and $\mathbf{\Pi}_i(z) = (\Psi_{i0}(z),\cdots,\Psi_{iN}(z))^\tau, i=0,1$, $\mathbf{\Pi}_2(z) = (\Psi_{20}(z),\cdots,\Psi_{2v}(z))^\tau$,
$\mathcal{P}_i = (p_{i,0,0},\cdots,p_{i,0,N})^\tau$, $i=0,1$, $\mathcal{P}_2 = (p_{2,0,0},\cdots,p_{2,0,v})^\tau$.

Theorem 2. $det\mathcal{M}(z)$ has exactly $N+v+3$ zeros $z_i, i=0,\cdots,N+v+2$ in $|z|\leq 1$. Especialy, $z_0=1$.

Using the $N+v+2$ zeros $z_i, i=1,\cdots,N+v+2$, the normalizing condition $\Psi_0(1,1)+\Psi_1(1,1)+\Psi_2(1,1)+p_{0,0}+p_{0,1}+p_{1,0}=1$ and the stationary equations:
$$(\lambda_1+\lambda_2+\mu_1)p_{1,0} = \lambda_1 p_{0,0}+2\mu_1 p_{2,0,0}+\mu_2 p_{1,0,0},$$
$$(\lambda_1+\lambda_2+\mu_2)p_{0,1} = \lambda_2 p_{0,0}+\mu_1 p_{1,0,0}+2\mu_2 p_{0,0,0}, \qquad (\lambda_1+\lambda_2)p_{0,0} = \mu_1 p_{1,0}+\mu_2 p_{0,1},$$
$$(\lambda_1+\lambda_2+2\mu_2)p_{0,0,m} = \lambda_2 p_{0,0,m-1}+\mu_1 p_{1,0,m+1}+2\mu_2 p_{0,0,m+1}, \quad 0<m\leq N-1$$
$$(\lambda_1+\lambda_2+2\mu_2)p_{0,0,0} = \lambda_2 p_{0,1}+\mu_1 p_{1,0,1}+2\mu_2 p_{0,0,1},$$
we obtain a matrix system with $2N+v+6$ equations for the unknown constants $\mathbf{P}, \hat{\mathcal{P}}$ as follows,
$$\mathcal{Q}[\mathbf{P}^\tau, \hat{\mathcal{P}}^\tau]^\tau = \boldsymbol{\nu} \tag{3.4}$$
where $\boldsymbol{\nu} = (0,\cdots,0,1)^\tau$ is an $(2N+v+6)$-dimensional vector. Solving the system and substituting the solution into (3.3), we obtain the one-dimensional generating functions $\psi_{im}(z), i=0,1; 0\leq m\leq N$ and $\psi_{2m}(z), 0\leq m\leq v$, and then substituting these functions into (2.3)–(2.5), we can finally determine the two-dimensional generating functions $\Psi_i(z,w), 0\leq i\leq 2$.

4. The mean queue lengths and mean waiting times

The mean queue lengths $E[Q_i]$ and mean waiting times $E[W_i]$ for the queue $Q_i, i=1,2$ can be calculated by the following formulae,
$$E[Q_1] = \sum_{i=0}^{2}\frac{\partial}{\partial z}\Psi_i(z,1)|_{z=1}, \qquad E[Q_2] = \sum_{i=0}^{2}\frac{\partial}{\partial w}\Psi_i(1,w)|_{w=1}, \tag{4.1}$$
$$E[W_1] = \frac{E[Q_1]}{\lambda_1}, \qquad E[W_2] = \frac{E[Q_2]}{\lambda_2}. \tag{4.2}$$

References

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