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Optimal Life Insurance and Portfolio Choice in a Life Cycle

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1 Introduction

This paper considers an optimal life insurance for a householder subject to mortality risk. The household receives a wage income continuously, which is terminated by either the householder's death or retirement, whichever happens first. In order to hedge the risk to lose the income by an unpredictable death, the household enters a life insurance contract by paying a premium to an insurance company. The household may also invest their wealth into a financial market. The problem is to determine an optimal insurance/investment strategy in order to maximize the expected total, discounted utility from consumption and terminal wealth.

2 The Model

Let T, $T > 0$, be the maturity of an insurance contract. We consider a continuous-time economy in $\mathcal{T} \stackrel{\text{def}}{=} [0, T]$, consisting of the insurance contract, and a frictionless and perfectly competitive financial market. On a given $(\Omega, \mathcal{F}, P)$,

$Z = \{Z(t); t \in \mathcal{T}\} \stackrel{\text{def}}{=}$ the standard Brownian motion,

$N = \{N(t); t \in \mathcal{T}, N(0) = 0\} \stackrel{\text{def}}{=}$ a Poisson process with intensity process $\lambda = \{\lambda(t), t \in \mathcal{T}\}$,

$\mathbb{F}^Z = \{\mathcal{F}_t^Z = \sigma\{Z(s); s \le t\}; t \in \mathcal{T}\}$,

$\mathbb{F}^N = \{\mathcal{F}_t^N = \sigma\{1_{\{\tau \le s\}}; s \le t\}; t \in \mathcal{T}\}$,

$\mathbb{F} = \{\mathcal{F}_t = \mathcal{F}_t^Z \vee \mathcal{F}_t^N; t \in \mathcal{T}\}$.

$\tau = \inf\{t > 0; N(t) = 1\} \stackrel{\text{def}}{=}$ the time of the householder's death.

We assume that λ is predictable w.r.t. $\mathbb{F}^Z$, that $\mathbb{F}$ and $\mathbb{F}^N$ satisfy the usual conditions, and that

$$P\{\tau > t | \lambda\} = \exp\left\{-\int_0^t \lambda(u) du\right\}, \quad t \in \mathcal{T}.$$

$P_0(t) \stackrel{\text{def}}{=}$ time t price of the riskless security;

$$\frac{dP_0(t)}{P_0(t)} = r(t)dt, \quad t \in \mathcal{T}, \quad P_0(0) = p_0,$$

where $r(t)$ is a positive, predictable proc. w.r.t. $\mathbb{F}^Z$.

$P_1(t) \stackrel{\text{def}}{=}$ time t price of the risky security;

$$\frac{dP_1(t)}{P_1(t)} = \mu(t)dt + \sigma(t)dZ(t), \quad t \in \mathcal{T}, \quad P_1(0) = p_1,$$

where $\mu(t)$ and $\sigma(t)$ are progressively measurable processes w.r.t. $\mathbb{F}^Z$.

The life insurance contract is such that an insurance company pays $\theta(t)$ at time $t \iff \tau = t$, $t \le T$, on the other, the household pays $\bar{p}$ at time 0.

$y = \{y(t), t \in \mathcal{T}\} \stackrel{\text{def}}{=}$ the *income process.*

$c = \{c(t), t \in \mathcal{T}\} \stackrel{\text{def}}{=}$; the *consumption process.*

$w(t) \stackrel{\text{def}}{=}$ the amount invested into the risky security at time t.

Given $w = \{w(t); t \in \mathcal{T}\}$, c, $\theta = \{\theta(t); t \in \mathcal{T}\}$, and y, the *wealth process* $W = \{W(t); t \in \mathcal{T}\}$ is defined by

$$\begin{aligned} W(t) &= W_0 - \bar{p} \\ &+ \int_0^t \left(r(s)W(s) + y(s)1_{\{N(s-)=0\}} - c(s)\right) ds \\ &+ \int_0^t w(s) \left[(\mu(s) - r(s))ds + \sigma(s)dZ(s)\right] \\ &+ \int_0^t \theta(s)1_{\{N(s-)=0\}}dN(s) - C(t), \end{aligned} \tag{1}$$

where W_0 is a positive constant, and $C = \{C(t); t \in \mathcal{T}, C(0) = 0\}$ is a nonnegative, nondecreasing process.

Definition 2.1 $(c, W(T))$ is *feasible* if $c(t) \ge 0$ and $W(t) > -\infty$ for $t \in \mathcal{T}$ and $W(T) \ge 0$.

$u_1 : (0, \infty) \to \mathbb{R}$; the utility function for consumption.

$u_2 : (0, \infty) \to \mathbb{R}$; the utility function for the terminal wealth.

$e^{-\int_0^t \rho(s)ds}$, $t \in \mathcal{T}$; *time-discount factor.*

(MP) Given ρ and $u_i(x)$, $i = 1, 2$, find an optimal consumption/portfolio process $(\hat{c}, \hat{w})$ as well as an optimal insurance process $\hat{\theta}$ to

$$\begin{aligned} \text{maximize} \quad & U(c, W(T)) \\ & = E\left[\int_0^T e^{-\int_0^t \rho(s)ds} u_1(c(t))dt \right. \\ & \quad \left. + e^{-\int_0^T \rho(s)ds} u_2(W(T))\right] \\ \text{s.t.} \quad & (c, W(T)) \in \mathcal{C}, \end{aligned}$$

where

$$\mathcal{C} \stackrel{\text{def}}{=} \left\{ \text{feasible } (c, W(T)); E\left[\int_0^T e^{-\int_0^t \rho(s)ds} u_1^-(c(t))dt \right.\right.$$

$$\left.\left. + e^{-\int_0^T \rho(s)ds} u_2^-(W(T))\right] < \infty \right\}.$$

3 Main Results

Let $\phi(t)$ be the state price density at time t;

$$\phi(0) = 1, \qquad 0 < \phi(t) < \infty,$$
$$E_t[\phi(s)P_j(s)] = \phi(t)P_j(t),\ j = 0, 1, \quad \forall s, t \in \mathcal{T}; s > t,$$

where E_t denotes the conditional expectation given $\mathcal{F}_t$ under P with $E = E_0$.

Lemma 3.1
(i) *The state price density is represented as*

$$\phi(t) = \beta(t)\phi^Z(t)\phi^N(t), \quad t \in \mathcal{T},$$

where

$$\beta(t) = \exp\left\{-\int_0^t r(s)ds\right\},$$
$$\phi^N(t) = \left(\frac{\psi(\tau)}{\lambda(\tau)}1_{\{\tau\le t\}} + 1_{\{\tau>t\}}\right)e^{\int_0^{t\wedge\tau}(\lambda(s)-\psi(s))ds},$$
$$\phi^Z(t) = \exp\left\{-\int_0^t \xi(s)dZ(s) - \frac{1}{2}\int_0^t \xi^2(s)ds\right\};$$
$$\xi(t) = \frac{\mu(t) - r(t)}{\sigma(t)}.$$

Here, $\psi = \{\psi(t), t \in \mathcal{T}\}$ is a positive, predictable process w.r.t. $\mathbb{F}^Z$.
(ii) *The process ψ represents the intensity process under the equivalente martingale measure Q.*

To specify ψ, we introduce a fictitious risky security as in Kartzas and Shreve (1998). That is, for $v \in \mathbb{R}$, suppose that there exists a fictitious risky-security whose price proc;

$$P_2(t) = p_2 e^{vt}1_{\{\tau>t\}} + e^{v\tau + \int_\tau^t r(u)du}1_{\{\tau\le t\}},\ t \ge 0.$$

To make the dependence of $v \in \mathbb{R}$ explicit, we denote the state price density by $\phi_v(t) = \beta(t)\phi^Z(t)\phi_v^N(t), \quad t \in \mathcal{T}$.

$Q_v \stackrel{\text{def}}{=} \tilde{Q}$ associated with the fictitious market.

$\psi_v = \{\psi_v(t), t \in \mathcal{T}\} \stackrel{\text{def}}{=}$ the intensity process under Q_v.

$w_2(t) \stackrel{\text{def}}{=}$ the amount invested into the fictitious security at time t.

To solve (MP), we consider the following problem:

$$\textbf{(TP)} \quad \min_{(\zeta,v)\in(0,\infty)\times\mathbb{R}} J(\zeta, v),$$

where

$$J(\zeta, v)$$
$$= E\left[\int_0^T \tilde{u}_1(\zeta\phi_v(t), t)dt + \tilde{u}_2(\zeta\phi_v(T), T)\right.$$
$$\left. + \zeta\left(W_0 + \int_0^T \phi_v(t)(y(t) + \psi_v(t)\theta(t))1_{\{N(t-)=0\}}dt\right)\right]$$

with

$$\tilde{u}_i(z, t) = \sup_{c\ge 0}\left[e^{-\int_0^t \rho(s)ds}u_i(c) - zc\right],\ t \in \mathcal{T},\ i = 1, 2.$$

$I_i(x, t) \stackrel{\text{def}}{=}$ the inverse function of $\frac{d}{dx}\left[u_i(x)e^{-\int_0^t \rho(s)ds}\right]$ w.r.t. x for each $i = 1, 2$ and $t \in \mathcal{T}$.

Theorem 3.1 *For $\forall c \in (0, \infty)$, suppose that $\exists \alpha \in (0, 1)$, $\gamma \in (0, \infty)$; $\alpha u_i'(c) \ge u_i'(\gamma c)$. Let (ζ^*, v^*) be a solution to* (TP)*. Then, $\hat{c}$ and $\hat{W}$ are given by*

$$\hat{c}(t) = I_1(\zeta^*\phi_{v^*}(t), t),$$
$$\hat{W}(t) = \frac{1}{\beta(t)}E_t^{Q_{v^*}}\left[\int_t^T \beta(s)\left(\hat{c}(s) - (y(s) + \psi_{v^*}(s)\theta(s))1_{\{N(s-)=0\}}\right)ds + \beta(T)\hat{W}(T)\right], \quad t \in \mathcal{T},$$

with $\hat{W}(T) = I_2(\zeta^*\phi_{v^*}(T), T)$. ζ^* satisfies

$$E^{Q_{v^*}}\left[\int_0^T \beta(t)I_1(\zeta^*\phi_{v^*}(t), t)dt + \beta(T)I_2(\zeta^*\phi_{v^*}(T), T)\right]$$
$$= W_0 + E^{Q_{v^*}}\left[\int_0^T \beta(t)y(t)1_{\{N(t-)=0\}}dt\right].$$

$(\hat{w}, \hat{\theta})$ is given by the stochastic representation of

$$E_t^{Q_{v^*}}\left[\int_0^T \beta(s)\left(\hat{c}(s) - (y(s) + \psi_{v^*}(s)\theta(s))1_{\{N(s-)=0\}}\right)ds + \beta(T)\hat{W}(T)\right], \quad t \in \mathcal{T},$$

and (1).

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