# Mean Waiting Times in Markovian Polling Systems

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## 1. Introduction.

Polling systems with various polling orders have been studied extensively. Deterministic polling orders are considered in [1]. *Random polling* systems are the systems with the random polling orders [3]. *Markovian polling* systems with the Markovian polling orders are investigated in [2, 4]. We make another approach to the analysis of the Markovian polling systems with infinite buffer capacities and obtain the mean waiting times.

## 2. Model description.

A single server serves $J$ classes of customers at $J$ stations. Customers arrive at station $i$ from outside the system (called $i$-customers) according to a Poisson process with rate $\lambda_i$ ($i = 1,\ldots,J$; $\lambda \equiv \sum_{i=1}^{J} \lambda_i$). Service times $S_i$ of $i$-customers are independently, identically and arbitrarily distributed with mean $\overline{s_i}$ and second moment $\overline{s_i^2}$. The server serves customers according to a predetermined scheduling algorithm where $i$-customers are admitted into the service facility in either a *gated* fashion ($i \in \Pi_G$) or an *exhaustive* fashion ($i \in \Pi_E$). The other customers should wait for service in the waiting rooms. The server utilizations are defined by $\rho_i \equiv \lambda_i \overline{s_i}$ and $\rho \equiv \sum_{i=1}^{J} \rho_i$. After completing services of all customers in the service facility at station $i$, the server selects station $j$ with probability $p_{ij}$ ($i,j = 1,\ldots,J$). Let $P \equiv (p_{ij} : i,j = 1,\ldots,J)$. An arbitrarily distributed switchover time $S_{ij}^o$ with mean $\overline{s_{ij}^o}$ and second moment $\overline{s_{ij}^{o2}}$ is incurred at every time when the server switches from station $i$ to station $j$.

Let $\Pi$ and $\Pi^s$ be the sets of service periods and of switchover periods, respectively. For any time $t$, let $\kappa(t) \in \Pi \cup \Pi^s$ denote the period, and let $r(t)$ denote a remaining service time of a customer being served if $\kappa(t) \in \Pi$, or a remaining length of a switchover period if $\kappa(t) \in \Pi^s$. The number of $i$-customers in the service facility (who are not being served) is denoted by $g_i(t)$, and the number of $i$-customers in the waiting room is denoted by $n_i(t)$ ($i = 1,\ldots,J$). Let $g(t) \equiv (g_1(t),\ldots,g_J(t))$ and $n(t) \equiv (n_1(t),\ldots,n_J(t))$. The other informations at time $t$ are accumulated in $L(t)$. Then we define the stochastic process $\mathcal{Q} = \{Y(t) = (\kappa(t), r(t), g(t), n(t), L(t)) : t \geq 0\}$ with state space $\mathcal{E}$. The $e^{th}$ customer ($c^e$) arrives from outside the system at epoch $\tau_0^e$ ($e = 1,2,\ldots$). Then let $\tau_k^e$ be the time epoch just when the server visits a station for the $k^{th}$ time counting from $c^e$'s arrival epoch ($k = 1,2,\ldots$). Let $Z^e(t)$ denote his station staying at time $t$.

**The performance measures.**

For $t \geq 0$ and $i = 1,\ldots,J$, let $C_{Wi}^e(t) \equiv 1$ if $c^e$ stays in the waiting room at station $i$ at time $t$, or $\equiv 0$ otherwise; and let $C_{Fi}^e(t) \equiv 1$ if $c^e$ waits for service in the service facility as an $i$-customer at time $t$, or $\equiv 0$ otherwise. Let

$$H_i(Y,j,e,l,\kappa_0) \equiv E\left[\int_{\tau_l^e}^{\infty} C_{Wi}^e(t)1\{\kappa(t)=\kappa_0\}dt | A_l^e(Y,j)\right], \qquad H_i^e(\kappa_0) \equiv \int_0^{\infty} C_{Wi}^e(t)1\{\kappa(t)=\kappa_0\}dt,$$

$$H_i^0(Y,j,e,l,\kappa_0) \equiv E\left[\int_{\tau_l^e}^{\tau_{l+1}^e} C_{Wi}^e(t)1\{\kappa(t)=\kappa_0\}dt | A_l^e(Y,j)\right], \tag{1}$$

$$F_i(Y,j,e) \equiv E\left[\int_{\tau_0^e}^{\infty} C_{Fi}^e(t)dt | A_0^e(Y,j)\right], \qquad F_i^e \equiv \int_0^{\infty} C_{Fi}^e(t)dt, \tag{2}$$

for $Y \in \mathcal{E}; i,j = 1,\ldots,J; \kappa_0 \in \Pi \cup \Pi^s$ and $l = 0,1,2,\ldots$ where $A_l^e(Y,j) \equiv \{Y(\tau_l^e) = Y, Z^e(\tau_l^e) = j\}$. The performance measure $H_i(\cdot)$ denotes the conditional expected waiting time of cutomers in the waiting room of station $i$, and $F_i(\cdot)$ denotes the conditional expected waiting time of cutomers in the service facility of station $i$. Then it can be shown that

$$H_i(Y,j,e,l,\kappa_0) = \begin{cases} H_i^0(Y,j,e,l,\kappa_0) + E[H_i(Y(\tau_{l+1}^e),j,e,l+1,\kappa_0)|Y(\tau_l^e) = Y, Z^e(\tau_l^e) = j], \\ \qquad \text{if } (\kappa \neq j) \text{ or } (\kappa = j, l = 0, j \in \Pi_G), \\ 0, \qquad \text{if } (\kappa = j, l = 0, j \in \Pi_E) \text{ or } (\kappa = j, l > 0, j \in \Pi_E \cup \Pi_G). \end{cases} \tag{3}$$

## 3. Expressions of the performance measures.

The expressions of the above two performance measures are given by

$$H_j^0(Y,j,e,l,\kappa_0) = \begin{cases} r\varphi^0(\kappa,j,\kappa_0) + (g,n)h_{00}^0(\kappa,j,\kappa_0) + p_{\kappa_0}h_{01}^0(\kappa,j,\kappa_0), & l = 0, \kappa \in \Pi, \\ r\varphi^0(\kappa,j,\kappa_0), & l = 0, \kappa \in \Pi^s, \\ (g,n)h_{10}^0(\kappa,j,\kappa_0) + p_{\kappa_0}h_{11}^0(\kappa,j,\kappa_0), & l > 0, \kappa \in \Pi, \\ 0, & l > 0, \kappa \in \Pi^s, \end{cases} \tag{4}$$

$$F_j(Y,j,e) = r\psi(\kappa,j) + (g,n)f(\kappa,j), \tag{5}$$

for $Y=(\kappa,r,g,n,L)\in\mathcal{E}; j=1,\ldots,J; l=0,1,2,\ldots$ and $\kappa_0\in\Pi\cup\Pi^s$. Further the expected numbers of customers in the system at a beginning epoch of a service period conditioned on the system state at its previous epoch are given by

$$E[(g(\tau_{l+1}^e),n(\tau_{l+1}^e))|\kappa(\tau_{l+1}^e)=k,A_l^e(Y,j)] \;=\; \begin{cases} rv(\kappa)+(g,n)U_0(\kappa)+u_0(j,\kappa,k), & l=0,\kappa\in\Pi,\\ rv+(g,n)U_0+e_j, & l=0,\kappa\in\Pi^s,\\ (g,n)U_1(\kappa)+u_1(\kappa,k), & l>0,\kappa\in\Pi,\\ 0, & l>0,\kappa\in\Pi^s,\end{cases} \qquad (6)$$

for $Y=(\kappa,r,g,n,L)\in\mathcal{E}$ and $j,k\in\Pi$ where $A_l^e(Y,j)\equiv\{Y(\tau_l^e)=Y,Z^e(\tau_l^e)=j\}$. The coefficients in the above expressions (4), (5) and (6) can be calculated from the known quantities given in the last section.

Then for any $Y=(\kappa,r,g,n,L)\in\mathcal{E}; j=1,\ldots,J; l=0,1,2,\ldots$ and $\kappa_0\in\Pi\cup\Pi^s$, we define

$$\hat{H}_j(Y,j,e,l,\kappa_0) \;\equiv\; \begin{cases} r\varphi(\kappa,j,\kappa_0)+(g,n)h_{00}(\kappa,j,\kappa_0)+h_{01}(\kappa,j,\kappa_0), & l=0,\\ (g,n)h_{10}(\kappa,j,\kappa_0)+h_{11}(\kappa,j,\kappa_0), & l>0.\end{cases} \qquad (7)$$

The constants $h_{00}(\cdot),h_{10}(\cdot)\in\mathcal{R}^{2J\times 1}$ and $\varphi(\cdot),h_{01}(\cdot),h_{11}(\cdot)\in\mathcal{R}$ are obtained by solving $J$ sets of linear equations with $O(J^2)$ unknowns whose coefficients consist of the constants given in (4) and (6).

**Proposition 1.** $\hat{H}_j$ $(j=1,\ldots,J)$ defined in (7) satisfy the equation (3) for $i=j$. □

Since uniqueness of the solution can be shown under some assumptions, the functions $\hat{H}_j$ defined in (7) become the performance measures $H_j$ in (1) $(j=1,\ldots,J)$. We further note that these performance measures are linear functions of components $r$ and $(g,n)$ of the system state $Y=(\kappa,r,g,n,L)\in\mathcal{E}$.

## 4. Steady state values.

In this section, we obtain the *mean waiting times* $\bar{D}_j$ for all classes of customers $(j=1,\ldots,J)$. Now let

$$\bar{H}_j(\kappa,\kappa_0)\equiv\lim_{N\to\infty}(1/N)\sum_{e=1}^N E[H_j^e(\kappa_0)\mathbf{1}\{\kappa(\tau_0^e)=\kappa\}|Z^e(\tau_0^e)=j], \qquad (8)$$

$$\bar{F}_j(\kappa)\equiv\lim_{N\to\infty}(1/N)\sum_{e=1}^N E[F_j^e\mathbf{1}\{\kappa(\tau_0^e)=\kappa\}|Z^e(\tau_0^e)=j], \qquad (j\in\Pi;\ \kappa,\kappa_0\in\Pi\cup\Pi^s), \qquad (9)$$

be the average values of the performance measures. Further we define the average values of the system state:

$$\tilde{Y}^\kappa=(\kappa\tilde{q}^\kappa,\tilde{r}^\kappa,\tilde{g}^\kappa,\tilde{n}^\kappa,\tilde{L}^\kappa)\equiv\lim_{t\to\infty}(1/t)\int_0^t E[Y(s)\mathbf{1}\{\kappa(s)=\kappa\}]ds, \qquad (\kappa\in\Pi\cup\Pi^s). \qquad (10)$$

Let $\pi=(\pi_1,\ldots,\pi_J)$ be the steady state probability vector of a Markov chain generated by the transition probability matrix $P$. Then we have $\tilde{q}^\kappa=\lambda_\kappa\overline{s_\kappa}$ and $\tilde{r}^\kappa=\lambda_\kappa\overline{s_\kappa^2}/2$ for $\kappa\in\Pi$, and $\tilde{q}^{(i,j)}=(1-\rho)\pi_i p_{ij}\overline{s_{ij}^o}/\overline{S^o}$ and $\tilde{r}^{(i,j)}=(1-\rho)\pi_i p_{ij}\overline{s_{ij}^{o2}}/(2\overline{S^o})$ for $(i,j)\in\Pi^s$, where $\overline{S^o}=\sum_{i=1}^J\sum_{j=1}^J\pi_i p_{ij}\overline{s_{ij}^o}$. From the generalized version of the Little's formula, the PASTA property, and the expressions (5) and (7), we have a set of equations:

$$\tilde{n}_j^{\kappa_0} \;=\; \lambda_j\sum_{\kappa\in\Pi\cup\Pi^s}\bar{H}_j(\kappa,\kappa_0)=\lambda_j\sum_{\kappa\in\Pi\cup\Pi^s}\{\tilde{r}^\kappa\varphi(\kappa,j,\kappa_0)+(\tilde{g}^\kappa,\tilde{n}^\kappa)h_{00}(\kappa,j,\kappa_0)+\tilde{q}^\kappa h_{01}(\kappa,j,\kappa_0)\}, \qquad (11)$$

$$\tilde{g}_j \;=\; \lambda_j\sum_{\kappa\in\Pi\cup\Pi^s}\bar{F}_j(\kappa)=\lambda_j\sum_{\kappa\in\Pi\cup\Pi^s}\{\tilde{r}^\kappa\psi(\kappa,j)+(\tilde{g}^\kappa,\tilde{n}^\kappa)f(\kappa,j)\}, \qquad (12)$$

for $j\in\Pi$ and $\kappa_0\in\Pi\cup\Pi^s$. ($\tilde{g}_j^\kappa=\tilde{g}_j$ if $\kappa=j\in\Pi$, or $\tilde{g}_j^\kappa=0$ otherwise.)

**Proposition 2.** The mean waiting times are given by

$$\bar{D}_j \;\equiv\; \lim_{N\to\infty}\frac{1}{N}\sum_{e=1}^N E\left[F_j^e+\sum_{\kappa_0\in\Pi\cup\Pi^s}H_j^e(\kappa_0)|Z^e(\tau_0^e)=j\right]=\frac{1}{\lambda_j}\left(\tilde{g}_j+\sum_{\kappa_0\in\Pi\cup\Pi^s}\tilde{n}_j^{\kappa_0}\right), \quad j=1,\ldots,J. \;□ \qquad (13)$$

**Note:** Although the above equations (11) and (12) have $O(J^3)$ unknowns, we can easily reduce them $O(J^2)$ unknowns by defining $\tilde{n}^{\cdot k_1}\equiv\sum_{k_0\in\Pi}\tilde{n}^{(k_0,k_1)}$ and by arranging these equations. □

# References


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