Mathematical Structure of Dominant AHP and Concurrent Convergence Method

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1 Introduction

This study discusses about mathematical structure of dominant AHP and concurrent convergence method(CCM) which are originally developed by Kinoshita and Nakanishi [1]. They introduce a new concept of regulating alternative into an analyzing tool for an evaluation problem with a criterion set J and an alternative set I. The original idea of dominant AHP and CCM is unique but dominant AHP and CCM are not sufficiently analyzed in mathematical theory.

2 Mathematical description of dominant AHP and CCM

The single regulating alternative is called dominant one. Mathematical description of the dominant AHP is as follows:

Step0 : A decision maker(DM) selects a regulating alternative from the alternative set I. Let alternative k be the regulating alternative.

Step1 : From the viewpoint of every criterion $j \in J$, DM evaluate relative importance of all alternatives and quantifies the evaluation values of all alternatives. Let a_{ij} be the evaluation value of alternative i from criterion j and let A be an $|I| \times |J|$ evaluation matrix whose (i,j) element is a_{ij}.

Step2 : From the viewpoint of regulating alternative k, DM evaluates relative importance of all criteria and quantifies the evaluation values of all criteria. Let $\boldsymbol{b}^k$ be a $|J|$ dimensional vector whose jth element is the evaluation value of criteria j from regulating alternative k.

Step 3: Let A_k be a $|J| \times |J|$ diagonal matrix whose (j,j) element is a_{kj}. Calculate $AA_k^{-1}\boldsymbol{b}^k$ and define the ith element of $AA_k^{-1}\boldsymbol{b}^k$ as the overall evaluation value of alternative i.

Let $\hat{\boldsymbol{b}}^i$ be an $|J|$ dimensional vector whose jth element is the unknown evaluation value of the criteria j from the alternative $i \neq k$, then, Kinoshita and Nakanishi [1] propose

$$\hat{\boldsymbol{b}}^i = \frac{A_i A_k^{-1}\boldsymbol{b}^k}{\boldsymbol{e}^\top A_i A_k^{-1}\boldsymbol{b}^k} \tag{1}$$

for all $i \in I \setminus \{k\}$, where $\boldsymbol{e}$ is all one vector and $^\top$ is denoted by the transpose operation. They define

$$AA_i^{-1}\hat{\boldsymbol{b}}^i \tag{2}$$

as the overall evaluation vector derived from alternative i and they point out that $AA_i^{-1}\hat{\boldsymbol{b}}^i$ coincides with $AA_k^{-1}\boldsymbol{b}^k$ without multiplication of scalar for all $i \in I \setminus \{k\}$.

CCM is an analyzing tool for the case of multiple regulating alternatives. Let K be an index set of regulating alternatives, then $\boldsymbol{b}^k$ of regulating alternative $k \in K$ can be given by Step 2 and $|K|$ types of A, say $\left\{A^{(k)} | k \in K\right\}$, can be given by Step 1. In the first stage of CCM, we merge $\left\{A^{(k)} \mid k \in K\right\}$ into a positive matrix A. In the second stage, we employ a following iterative method whose convergence is not guaranteed theoretically:

Algorithm 1

Step 0: For a given set of the evaluation vectors of criteria, $\left\{\boldsymbol{b}^k \mid k \in K\right\}$, in the first stage and let $\boldsymbol{p}_0^k := A_k^{-1}\boldsymbol{b}^k$ for all $k \in K$. Let $t := 0$ and go to Step 1.

Step 1: Let $\boldsymbol{p}_{t+1}^k := \frac{1}{|K|}\sum_{l \in K} \frac{\boldsymbol{p}_t^l}{\boldsymbol{e}^\top A_k \boldsymbol{p}_t^l}$ for all $k \in I$.

Step 2: If $\max\left\{ \|\boldsymbol{p}_{t+1}^k - \boldsymbol{p}_t^k\| \mid k \in K \right\} = 0$ then set $\bar{\boldsymbol{b}}^i = \frac{1}{|K|}\sum_{l \in K} \frac{A_i \boldsymbol{p}_{t+1}^l}{\boldsymbol{e}^\top A_i \boldsymbol{p}_{t+1}^l}$ for all $i \in I$ and $\bar{\boldsymbol{p}}^k = \boldsymbol{p}_{t+1}^k$ for all $k \in K$ and stop. Otherwise, update $t := t+1$ and go to Step 1.

3 Structure of dominant AHP

We focus on only the directions of the overall evaluation vectors, $AA_k^{-1}\boldsymbol{b}^k$ and $AA_i^{-1}\hat{\boldsymbol{b}}^i$, and the evaluation vectors of criteria, $\boldsymbol{b}^k$ and $\hat{\boldsymbol{b}}^i$. So, if a vector $\boldsymbol{a}$

coincides with a vector $\boldsymbol{b}$ without a scalar multiplication, we say that $\boldsymbol{a}$ has the same direction as $\boldsymbol{b}$. For a $|J|$-dimensional vector $\boldsymbol{b}$, we define $V_i(\boldsymbol{b}) = AA_i^{-1}\boldsymbol{b}$ and $B_i^k(\boldsymbol{b}) = A_iA_k^{-1}\boldsymbol{b}$ for all $i,k \in I$. Then, the overall evaluation vector by evaluation rule (2) is the function value $V_i(\hat{\boldsymbol{b}}^i)$. In the sequel, all proofs of lemmas and theorems are omitted.

Theorem 1 *Let $\boldsymbol{b}$ be a $|J|$-dimensional vector, then $V_k(\boldsymbol{b}) = V_i(B_i^k(\boldsymbol{b}))$ for all $i,k \in I$. Suppose that $\hat{\boldsymbol{b}}^i$ is defined by (1) for all $i \in I \setminus \{k\}$, then $V_k(\boldsymbol{b}^k)$ has the same direction as $V_i(\hat{\boldsymbol{b}}^i)$.*

Theorem 2 *Let $\boldsymbol{b}$ be a $|J|$-dimensional vector, then $B_i^k(B_k^i(\boldsymbol{b})) = \boldsymbol{b}$ for all $i,k \in I$. Suppose that $\hat{\boldsymbol{b}}^i$ is defined by (1) for all $i \in I \setminus \{k\}$, then $B_k^i(\hat{\boldsymbol{b}}^i)$ has the same direction as $\boldsymbol{b}^k$.*

The properties in the above theorems might not only hold under the pair of (1) and (2).

Theorem 3 *Let $\boldsymbol{b}$ be a $|J|$-dimensional vector. For two $|J| \times |J|$ matrices M and N let $B_i^k(\boldsymbol{b}) = A_iMA_k^{-1}\boldsymbol{b}$ and $V_i(\boldsymbol{b}) = ANA_i^{-1}\boldsymbol{b}$ for all $i,k \in I$. Suppose that there exists a nonzero scalar λ such that $NM = \lambda N$, then $V_k(\boldsymbol{b})$ has the same direction as $V_i(B_i^k(\boldsymbol{b}))$ for all $i,k \in I$. Moreover, if M^2 is a multiplication of the unit matrix, then $B_i^k(B_k^i(\boldsymbol{b}))$ has the same direction as $\boldsymbol{b}$.*

Corollary 4 *Consider that $B_i^k(\cdot)$ and $V_i(\cdot)$ of Theorem 3. Suppose that there exists a nonzero scalar λ such that $NM = \lambda N$ and let $\hat{\boldsymbol{b}}^i = B_i^k(\boldsymbol{b}^k)$, then $V_i(\hat{\boldsymbol{b}}^i)$ has the same direction as $V_k(\boldsymbol{b}^k)$. Moreover, if M^2 is a multiplication of the unit matrix, then $B_k^i\left(\hat{\boldsymbol{b}}^i\right)$ has the same direction as $\boldsymbol{b}^k$.*

4 Mathematical Structure of CCM

We consider the case that $|K| \geq 2$.

Lemma 5 *The iterate $\boldsymbol{p}_t^k$ is a positive vector for all $k \in K$ and $t = 0,1,\ldots$.*

Lemma 6 *$\boldsymbol{e}^\top A_k\boldsymbol{p}_t^k = 1$ for all $k \in K$ and $t = 0,1,\ldots$.*

Consider the convex cone $Cone\left(\left\{\boldsymbol{p}_{t+1}^k | k \in K\right\}\right)$ which is generated by the vectors $\left\{\boldsymbol{p}_{t+1}^k \mid k \in K\right\}$. For a set D we denote the relative interior and relative boundary of D by $\boldsymbol{ri}D$ and $\boldsymbol{bd}D$, respectively.

Lemma 7 *Let $\boldsymbol{R}$ be an extreme ray set of $Cone\left(\left\{\boldsymbol{p}_t^k | k \in K\right\}\right)$. If $\dim \boldsymbol{R} = 1$, then Algorithm 1 stops. Otherwise, for all $k \in K$ and $t = 0,1,\ldots$, $\boldsymbol{p}_{t+1}^k \in \boldsymbol{ri}Cone\left(\left\{\boldsymbol{p}_t^k | k \in K\right\}\right)$ and $\boldsymbol{p}_{t+1}^k \notin \boldsymbol{R}$.*

Lemma 8 *$Cone\left(\left\{\boldsymbol{p}_{t+1}^k | k \in K\right\}\right) \subset Cone\left(\left\{\boldsymbol{p}_t^k | k \in K\right\}\right)$ and $Cone\left(\left\{\boldsymbol{p}_{t+1}^k \mid k \in K\right\}\right) \cap \boldsymbol{bd}Cone\left(\left\{\boldsymbol{p}_t^k \mid k \in K\right\}\right)$ is the origin for $t = 1,2,\ldots,$*

Lemma 8 means that $Cone\left(\left\{\boldsymbol{p}_t^k \mid k \in K\right\}\right)$ shrinks monotonically for $t = 0,1,\ldots$.

Lemma 9 *Let $S^k = \left\{A_k^{-1}\boldsymbol{b} | \boldsymbol{b} \geq 0,\ \boldsymbol{e}^\top\boldsymbol{b} = 1\right\}$ for all $k \in K$. $\boldsymbol{p}_t^k \in S^k$ for all $k \in K$ and $t = 0,1,\ldots$. There exists an index set T and an accumulation point $\hat{\boldsymbol{p}}_k$ for all $k \in K$ such that $\lim_{t\in T, t\to\infty} \boldsymbol{p}_t^k = \hat{\boldsymbol{p}}^k$.*

Lemma 10 *Suppose that the index set T and $|K|$ points $\left\{\hat{\boldsymbol{p}}^k \mid k \in K\right\}$ satisfy $\lim_{t\in T, t\to\infty} \boldsymbol{p}_k^t = \hat{\boldsymbol{p}}^k$ for all $k \in K$, then $Cone\left(\left\{\hat{\boldsymbol{p}}^k | k \in K\right\}\right)$ is a half-line.*

The following lemma guarantees the existence of a limit point of the sequence $\{\boldsymbol{p}_t^k \mid t = 0,1,\ldots\}$ for all $k \in K$.

Lemma 11 *If Algorithm 1 repeats infinitely, there exist a half line $\boldsymbol{H}$ such that $\lim_{t\to\infty} \boldsymbol{p}_t^k \in \boldsymbol{H}$ for all $k \in K$. Let $\hat{\boldsymbol{p}}^k = \lim_{t\to\infty} \boldsymbol{p}_t^k$, then $\hat{\boldsymbol{p}}^k$ has the same direction as $\hat{\boldsymbol{p}}^l$ for all $k,l \in K$.*

When Algorithm 1 converges within the finite number of iterations, the point set $\{\bar{\boldsymbol{p}}^k \mid k \in K\}$ of Step 2 has the same property as stated in Lemma 11.

Lemma 12 *Suppose that Algorithm 1 stops within the finite number of iteration and let $\bar{\boldsymbol{p}}^k$ be defined by Step 2 of Algorithm 1 for all $k \in K$, then $\bar{\boldsymbol{p}}^k$ has the same direction as $\bar{\boldsymbol{p}}^l$ for all $k,l \in K$.*

Theorem 13 *CCM has a limit point set $\left\{\bar{\boldsymbol{b}}^i \mid i \in I\right\}$. Let $AA_i^{-1}\bar{\boldsymbol{b}}^i$ be the overall evaluation vector of alternative i, then the overall evaluation vector of alternative i has the same direction as that of alternative l for all $i,l \in I$.*

5 Conclusion

This study shows by the mathematical description that dominant AHP consists of a pair of simple evaluation rules (1) and (2). We have shown the convergence of CCM, which will extend CCM into an analyzing tool for more complex evaluation problem such as Group AHP and Interval AHP.

References

[1] E. Kinoshita and M. Nakanishi: Proposal of new AHP model in light of dominative relationship among alternatives. *Journal of the Operations Research Society of Japan*, **42**(1999) 180-198.