A Generalized Discrete-Time Order-Replacement Model

Tadashi Dohi[†] (01307065), Kazuki Iwamoto[†], Naoto Kaio[‡] (01105445) and Shunji Osaki[††] (01002263)
[†]Hiroshima University, [‡]Hiroshima Shudo University, [††]Nanzan University

1. Summary

A discrete-time order-replacement model was first considered by Kaio and Osaki [1]. They reformulated a classical order-replacement model in discrete-time setting and characterized the optimal (regular) ordering time which minimizes the expected cost per unit time in the steady state. Also, the same authors [2] extended the earlier model [1] by taking account of the minimal repair. In this article, we generalize the Kaio and Osaki model [1] from the different point of view. More precisely, we treat a generalized order-replacement model with more complex cost structure and two decision variables [3]. Based on the discrete probabilistic argument, we derive the optimal timming to deliver a spare unit preventively by a regular order, so as to minimize the expected cost per unit time in the steady state.

2. Model Description

For a discrete time index $n = 0, 1, 2, \cdots$, consider a order-replacement problem for one-unit systems where each failed unit scrapped and each spare unit is provided, after a lead time, by an order. Let $P(n)$ be the unit failure time distribution with p.m.f. $p(n)$ and finite mean λ (> 0). The original unit begins operating at time $n = 0$. If the original unit does not fail up to a prespecified time $n_0 \in [0, \infty)$, the regular order for a spare is made at that time and the spare is delivered after a lead time L_2 with p.m.f. $g_2(n)$ and finite mean $1/\mu_2$ (> 0). Then, if the original unit has already failed by $n = n_0 + L_2$, the delivered spare takes over its operation from the delivery point. In this situation, if the original unit is still operating, the spare is put into the inventory and the original one is replaced/exchanged by the spare in the inventory when it fails/passes an allowable period $n_1 \in [0, \infty)$ after the spare is delivered, whichever occurs first. It is assumed that the spare in the inventory does not fail or deteriorate with probability one.

On the other hand, if the original unit fails before the time n_0, an expedited order is made immediately at the failure time point and the spare takes over its operation just after it is delivered after a lead time L_1 with p.m.f. $g_1(n)$ and finite mean $1/\mu_1$ (> 0). In this situation, the regular order is not made. Define the time interval from one replacement or exchange of the unit to the following replacement or exchange as one cycle, where the same cycle repeats itself continually. Let k_s (> 0) and k_i (> 0) denote the shortage and inventory holding costs per unit time, respectively. Also, we define the fixed costs associated with expedited and regular orders by c_1 (> 0) and c_2 (> 0). Then, the problem is to seek the optimal pair (n_0^*, n_1^*) minimizing the expected cost per unit time in the steady state $C(n_0, n_1)$.

We make the following assumptions:

(A-1) $k_s/\mu_1 + c_1 > k_s/\mu_2 + c_2$,

(A-2) $1/\mu_2 \geq 1/\mu_1$,

(A-3) $k_s > C(n_0, n_1)$ for all $n_0, n_1 \in [0, \infty)$.

3. Analysis & Main Results

The expected cost per unit time in the steady state is given by

$$\begin{aligned} C(n_0, n_1) &= \lim_{n\to\infty} \frac{\mathrm{E}\Big[\text{total cost incurred for } (0, n]\Big]}{n} \\ &= V(n_0, n_1)/T(n_0, n_1), \end{aligned} \tag{1}$$

where

$$\begin{aligned} V(n_0, n_1) &= k_s\Big\{\sum_{l_1=0}^{\infty}\sum_{n=0}^{n_0-1} l_1 p(n) g_1(l_1) \\ &\quad + \sum_{l_2=0}^{\infty}\sum_{n=n_0}^{n_0+l_2-1} (n_0 + l_2 - n) p(n) g_2(l_2)\Big\} \\ &\quad + k_i\Big\{\sum_{l_2=0}^{\infty}\sum_{n=n_0+l_2}^{n_0+l_2+n_1-1} (n - n_0 - l_2) p(n) \\ &\quad \times g_2(l_2) + \sum_{l_2=0}^{\infty}\sum_{n=n_0+l_2+n_1}^{\infty} n_1 p(n) g_2(l_2)\Big\} \\ &\quad + c_1 P(n_0 - 1) + c_2 \overline{P}(n_0 - 1), \end{aligned} \tag{2}$$

$$\begin{aligned} T(n_0, n_1) &= \sum_{l_1=0}^{\infty}\sum_{n=0}^{n_0-1} (n + l_1) p(n) g_1(l_1) \\ &\quad + \sum_{l_2=0}^{\infty}\sum_{n=n_0}^{n_0+l_2-1} (n_0 + l_2) p(n) g_2(l_2) \end{aligned}$$

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$$+\sum_{l_2=0}^{\infty}\sum_{n=n_0+l_2}^{n_0+l_2+n_1-1} np(n)g_2(l_2)$$
$$+\sum_{l_2=0}^{\infty}\sum_{n=n_0+l_2+n_1}^{\infty} (n_0+l_2+n_1)p(n)$$
$$\times g_2(l_2), \quad (3)$$

and in general $\overline{\psi}(\cdot) = 1-\psi(\cdot)$ denotes the suvivor function.

Define the function $q(n_0) = k_i T(n_0, n_1) - V(n_0, n_1)$ which is independent of n_1. The following result will be useful to reduce the underlying two-dimensional optimization problem to a simple one-dimensional one.

Theorem 1: For an arbitrary regular ordering time n_0, if $q(n_0) \le 0$ then $n_1^* \to \infty$, otherwise $n_1^* = 0$.

When $n_1 \to \infty$, the expected cost function $C(n_0,\infty) = V(n_0,\infty)/T(n_0,\infty)$ can be easily derived from Eq.(1). Taking the difference of $C(n_0,\infty)$ with respect to n_0, we define the function

$$q_\infty(n_0) = \Big\{r(n_0)\Big[k_s(1/\mu_1 - 1/\mu_2) + c_1 - c_2\Big] + (k_s+k_i)R(n_0) - k_i\Big\}T(n_0,\infty) - \Big\{(1/\mu_1 - 1/\mu_2)r(n_0) + R(n_0)\Big\}V(n_0,\infty), \quad (4)$$

where $r(n) = p(n)/\overline{P}(n)$ and $R(n) = \sum_{l_2=0}^{\infty}[P(n+l_2) - P(n)]g_2(l_2)/\overline{P}(n)$.

Theorem 2: (1) Under the assumptions (A-1)-(A-3), suppose that the function $r(n)$ is a strictly increasing function of n.

(i) If $q_\infty(0) < 0$ and $q_\infty(\infty) > 0$, there exists (at least one, at most two) optimal ordering time n_0^* $(0 < n_0^* < \infty)$ satisfying $q_\infty(n_0^*-1) < 0$ and $q_\infty(n_0^*) \ge 0$. Then the corresponding minimum expected cost per unit time in the steady state has the following upper and lower bounds:

$$U_\infty(n_0^*-1) < C(n_0^*,\infty) \le U_\infty(n_0^*), \quad (5)$$

where

$$U_\infty(n) = \Big\{r(n)\Big[k_s(1/\mu_1 - 1/\mu_2) + c_1 - c_2\Big] + (k_s+k_i)R(n) - k_i\Big\}\Big/\Big\{(1/\mu_1 - 1/\mu_2)r(n) + R(n)\Big\}. \quad (6)$$

(ii) If $q_\infty(0) \le 0$, the optimal ordering time is $n_0^* = 0$ with $C(n_0^*,\infty) = C(0,\infty)$, *i.e.*, it is optimal to order a new spare unit regularly at time 0.

(iii) If $q_\infty(0) \ge 0$, the optimal ordering time is $n_0^* \to \infty$ with $C(n_0^*,\infty) = C(\infty,\infty)$, *i.e.*, it is optimal to deliver a new spare unit by only an expedited order after the original unit failed.

(2) Under the assumptions (A-1)-(A-3), suppose that the function $r(n)$ is a decreasing function of n. Then, the optimal regular ordering time is $n_0^* = 0$ or $n_0^* \to \infty$.

On the other hand, when $n_1 = 0$, we can obtain the expected cost function $C(n_0,0) = V(n_0,0)/T(n_0,0)$ from Eq.(1). Taking the difference of $C(n_0,0)$ with respect to n_0, we define the function:

$$q_0(n_0) = \Big\{r(n_0)\Big[k_s(1/\mu_1 - 1/\mu_2) + c_1 - c_2\Big] + k_s \times R(n_0)\Big\}T(n_0,0) - \Big\{(1/\mu_1 - 1/\mu_2) \times r(n_0) + 1\Big\}V(n_0,0). \quad (7)$$

Theorem 3: (1) Under the assumptions (A-1) and (A-2), suppose that the function $r(n)$ is a strictly increasing function of n.

(i) If $q_0(0) < 0$ and $q_0(\infty) > 0$, there exists (at least one, at most two) optimal ordering time n_0^* $(0 < n_0^* < \infty)$ satisfying $q_0(n_0^*-1) < 0$ and $q_0(n_0^*) \ge 0$. Then the corresponding minimum expected cost per unit time in the steady state has the following upper and lower bounds:

$$U_0(n_0^*-1) < C(n_0^*,0) \le U_0(n_0^*), \quad (8)$$

where

$$U_0(n) = \Big\{r(n)\Big[k_s(1/\mu_1 - 1/\mu_2) + c_1 - c_2\Big] + k_s R(n)\Big\}\Big/\Big\{(1/\mu_1 - 1/\mu_2)r(n) + 1\Big\}. \quad (9)$$

(ii) If $q_0(0) \le 0$, the optimal ordering time is $n_0^* = 0$ with $C(n_0^*,0) = C(0,0)$.

(iii) If $q_0(0) \ge 0$, the optimal ordering time is $n_0^* \to \infty$ with $C(n_0^*,0) = C(\infty,0)$.

(2) Under the assumptions (A-1) and (A-2), suppose that the function $r(n)$ is a decreasing function of n. Then, the optimal regular ordering time is $n_0^* = 0$ or $n_0^* \to \infty$.

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