2003年日本オペレーションズ・リサーチ学会
春季研究発表会

A Discrete-Time Consumption and Wealth Model with Uncertainty

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In this talk, we present a portfolio model on the basis of a sequence of fuzzy-valued random variables discussed at the annual meeting of OR Society Japan in 1998.

We consider a portfolio model with a bond and n stocks, where there is no arbitrage opportunities under uncertainty of stock prices. Let $\mathbb{T} := \{0, 1, 2, \cdots, T\}$, and $\mathbb{R}$ denotes the set of all real numbers. Let $(\Omega, \mathcal{M}, P)$ be a probability space, where $\mathcal{M}$ is a σ-field of Ω and P is a non-atomic probability measure. Take a probability space $\Omega := (\mathbb{R}^{n+1})^{T+1}$. Let a positive number r_t be an *interest rate* of a bond price at time $t(= 1, 2, \cdots, T)$, and put a *bond price process* $\{S_t^0\}_{t=0}^T$ by $S_0^0 = 1$ and $S_t^0 := \prod_{s=1}^t (1 + r_s)$ for $t = 1, 2, \cdots, T$. We define *stock price processes* $\{S_t^i\}_{t=0}^T$ for $i = 1, 2, \cdots, n$ as follows: An initial stock price S_0^i is a positive constant and stock prices at positive time t are given by $S_t^i := S_0^i \prod_{s=1}^t (1+Y_s^i)$ for $t = 1, 2, \cdots, T$, where $\{Y_t^i\}_{t=1}^T$ is a uniform integrable sequence of i.i.d. real random variables on $(-\infty, r_t + 1]$ such that $E(Y_t^i) = r_t$ for all $t = 1, 2, \cdots, T$. A sequence of σ-fields $\{\mathcal{M}_t\}_{t=0}^T$ on Ω is given by $\mathcal{M}_0 = \sigma\{\emptyset, \Omega\}$ and $\mathcal{M}_t = \sigma\{Y_s^i \mid i = 1, 2, \cdots, n; s = 1, 2, \cdots, t\}$ for $t = 1, 2, \cdots, T$.

Let $i = 1, 2, \cdots, n$, and let $\{\delta_t^i\}_{t=0}^T$ be a stochastic process such that $0 < \delta_t^i \le S_t^i$. We give a fuzzy stochastic process $\{\tilde{S}_t^i\}_{t=0}^T$ of the stock prices by fuzzy random variables

$$\tilde{S}_t^i(\omega)(x) := L((x - S_t^i(\omega))/\delta_t^i(\omega)), \quad x \in \mathbb{R} \quad \text{for } t \in \mathbb{T},\ \omega \in \Omega, \tag{1}$$

where $L(\cdot) := \max\{1 - |\cdot|, 0\}$ is the triangle type shape function. Hence, $\delta_t^i(\omega)$ is a spread of triangular fuzzy numbers $\tilde{S}_t^i(\omega)$ and corresponds to the amount of fuzziness in the stock price process $\{S_t^i\}_{t=0}^T$.

Assumption S. Let $i = 1, 2, \cdots, n$. The stochastic process $\{\delta_t^i\}_{t=0}^T$ is represented by $\delta_t^i(\omega) := \eta^i S_t^i(\omega)$, for $t \in \mathbb{T}$ and $\omega \in \Omega$, where η^i is a constant satisfying $0 < \eta^i < 1$.

We also represent the bond price process $\{\tilde{S}_t^0\}_{t=0}^T$ by the crisp number $\tilde{S}_t^0(x) := 1_{\{S_t^0\}}(x)$ for $t \in \mathbb{T}$ and $x \in \mathbb{R}$, where $1_{\{\cdot\}}$ denotes the characteristic function of a set. We consider a case where their short sales are allowed. A *trading strategy* $\pi = \{\pi_t\}_{t=0}^T = \{(\pi_t^0, \pi_t^1, \cdots, \pi_t^n)\}_{t=0}^T$ is a $\mathbb{R}^{n+1}$-valued $\mathcal{M}_t$-predictable process such that $\sum_{t=0}^T E(|\pi_t^0|) < \infty$, $\sum_{t=0}^T E(|\pi_t^i|S_t^i) < \infty$ and $\sum_{t=0}^{T-1} E(|\pi_{t+1}^i|S_t^i) < \infty$ for all $i = 0, 1, \cdots, n$. Here π_t^0 means *the amount of the bond* $\tilde{S}_t^0$ and π_t^i means *the amount of the stock* $\tilde{S}_t^i$. Define a *fuzzy wealth process* $\{\tilde{V}_t\}_{t=0}^T$ and *consumption process* $\{\tilde{C}_t\}_{t=0}^{T-1}$ by fuzzy random variables

$$\tilde{V}_t := \pi_t^0 \tilde{S}_t^0 + \sum_{i=1}^n \pi_t^i \tilde{S}_t^i \ (t \in \mathbb{T}), \quad \tilde{C}_t := (\pi_t^0 - \pi_{t+1}^0)\tilde{S}_t^0 + \sum_{i=1}^n (\pi_t^i - \pi_{t+1}^i)\tilde{S}_t^i \ (0 \le t \le T-1). \tag{2}$$

Let γ be an $\mathcal{M}_T$-adapted real random variable which is independent to $\mathcal{M}_{T-1}$. Then we put a crisp random variable $\tilde{C}_T(\omega) := 1_{\{\gamma(\omega)\}}$ for $\omega \in \Omega$, and it is called a *terminal consumption*. A trading strategy π is called *admissible* if $\tilde{C}_t(\omega) \succeq \tilde{0}$ for all $t \in \mathbb{T}$ and $\omega \in \Omega$, where $\tilde{0} = 1_{\{0\}}$ is the crisp number zero and $\succeq$ is the fuzzy max order.

Hence we consider utility estimation of consumption and wealth in the portfolio model. Let $\overline{\mathbb{R}} := [-\infty, \infty)$. A map $U_1 : \mathbb{T} \times \mathbb{R} \mapsto \overline{\mathbb{R}}$ is called a *consumption utility function* if $U_1(t, \cdot)$ is continuous, increasing and strictly concave on $(0, \infty)$ such that $\lim_{c\to\infty} U_1(t, c) = \infty$, $\lim_{c\downarrow 0} U_1(t, c) = -\infty$ and $U_1(t, c) = -\infty$ if $c \le 0$ for all $t \in \mathbb{T}$. A map $U_2 : \mathbb{R} \mapsto \overline{\mathbb{R}}$ is also called a time-invariant *wealth utility function* if it is satisfies the same conditions. Since the consumption process $\tilde{C}_t$ and the fuzzy wealth process $\tilde{V}_t$ take fuzzy values, we introduce their fuzzy utilities $\tilde{U}_1(t, \cdot) : \mathcal{R} \mapsto \mathcal{R}$ and $\tilde{U}_2 : \mathcal{R} \mapsto \mathcal{R}$ by

$$\tilde{U}_1(t, \tilde{a})(y) := \sup_{x : U_1(t,x)=y} \tilde{a}(x) \quad \text{and} \quad \tilde{U}_2(\tilde{a})(y) := \sup_{x : U_2(x)=y} \tilde{a}(x), \tag{3}$$

$y \in \overline{\mathbb{R}}$ for $t \in \mathbb{T}$ and $\tilde{a} \in \mathcal{R}$. Let $\pi = (\pi^1, \pi^2, \cdots, \pi^n)$ be an admissible trading strategy satisfying $\sum_{i=0}^n x^i \pi_0^i = w$ for initial securities prices $x = (x^0, x^1, \cdots, x^n)$ and initial wealth w such that $x^i > 0$ $(i = 0, 1, \cdots, n)$ and $w > 0$. Then, for a consumption $\tilde{C}_T$ satisfying $\tilde{C}_T \preceq \tilde{W}_T$, *the mean value of the total expected utilities* of the consumption process $\{\tilde{C}_t\}_{t=0}^T$ and the terminal wealth $\tilde{W}_T$ is given by

$$J(x, w, \pi, \tilde{C}_T) := \tilde{E}\left(E_{x,w}\left(\sum_{t=0}^T \tilde{U}_1(t, \tilde{C}_t) + \tilde{U}_2(\tilde{W}_T)\right)\right), \tag{4}$$

where $E_{x,w}(\cdot)$ is the expectation with a trading strategy π satisfying $\sum_{i=0}^n x^i \pi_0^i = w$ for initial securities prices x and an initial wealth w, and the mean value of a fuzzy number $\tilde{a} \in \mathcal{R}$ is given by $\tilde{E}(\tilde{a}) := \int_0^1 g(\tilde{a}_\alpha)\, d\alpha$ with a map g such that $g([x, y]) := \lambda x + (1 - \lambda) y$ for bounded closed intervals $[x, y]$ and a constant λ $(0 \le \lambda \le 1)$. Hence, g is called a *λ-weighting function*, and λ is called a *pessimistic-optimistic index* in the investor's decision making. Define the *optimal total expected utility* by $J(x, w) := \sup_{\pi, \tilde{C}_T} J(x, w, \pi, \tilde{C}_T)$ for an initial securities price x and an initial wealth w, where π and $\tilde{C}_T$ are taken over admissible trading strategies and admissible terminal consumptions satisfying the initial condition $\sum_{i=0}^n x^i \pi_0^i = w$. An optimal consumption and wealth problem is given as follows.

Problem P. Let $\lambda \in [0, 1]$ and let x be initial securities prices. Maximize the total expected utility $J(x, w, \pi, \tilde{C}_T)$ by admissible trading strategies π and admissible terminal consumptions $\tilde{C}_T$.

Let $t \in \mathbb{T}$. Now we define the *optimal total expected utility after time t* by

$$v_t(x, u) := \max_{(\pi_t, \cdots, \pi_T, \tilde{C}_T) \in \mathcal{A}(t)} E_{t,x,u,\pi}\left(\sum_{s=t}^T \int_0^1 g(\tilde{U}_1(s, \tilde{C}_s)_\alpha)\, d\alpha + \int_0^1 g(\tilde{U}_2(\tilde{W}_s)_\alpha)\, d\alpha\right) \tag{5}$$

for securities prices x, current initial trading strategies $u = (u^0, u^1, \cdots, u^n) \in \mathbb{R}^{n+1}$, an admissible trading strategy $\pi = (\pi^0, \pi^1, \cdots, \pi^n)$, an admissible terminal consumption $\tilde{C}_T$ such that $\pi_t^i = u^i$ for $i = 0, 1, \cdots, n$ holds at time t, where $E_{t,x,u,\pi}(\cdot)$ is the expectation with a trading strategy π satisfying $\pi_t^i = u^i$ $(i = 0, 1, \cdots, n)$.

Theorem 1 (Optimality equation). *The optimal total expected utility is a solution of the following recursive equation:*

$$v_t(x, u) = \max_{\pi_{t+1}} E_{t,x,u,\pi}(C_t + v_{t+1}(Z_{t+1}, \pi_{t+1})) \tag{6}$$

for $t = 0, 1, \cdots, T-1$, securities prices $x = (x^0, x^1, \cdots, x^n)$ and current initial trading strategies u; and at the terminal time T it holds that

$$v_T(x, u) = \max_{\tilde{C}_T} E_{x,u}(C_T + R_T), \tag{7}$$

where Z_{t+1} is given by $Z_{t+1} := (x^0(1 + r_{t+1}), x^1(1 + Y_{t+1}^1), \cdots, x^n(1 + Y_{t+1}^n))$. Let π^* and $\tilde{C}_T^*$ be an admissible trading strategy and an admissible consumption attaining the maxima in (6) and (7). Then π^* and $\tilde{C}_T^*$ are optimal for Problem P: For an initial securities price $x = (x^0, x^1, \cdots, x^n)$ and an initial wealth w, it holds that

$$J(x, w) = \max_{u : xu' = w} v_0(x, u), \tag{8}$$

where $xu' = \sum_{i=0}^n x^i u^i = w$ with the transpose u' of $u = (u^0, u^1, \cdots, u^n)$.

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