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The Break Minimization Problem

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In this abstract, we prove a previously proposed conjecture about the *break minimization problem*, which is a problem in the area of sports scheduling.

We consider a round-robin tournament that satisfies the following properties:

- the number of teams is $2n$ and that of slots is $2n-1$;
- each team plays one game in each slot;
- each team plays every other team once;
- each team has its home and each game is held at the home of one of the playing two teams.

Figure 1 is a schedule of a tournament satisfying these properties. In the figure, each game with '@' means that the game is held at the home of the opponent; without '@' means that the game is held at the home of the team corresponding to the row. For example, team 4 plays team 2 at the home of team 2 in slot 3.

If a team plays either both at home or both at away in slots s and $s+1$, it is said that the team has a break at slot $s+1$. In a schedule, a break is expressed as an underline at a slot where a break occurs. For example, in Fig. 1, team 3 plays at home in slots 1 and 2, and we say that team 3 has a break at slot 2. In total, the schedule has six breaks.

Given a schedule without a home-away assignment, an organizer of the tournament should decide a home-away assignment, which the number of breaks depends on. For a practical reason, an organizer generally prefers a home-away assignment in which the number of breaks is small. In this context, the break minimization problem is defined as follows.

<u>Break Minimization Problem</u>

Input: A schedule without a home-away assignment.

Output: A home-away assignment consistent to the given input and in which the number of breaks is minimized.

(team)		1	2	3	4	5 (slot)
1	:	@6	@$\underline{3}$	@$\underline{5}$	2	@4
2	:	@5	6	$\underline{4}$	@1	3
3	:	4	$\underline{1}$	@6	5	@2
4	:	@3	5	@2	6	$\underline{1}$
5	:	2	@4	1	@3	@$\underline{6}$
6	:	1	@2	3	@4	5

Figure 1. A schedule with six breaks

The schedule without a home-away assignment of Fig. 2 is an input for the break minimization problem. Although the schedule of Fig. 1 shows a feasible home-away assignment for the input, it is suboptimal. The schedule of Fig. 2 is an optimal solution, whose optimal value is four.

There are some previous results on the break minimization problem. Régin solved up to 20 teams instances with constraint programming [5]. Trick proposed integer programming formulations and solved instances up to 22 teams [6]. Elf, Jünger and Rinaldi formulated this problem as MAX CUT, and solved instances up to 26 teams [3]. The authors formulated this problem as MAX RES CUT, and proposed an algorithm based on positive semidefinite programming relaxation [4].

There are some open problems about the break minimization problem. Although it is conjectured that the break minimization problem is NP-hard, the complexity status of this problem is not yet determined. Concerning the complexity, Elf et al. reported the following result [3]: their instances of the break minimization problem were solved very quickly with their method when the instances had the optimal value $2n-2$. (The value $2n-2$ is a lower bound of the objective value for any instance of $2n$ teams, because a schedule of $2n$ teams has at least $2n-2$ breaks [2].) According to their experience, they conjectured that the break minimization problem is solvable in polynomial time if a given instance of $2n$ teams has the optimal value $2n-2$.

We prove their conjecture affirmatively by showing that the following problem P1 can be solved in polynomial time.

<u>Problem P1</u>

Input: A schedule of $2n$ teams and without a home-away assignment.

Output: A home-away assignment with $2n-2$ breaks, if exists; else infeasible.

In the following, we show that Problem P1 is solvable in $O(n^3)$ steps. For Problem P1, we define Subproblem P1(k) as follows ($k \in T$, where T is a set of teams, i.e. $\{1,2,\ldots,2n\}$). It is not difficult to see

		1	2	3	4	5
1	:	6	3	5	2	4
2	:	5	6	4	1	3
3	:	4	1	6	5	2
4	:	3	5	2	6	1
5	:	2	4	1	3	6
6	:	1	2	3	4	5

		1	2	3	4	5
1	:	@6	3	@5	2	@4
2	:	@5	6	4	@1	3
3	:	4	@1	@6	5	@2
4	:	@3	5	@2	6	1
5	:	2	@4	1	@3	@6
6	:	1	@2	3	@4	5

Figure 2. A schedule without a home-away assignment and an optimal assignment.

that Problem P1 is feasible if and only if at least one of P1(1), P1(2), ..., and P1($2n$) is feasible.

Subproblem P1(k)

Input: The same input as that of Problem P1.

Output: A home-away assignment with $2n-2$ breaks and in which team k has no break and plays at home in slot 1, if exists; else infeasible.

The feasibility of Subproblem P1(k) is equivalent to that of Subproblem P1′(k) defined below. In addition, a feasible home-away assignment of P1(k) can be constructed from that of P1′(k), by alternating home with away in all even slots. (More generally, the following statement holds: by the above mentioned alternation, an optimal solution of the break minimization problem is obtained from that of the break maximization problem and vice versa.)

Subproblem P1′(k)

Input: The same input as that of Problem P1.

Output: A home-away assignment with $2n(2n-2)-(2n-2)$ breaks and in which team k has $2n-2$ breaks and plays at home in slot 1, if exists; else infeasible. (In other words, team k plays only at home and every other team has at most one "non-break.")

Now we formulate Subproblem P1′(k) ($k \in T$) as 2SAT. Let S be a set of slots, i.e. $\{1, 2, \ldots, 2n-1\}$. We define a Boolean variable $x_{t,s}$ ($t \in T$, $s \in S$) as follows: a variable $x_{t,s}$ is FALSE if and only if team t plays at home in slot s. Then, an instance of Subproblem P1′(k) can be transformed as follows.

$$
\begin{array}{lll}
\text{Find} & x_{t,s} \in \{\texttt{TRUE}, \texttt{FALSE}\} & (\forall t \in T, \forall s \in S) \\
\text{s. t.} & x_{k,s} = \texttt{FALSE} & (\forall s \in S), \\
& x_{t,s} \neq x_{\tau(t,s),s} & (\forall t \in T, \forall s \in S), \\
& \neg x_{t,s} \vee x_{t,s+1} & (\forall t \in T \setminus \{k\}, \\
& & \ \forall s \in S,\ s < s_{k,t}), \\
& x_{t,s-1} \vee \neg x_{t,s} & (\forall t \in T \setminus \{k\}, \\
& & \ \forall s \in S,\ s > s_{k,t}), \\
& x_{t,1} \vee x_{t,2n-1} & (\forall t \in T \setminus \{k\}),
\end{array}
$$

where

$\tau(t,s)$: the team which team t plays at slot s in the input of P1′(k);

$s_{k,t}$: the slot at which team k plays team t in the input of P1′(k).

Each of the constraints can be represented as clause(s) with two literals; the number of variables and that of clauses with two literals are both $O(n^2)$. Since 2SAT with p literals and q clauses is solvable in $O(p+q)$ steps [1], Problem P1′(k) can be solved in $O(n^2)$ steps, and hence Problem P1 is solvable in $O(n^3)$ steps.

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