日本オペレーションズ・リサーチ学会
2004年春季研究発表会

Numerical Exploration of Dynamic Behavior of the Ornstein-Uhlenbeck Process via Ehrenfest Process Approximation

01204710 筑波大学 住田 潮 SUMITA Ushio
02103210 筑波大学 後藤順哉 GOTOH Jun-ya
筑波大学 金 輝 *JIN Hui

1 Ornstein-Uhlenbeck Process

The Ornstein-Uhrenbeck (O-U) process is of practical importance in many application areas such as statistics, meteorology and financial engineering. The O-U process $\{X_{\mathrm{OU}}(t) : t \geq 0\}$ is a Markov diffusion process on the real continuum $-\infty < x < \infty$. Its probability density function $f(x,t) = \frac{d}{dx}\mathrm{P}\left[X_{\mathrm{OU}}(t) \leq x\right]$ is governed by the forward diffusion equation

$$\frac{\partial}{\partial t} f(x,t) = \frac{\partial^2}{\partial x^2} f(x,t) + \frac{\partial}{\partial x}\left[x\, f(x,t)\right]. \qquad (1.1)$$

The O-U process is characterized by its Markov property, normal distribution, and exponential covariance function. A basic function describing this process is the conditional transition density $g(x_0, x, \tau) = \frac{d}{dx}\mathrm{P}\left[X(t+\tau) \leq x | X(t) = x_0\right]$ given by

$$g(x_0, x, \tau) = \frac{1}{\sqrt{2\pi}\sqrt{1-e^{-2\tau}}} \exp\left\{-\frac{\left(x - x_0\, e^{-\tau}\right)^2}{2\left(1-e^{-2\tau}\right)}\right\}, \qquad (1.2)$$

with $-\infty < x < \infty$.
While transition probabilities of the O-U process are readily accessible, quantifying its dynamic behavior is numerically cumbersome.

2 Convergence of the Ehrenfest Process to the O-U Process

We consider $2V$ independent and identical Markov chains $\{J_j(t) : t \geq 0\}$, $j = 1, ..., 2V$, in continuous time on $\{0, 1\}$ governed by transition rates $\nu_{01} = \nu_{10} = \frac{1}{2}$. Let $\{\, N_{2V}(t) : t \geq 0 \,\}$ be defined by

$$N_{2V}(t) \stackrel{\mathrm{def}}{=} \sum_{j=1}^{2V} J_j(t). \qquad (2.1)$$

Then $\{\, N_{2V}(t) : t \geq 0 \,\}$ is a birth-death process on $\mathcal{N} = \{0, 1, ..., 2V\}$ governed by the upward transition rates $\lambda_n = \frac{1}{2}(2V - n)$ and the downward transition rates $\mu_n = \frac{n}{2}$, $n \in \mathcal{N}$. We note that $\nu_n \stackrel{\mathrm{def.}}{=} \lambda_n + \mu_n = V$, $n \in \mathcal{N}$, which is independent of state n. This birth-death process is called the Ehrenfest process.
Let $\{\, X_V(t) : t \geq 0 \,\}$ be a stochastic process defined by

$$X_V(t) \stackrel{\mathrm{def}}{=} \sqrt{\frac{2}{V}}\, N_{2V}(t) - \sqrt{2V}. \qquad (2.2)$$

We note that $\{\, X_V(t) : t \geq 0 \,\}$ has a discrete support on $\{\, r(0), ..., r(2V) \,\}$ where

$$r(n) = \sqrt{\frac{2}{V}}\, n - \sqrt{2V}, \quad n = 0, 1, ... \qquad (2.3)$$

Clearly $r(n+1) - r(n) = \sqrt{\frac{2}{V}} \to 0$ as $V \to \infty$. When $N_{2V}(0)$ is chosen appropriately, $\{\, X_V(t) : t \geq 0 \,\}$ converges in law to $\{\, X_{\mathrm{OU}}(t) : t \geq 0 \,\}$ as $V \to \infty$.
It can be shown that the first passage time and the historical maximum of $\{X_V(t) : t \geq 0\,\}$ also converges in law to those of $\{\, X_{\mathrm{OU}}(t) : t \geq 0 \,\}$ as $V \to \infty$. Hence the dynamic behavior of $\{\, X_{\mathrm{OU}}(t) : t \geq 0 \,\}$ can be approximated by that of $\{\, X_V(t) : t \geq 0 \,\}$.

3 First Passage Time Structure of the Ehrenfest Process

Let T_{mn} $(m < n)$ be the first passage time of a general birth-death process with probability density function $s_{mn}(\tau)$ and its Laplace transform $\sigma_{mn}(s)$. For notational convenience, we denote $T_{m,m+1}$ by T_m^+, and $s_m^+(\tau)$ and $\sigma_m^+(s)$ are defined similarly. From the consistency relations, one has

$$\sigma_n^+(s) = \frac{\nu_n}{s + \nu_n}\left[\frac{\lambda_n}{\nu_n} + \frac{\mu_n}{\nu_n}\, \sigma_{n-1}^+(s)\, \sigma_n^+(s)\right], \quad n \geq 1, \qquad (3.1)$$

where $\nu_n = \lambda_n + \mu_n$.
Let $g_n(s)$ be a polynomial of order n defined by $\sigma_{0n}(s) = \frac{1}{g_n(s)}$, $n \geq 1$, $g_0(s) = 1$. From $\sigma_{0n}(s) = \sigma_{0\,n-1}(s)\, \sigma_{n-1}^+(s)$, $n \geq 0$, one then sees

$$g_{n+1}(s) = \frac{1}{\lambda_n}\left[(s + \nu_n)\, g_n(s) - \mu_n\, g_{n-1}(s)\right], \; n \geq 0, \qquad (3.2)$$

where $g_{-1}(s) = 0$ and $g_0(s) = 1$. It should be noted that

$$\sigma_n^+(s) = \frac{g_n(s)}{g_{n+1}(s)}, \quad n \geq 0. \tag{3.3}$$

As shown in Keilson [1], $\{g_n(s)\}$ are orthogonal polynomials. Consequently, from (3.3), $\sigma_n^+(s)$ can be written as

$$\sigma_n^+(s) = \sum_{j=0}^{n} r_{n+1,j} \frac{\alpha_{n+1,j}}{s+\alpha_{n+1,j}}, \tag{3.4}$$

where $-\alpha_{n+1,j}$ are the zeros of $g_{n+1}(s)$, $r_{n+1,j} = \lim_{s \to -\alpha_{n+1,j}} \frac{s+\alpha_{n+1,j}}{\alpha_{nj}} \frac{g_n(s)}{g_{n+1}(s)} \geq 0$ and $\sum_{j=0}^{n} r_{n+1,j} = \sigma_n^+(0+) = 1$.
In case of the Ehrenfest process, this becomes

$$g_{n+1}(s) = \frac{2}{2V-n}\left[(s+V)\,g_n(s) - \frac{n}{2}\,g_{n-1}(s)\right], \tag{3.5}$$

with $g_{-1}(s) = 0$ and $g_0(s) = 1$.

4 Dynamic Behavior and its Numerical Algorithm

In order to evaluate the first passage times $s_{mn}(\tau)$ $(m < n)$ with corresponding Laplace transforms $\sigma_{mn}(s) = \sigma_m^+(s) \cdots \sigma_{n-1}^+(s) = g_m(s)/g_n(s)$, the zeros of $g_n(s)$ are needed. These zeros in turn enables one to quantify the historical maximum. In principle, the zero search of $g_n(s)$ can be accomplished via a straightforward bisection approach since the zeros of $g_n(s)$ and $g_{n+1}(s)$ interleave because of the underlying orthogonality. Let $h_n(s) = g_n(s-V)$, $n \geq 0$, then the recursive formula can be rewritten as

$$h_{n+1}(s) = \frac{2}{2V-n}\left[s\,h_n(s) - \frac{n}{2}\,h_{n-1}(s)\right], \quad n \geq 0, \tag{4.1}$$

Clearly the zeros of $h_n(s)$ are symmetric about 0 while the zeros of $g_n(x)$ are symmetric about $-V$. Consequently the computational time of the zero search can be reduced approximately by a factor of 4. More specifically, one can write

$$\begin{cases} h_{2m}(s) = \sum_{j=0}^{m} w_{2m,2j}\, s^{2j}, \quad m \geq 0, \\ h_{2m+1}(s) = \sum_{j=0}^{m} w_{2m+1,2j+1}\, s^{2j+1}, \quad m \geq 0\ . \end{cases}$$

Since $h_{2m}(s)$ is an even function and $h_{2m+1}(s)$ is an odd function, it then follows from the recursive formula for $m \geq 0$, that

$$\begin{cases} w_{2m,0} = -\frac{2}{2(V-m)+1}\left(m-\frac{1}{2}\right) w_{2m-2,0}, \\ w_{2m,2j} = \frac{2}{2(V-m)+1}\{w_{2m-1,2j-1} \\ \qquad - \left(m-\frac{1}{2}\right) w_{2m-2,2j}\}, \; j = 1, ..., m-1, \\ w_{2m,2m} = \frac{2}{2(V-m)+1} w_{2m-1,2m-1}, \end{cases}$$

and

$$\begin{cases} w_{2m+1,2j+1} = \frac{w_{2m,2j} - m\,w_{2m-1,2j+1}}{V-m}, \\ \qquad j = 0, ..., m-1, \\ w_{2m+1,2m+1} = \frac{w_{2m,2m}}{V-m}, \end{cases}$$

where $h_0(s) = w_{0,0} = 1$.
For both $h_{2m}(s)$ and $h_{2m+1}(s)$, it suffices to search m zeros in $(0, \infty)$. For $h_n(s)$ with $1 \leq n \leq 4$, the zeros can be obtained explicitly by solving the underlying equations. For higher values of n, a straightforward bisection method can be employed by exploiting the fact that zeros of $h_{n+1}(s)$ interleave those of $h_n(s)$.
Let ξ_{nj} $(0 \leq j \leq n-1)$ be zeros of $h_n(s)$ and $-\alpha_{nj}$ $(0 \leq j \leq n-1)$ be zeros of $g_n(s)$, one then has $\alpha_{nj} = V - \xi_{nj}$, $0 \leq j \leq n-1$.
From $\sigma_{mn}(s) = \sigma_m^+(s) \cdots \sigma_{n-1}^+(s)$, one has that

$$\sigma_{mn}(s) = \frac{g_m(s)}{g_n(s)} = c_{mn}\frac{\prod_{j=0}^{m-1}(s+\alpha_{mj})}{\prod_{j=0}^{n-1}(s+\alpha_{n,j})}; \quad c_{mn} = \frac{\prod_{j=0}^{n-1}\alpha_{nj}}{\prod_{j=0}^{m-1}\alpha_{mj}}. \tag{4.2}$$

Since $\sigma_{mn}(s)$ is regular apart from singular points $-\alpha_{n,j}$, $0 \leq j \leq n-1$, this can be rewritten as

$$\sigma_{mn}(s) = \sum_{j=0}^{n-1} A_{mn:j}\frac{\alpha_{nj}}{s+\alpha_{nj}}; \quad A_{mn:k} = \frac{\prod_{j=0}^{m-1}(1-\frac{\alpha_{nk}}{\alpha_{mj}})}{\prod_{j=0, j\neq k}^{n-1}(1-\frac{\alpha_{nk}}{\alpha_{nj}})}. \tag{4.3}$$

In real domain, this leads to the probability function $s_{mn}(\tau)$ and its corresponding survival function $\overline{S}_{mn}(\tau) = \int_\tau^\infty s_{mn}(y)dy$ given as

$$s_{mn}(\tau) = \sum_{j=0}^{n-1} A_{mn:j} \cdot \alpha_{nj} e^{-\alpha_{nj}\tau}; \tag{4.4}$$

$$\overline{S}_{mn}(\tau) = \sum_{j=0}^{n-1} A_{mn:j} e^{-\alpha_{nj}\tau}. \tag{4.5}$$

References

[1] Keilson, J. (1979), *Markov chain models: rarity and exponentiality*, (Applied Mathematical Science Series, 28), Springer, New York.