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A Benefit–Cost Game: An Extension and an Application of the DEA Game

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1. Introduction

In the basic DEA game model [1], the score matrix X represents either the superiority (benefits) or the inferiority (costs) of players. However, there are occasions where two types of criteria are mixed. Thus, the merit of a player is evaluated by the difference between benefits and costs. In this study, we extend the basic model to a benefit–cost (B–C) game and introduce an application to the NATO's burden-sharing problem.

2. A benefit–cost game

Suppose that there are s criteria for representing benefits and m criteria for costs. Let y_{rk} ($r = 1, \cdots, s$) and x_{ik} ($i = 1, \cdots, m$) be the benefits and costs of player k ($k = 1, \cdots, n$), respectively. The merit of player k is evaluated by

$$(u_1 y_{1k} + \cdots + u_s y_{sk}) - (v_1 x_{1k} + \cdots + v_m x_{mk}),$$

where $u = (u_1, \cdots, u_s)$ and $v = (v_1, \cdots, v_m)$ are respectively the virtual weights for benefits and costs. We define the relative score of player k to the total scores as follows:

$$\frac{\sum_{r=1}^{s} u_r y_{rk} - \sum_{i=1}^{m} v_i x_{ik}}{\sum_{j=1}^{n} \left(\sum_{r=1}^{s} u_r y_{rj} - \sum_{i=1}^{m} v_i x_{ij}\right)}. \tag{1}$$

We assume player k wishes to maximize his score, subject to the condition that the merit of all players is nonnegative, – *i.e.*,

$$\sum_{r=1}^{s} u_r y_{rj} - \sum_{i=1}^{m} v_i x_{ij} \geq 0 \ (j = 1, \cdots, n). \tag{2}$$

We can express this situation by the linear program below:

$$\begin{aligned} \max_{u,v} \quad & \sum_{r=1}^{s} u_r y_{rk} - \sum_{i=1}^{m} v_i x_{ik} \qquad (3) \\ \text{subject to} \quad & \sum_{j=1}^{n} \left(\sum_{r=1}^{s} u_r y_{rj} - \sum_{i=1}^{m} v_i x_{ij} \right) = 1 \\ & \sum_{r=1}^{s} u_r y_{rj} - \sum_{i=1}^{m} v_i x_{ij} \geq 0, \\ & \qquad (j = 1, \cdots, n) \\ & u_r \geq 0 \ (\forall r), \quad v_i \geq 0 \ (\forall i). \end{aligned}$$

Following the same scenario as the basic DEA game model, we can develop coalitions and imputations of this benefit–cost (B–C) game, although the row-wise normalization is not available in this game. That is, a characteristic function of the coalition S is defined by the linear program below:

$$\begin{aligned} c(S) = \max_{u,v} \quad & \sum_{k \in S} \left(\sum_{r=1}^{s} u_r y_{rk} - \sum_{i=1}^{m} v_i x_{ik} \right) \qquad (4) \\ \text{subject to} \quad & \sum_{j=1}^{n} \left(\sum_{r=1}^{s} u_r y_{rj} - \sum_{i=1}^{m} v_i x_{ij} \right) = 1 \\ & \sum_{r=1}^{s} u_r y_{rj} - \sum_{i=1}^{m} v_i x_{ij} \geq 0, \\ & \qquad (j = 1, \cdots, n) \\ & u_r \geq 0 \ (\forall r), \quad v_i \geq 0 \ (\forall i). \end{aligned}$$

In the program (4), we keep the condition that the merit of all players is nonnegative. Since the constraints of program (4) are the same for all coalitions, we have the following proposition:

Proposition 1 *The B–C max game satisfies a sub-additivity property.*

As with the basic DEA game model, we can define the B–C minimum game by replacing *max* in (4) by *min*. This B–C *min* game satisfies a super-additivity property, and we arrive at the following proposition:

Proposition 2 *The Shapley values of the B–C max and min games are the same.*

Furthermore, similar to the basic DEA game model, we can confirm that the B–C *min* game is balanced and has a non-empty core, and that the B–C *max* and *min* games with three players are concave and convex respectively.

3. A property related to the player's scale

Now suppose two players A and B with their records (y_{rA}, x_{iA}) and (y_{rB}, x_{iB}) such that $t > 1$, $y_{rA} = ty_{rB}$ $(\forall r)$ and $x_{iA} = tx_{iB}$ $(\forall i)$. Then we have the following two lemmas:

Lemma 1 *For every coalition S excluding A and B, it holds $c(S \cup A) \geq c(S \cup B)$.*

Lemma 2 *It holds that $c(A) > c(B)$ unless $(y_{1B}, \cdots, y_{sB}) = \mathbf{0}$.*

From the lemmas 1 and 2, we have $c(S \cup A) - c(S) \geq c(S \cup B) - c(S)$ and $c(A) > c(B)$, respectively. Hence, we obtain the followong proposition:

Proposition 3 *If $t > 1$, $y_{rA} = ty_{rB}$ $(\forall r)$ and $x_{iA} = tx_{iB}$ $(\forall i)$ for the two players A and B, then the Shapley value for A is greater than that for B.*

4. An application to the NATO's problem

We apply our DEA B$-$C game to the NATO burden-sharing problem with the spirit of Kim and Hendry [2].

Kim and Hendry presented a new index – the 'net-burden index,' which is measured by the relationship between the 17 contribution (cost) categories and the seven benefit categories. In thier paper the quantitative format for the total 24 categories was arranged by applying AHP to some qualitative data. They finally assessed the 16 allied nations' 'net-burden' using DEA by treating each nation as a distinct DMU, the contribution categories as outputs, and the benefit categories as inputs.

We here pick up some clear categories and apply our DEA B$-$C game to their data. The following seven categories are used in our test.

Benefits

y_1: Protection from external threat
The degree of reliance on US (NATO) protection against an external threat.

y_2: Receipt of economic and military aid
Amount of US economic and military aid received: the sum of the amounts from 1948 to 1990.

y_3: Receipt of economic spin-offs (employment in defense industry)
The number of workers employed in the world's top 100 defense contractors: the average of the figures for the years 1988, 1989 and 1990.

Contributions

x_1: Defense efforts
D/GDP (%) figure: the average D/GDP figures for the years 1970, 1975, 1980, 1985 and 1990.

x_2: Technological capability
R&D spending share: the average R&D spending share for the years from 1960 to 1988.

x_3: Overseas deployment of troops
The number of troops deployed: the average number of troops deployed for the years 1970, 1975, 1980, 1985 and 1990.

x_4: Hosting of foreign forces
The number of troops stationed: the average number of US troops stationed for the years 1971, 1985 and 1990.

We compare between a 'single benefit – single contribution' case and a 'three benefits – four contributions' case. The results are shown in our presentation.

5. Conclusion

The midified B$-$C game might have even more practical relevance than the basic DEA game model. This game deals with two types of data category in common with DEA. However, there is a difference between the benefit-cost formulation in our scheme and the ratio formulation in DEA. The B$-$C game is an important area for future investigation.

References

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[3] Nakabayashi, K., (2004), "Egoist's Dilemma: A DEA Game Solution," Doctor thesis, National Graduate Institute for Policy Studies, Japan.