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A Homogeneous Model for Mixed Complementarity Problems over Symmetric Cones

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1 Introduction

In this article, we propose a homogeneous model for mixed complementarity problems (MCPs) over symmetric cones, and discuss its theoretical aspects. The model is a natural extension of the model in [1] for monotone complementarity problems over the nonnegative orthant in $\Re^n$. The analyses are based on [4] where the weighted paths for nonlinear monotone MCPs are studied. We show the existence of a path having the following properties: (a) The path is bounded and has a trivial starting point without any regularity assumption concerning the existence of feasible or strictly feasible solutions. (b) Any accumulation point of the path is a solution of the homogeneous model. (c) If the original problem is solvable, then every accumulation point of the path gives us a finite solution. (d) If the original problem is strongly infeasible, then every accumulation point of the path gives us a finite certificate proving infeasibility.

2 MCPs over symmetric cones

Let V be an n-dimensional real vector space and (V, γ) be a Euclidian Jordan algebra with an identity element e. We denote by K a symmetric cone of V which is a self-dual closed convex cone. It is known that a cone in V is symmetric if and only if it is the cone of squares of V given by $K = \{x \circ x : x \in V\}$ (cf. [2]).

The MCP over the symmetric cone is given by

$$\begin{aligned} &\text{Find} \quad (x, y, z) \in K \times K \times \Re^m, \\ &\text{s.t.} \quad F(x, y, z) = 0, \; x \circ y = 0. \end{aligned} \tag{1}$$

where F is a continuously differentiable function from $K \times K \times \Re^m$ to $V \times \Re^m$.

Observe the following nonlinear semidefinite program:

$$\text{Min} \;\; \theta(x) \text{ s.t. } G(x) \in -S^n_+, \;\; h(x) = 0 \tag{2}$$

where S^n_+ is the cone of semidefinite matrices in the set S^n of $n \times n$ symmetric matrices, $\theta : \Re^m \to \Re$, $G : \Re^m \to S^n$ and $h : \Re^m \to \Re^p$ are given smooth mappings. It has been shown that an optimality condition for (2) can be formulated as a problem of the form (1) under an appropriate constraint qualification[3].

We say that an MCP is

- *(asymptotically) feasible* iff there exists a bounded sequence $\{x^{(k)}, y^{(k)}, z^{(k)}\} \subseteq \text{int}K \times \text{int}K \times \Re^m$ such that $\lim_{k\to\infty} F(x^{(k)}, y^{(k)}, z^{(k)}) = 0$,
- *strongly infeasible* iff $0 \notin \text{cl}(F(K \times K \times \Re^m))$.

We use the following assumption on the map F.

Assumption 2.1 (i) *F is (x, y)-equilevel-monotone on its domain, i.e., for every (x, y, z) and (x', y', z') in the domain of F satisfying $F(x, y, z) = F(x', y', z')$, $\langle x - x', y - y' \rangle \geq 0$ holds.*

(ii) *F is z-bounded on its domain, i.e., for every $\{(x^{(k)}, y^{(k)}, z^{(k)})\}$ in the domain of F, if $\{(x^{(k)}, y^{(k)})\}$ and $\{F(x^{(k)}, y^{(k)}, z^{(k)})\}$ are bounded then the sequence $\{z^{(k)}\}$ is also bounded.*

(iii) *$F(x, y, z)$ is z-injective on its domain, i.e., if (x, y, z) and (x, y, z') lie in the domain of F and satisfy $F(x, y, z) = F(x, y, z')$ then $z = z'$.*

3 A homogeneous model

In this section, we give a homogenous model for solving MCPs where the map $F : K \times K \times \Re^m \to V \times \Re^m$ is of the form

$$F(x, y, z) = (y - \psi_1(x, z), \psi_2(x, z))$$

and $\psi := (\psi_1, \psi_2) : K \times \Re^m \to V \times \Re^m$. For the problem, we consider the following homogeneous model(HMCP):

$$\begin{aligned} &\text{Find} \quad (x, \tau, y, \kappa, z) \in (K \times \Re_{++}) \times (K \times \Re_+) \times \Re^m, \\ &\text{s.t.} \quad F_{\text{H}}(x, \tau, y, \kappa, z) = 0, (x, \tau) \circ_{\text{H}} (y, \kappa) = 0 \end{aligned}$$

where $F_{\text{H}} : (K \times \Re_{++}) \times (K \times \Re_+) \times \Re^m \to (V \times \Re) \times \Re^m$ and $(x, \tau) \circ_{\text{H}} (y, \kappa)$ are given by

$$F_{\text{H}}(x, \tau, y, \kappa, z) := \begin{pmatrix} y - \tau\psi_1(x/\tau, z/\tau) \\ \kappa + \langle \psi_1(x/\tau, z/\tau), x \rangle + \psi_2(x/\tau, z/\tau)^T z \\ \tau\psi_2(x/\tau, z/\tau) \end{pmatrix}$$

and

$$(x,\tau)\circ(y,\kappa) := \begin{pmatrix} x\circ y \\ \tau\kappa \end{pmatrix}.$$

We also define the scalar product $\langle (x,\tau),(y,\kappa)\rangle_{\mathrm{H}}$ by

$$\langle (x,\tau),(y,\kappa)\rangle_{\mathrm{H}} := \langle x,y\rangle + \tau\kappa.$$

For ease of notation, we use the following symbols: $V_{\mathrm{H}} := V\times\Re$, $K_{\mathrm{H}} := K\times\Re_{+}$, $x_{\mathrm{H}} := (x,\tau)\in V_{\mathrm{H}}$, $y_{\mathrm{H}} := (y,\kappa)\in V_{\mathrm{H}}$, and define $\psi_{\mathrm{H}} := (\psi_{\mathrm{H}1},\psi_{\mathrm{H}2})$ by

$$\psi_{\mathrm{H}1}(x_{\mathrm{H}},z) := \begin{pmatrix} \tau\psi_1(x/\tau,z/\tau) \\ -\langle\psi_1(x/\tau,z/\tau),x\rangle - \psi_2(x/\tau,z/\tau)^T z \end{pmatrix},$$
$$\psi_{\mathrm{H}2}(x_{\mathrm{H}},z) := \tau\psi_2(x/\tau,z/\tau).$$

We can easily see that $\mathrm{int}K_{\mathrm{H}} = \mathrm{int}K_{\mathrm{H}}\times\Re_{++}$ and

$$F_{\mathrm{H}}(x_{\mathrm{H}},y_{\mathrm{H}},z) = \begin{pmatrix} y_{\mathrm{H}} - \psi_{\mathrm{H}1}(x_{\mathrm{H}},z) \\ \psi_{\mathrm{H}2}(x_{\mathrm{H}},z) \end{pmatrix}.$$

Note that since $K_{\mathrm{H}} = \{x_{\mathrm{H}}^2 = (x^2,\tau^2) : x_{\mathrm{H}}\in V_{\mathrm{H}}\}$, the closed convex cone K_{H} is the symmetric cone of V_{H}.

4 Main results

The following theorems are the main results.

Theorem 4.1 *Suppose that ψ_{H} satisfies Assumption 2.1.*

(i) *The HMCP is asymptotically feasible and solvable.*

(ii) *The MCP has a solution if and only if the HMCP has an asymptotical solution $(x_{\mathrm{H}}^*,y_{\mathrm{H}}^*,z^*) = (x^*,\tau^*,y^*,\kappa^*,z^*)$ with $\tau^*>0$. In this case, $(x^*/\tau^*,y^*/\tau^*,z^*/\tau^*)$ is a solution of (CP).*

(iii) *The MCP is strongly infeasible if and only if the HMCP has an asymptotical solution $(x^*,\tau^*,y^*,\kappa^*,z^*)$ with $\kappa^*>0$.*

Theorem 4.2 *Suppose that ψ_{H} satisfies Assumption 2.1.*

(i) *The set*

$$P := \{(x_{\mathrm{H}}(t),y_{\mathrm{H}}(t),z(t)) : x_{\mathrm{H}}\circ_{\mathrm{H}} y_{\mathrm{H}} = te, F(x_{\mathrm{H}},y_{\mathrm{H}},z) = tF(e,e,0),\ t\in(0,1]\}$$

forms a bounded path $\in \mathrm{int}K_{\mathrm{H}}\times\mathrm{int}K_{\mathrm{H}}\times\Re^m$. Any accumulation point of the path P is an asymptotically complementary solution of (HMCP).

(ii) *If the HMCP has an asymptotically complementarity solution $(x_{\mathrm{H}}^*,y_{\mathrm{H}}^*,z^*) = (x^*,\tau^*,y^*,\kappa^*,z^*)$ with $\tau^*>0$ ($\kappa^*>0$), the any accumulation point $(x_{\mathrm{H}}(0),y_{\mathrm{H}}(0),z(0)) = (x(0),\tau(0),y(0),\kappa(0),z(0))$ of the bounded path P satisfies $\tau(0)>0$ ($\kappa(0)>0$).*

The above two theorems imply that if we have an accumulation point of the central path P, we can find that the original MCP falls into exactly one of the three categories in Table 1.

τ^*/κ^*	$=0$	>0
$=0$	other cases	strongly infeasible
>0	solvable	N/A

Note that the assumption in the theorems is slightly ambiguous since it is concerned with the homogeneous map ψ_{H}, but not with the map ψ appearing in the original MCP. The following proposition gives a class of ψs for which ψ_{H} satisfies Assumption 2.1.

Proposition 4.3 **(i)** *The map ψ_{H} is monotone on $\mathrm{int}K_{\mathrm{H}}\times\Re^m$ whenever ψ is monotone on $K\times\Re^m$.*

(ii) *The map F_{H} is $(x_{\mathrm{H}},y_{\mathrm{H}})$-everywhere-monotone on $\mathrm{int}K_{\mathrm{H}}\times\mathrm{int}K_{\mathrm{H}}\times\Re^m$ whenever ψ is monotone on $K\times\Re^m$.*

(iii) *The map F_{H} is z-bounded on $\mathrm{int}K_{\mathrm{H}}\times\mathrm{int}K_{\mathrm{H}}\times\Re^m$ whenever ψ is affine and z-bounded on $K\times\Re^m$.*

(iv) *The map F_{H} is z-injective on $\mathrm{int}K_{\mathrm{H}}\times\mathrm{int}K_{\mathrm{H}}\times\Re^m$ whenever ψ is z-injective on $K\times\Re^m$.*

References

[1] E. Andersen and Y. Ye. On a homogeneous algorithm for the monotone complementarity problems. *Mathematical Programming*, 84(1999)375–400.

[2] J. Faraut and A. Korányi, *Analysis on symmetric cones*, Oxford University Press, 1994.

[3] A. Shapiro. First and second order analysis of nonlinear semidefinite programs. *Mathematical Programming*, 77(1997)301-320.

[4] A. Yoshise. Interior Point Trajectories and a Homogeneous Model for Nonlinear Complementarity Problems over Symmetric Cones. DPS1098, IPPS, University of Tsukuba, 2004.