Journal of the Operations Research
Society of Japan
2009, Vol. 52, No. 1, 35-45

AN LP-BASED APPROACH TO THE RING LOADING PROBLEM WITH INTEGER DEMAND SPLITTING

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(Received May 31, 2007; Revised August 5, 2008)

Abstract We consider the Ring Loading Problem with integer demand splitting (RLP). The problem is given with a ring network, in which a required traffic requirement between each selected node pair must be routed on it. Each traffic requirement can be routed in both directions of the ring network while splitting each traffic requirement in two directions only by integer is allowed. The problem is to find an optimal routing of each traffic requirement which minimizes the maximum of traffic loads imposed on each link on the network. By characterizing extreme points of the LP relaxation of an IP formulation, we analyze the strength of the LP relaxation. Then we present a strengthened LP which provides enough information to determine the optimal objective value of RLP. Finally, we give an LP-based polynomial-time algorithm for the problem which can handle more general cases where nontrivial upper and lower bounds are imposed on the amount of traffic routed in one direction for some node pairs.

Keywords: Network flow, discrete optimization, integer programming, ring loading problem

1. Introduction

The advances in transmission technology can support dependable high speed telecommunication services. Especially, the fiber-optic technology, combined with the new intelligent synchronous digital network elements makes the networks highly reliable. Survivability, which is the ability to restore traffic demand when a failure happens in the network, is an important factor for the planning of reliable networks.

A ring network is a collection of nodes forming a closed loop, where each node is connected via a duplex communications facility. A Self-Healing Ring (SHR) is a ring network that provides redundant capacity and/or network equipment so disrupted services can be automatically restored following network failures. The Synchronous Optical Network (SONET) technology and associated high speed add/drop multiplexing technology make SHR's practical and economical for interoffice networking applications [6].

There are two types of SHR's, unidirectional self-healing rings (USHR's) and bidirectional self-healing rings (BSHR's). They both can restore 100% of traffic under single network failure. In USHR's, working traffic is carried around the ring in one direction, while a second communications ring is for protection only and transmits in the opposite direction of the working ring. In BSHR's, working traffic travels in both directions around the ring. BSHR's are further divided into 2-fiber and 4-fiber BSHR's. A 2-fiber BSHR uses half the capacity of the fiber system for working traffic and reserves the other half for protection, while a 4-fiber BSHR uses two separate fibers for working traffic and the other two fibers for protection. We refer the reader to Wu [6] for more details.

For given required traffic (demand) between each selected node pair, the capacity requirement of a SHR is defined to be the maximum of traffic loads imposed on links of it. The capacity requirement of a USHR is the sum of traffic requirements for all selected node pairs on it, since the routing is fixed. As stated above, in a BSHR, each traffic requirement can be routed in both directions around the ring. Hence, the capacity requirement of a BSHR varies according to the routing of traffic requirements.

The problem that we study in this paper, which we call the Ring Loading Problem with integer demand splitting (RLP), has been motivated by BSHR configuration. The problem is given with a ring network, in which a required traffic requirement between each selected node pair must be routed on it. Each traffic requirement can be routed in both directions of the ring network while splitting each traffic requirement in two directions only by integer is allowed. The problem is to find an optimal routing of each traffic requirement which minimizes the capacity requirement. Here, the capacity requirement (the maximum of traffic loads imposed on each link on the network).

RLP was previously studied by Lee and Chang [1], Myung [2], Vachani et al. [3], and Wang [4]. Lee and Chang [1] proposed a heuristic algorithm for RLP and have not mentioned the computational complexity of RLP. Vachani et al. [3] presented the first polynomial time algorithm for RLP. The complexity of their algorithm is $O(n^3)$, where n is the number of nodes (links) of the given cycle. Myung [2] gave an improved $O(n|K|)$ algorithm, where $|K|$ is the number of selected node pairs. Very recently, Wang [4] presented a linear time algorithm. The complexity of the algorithm is $O(|K|)$.

If there exists an efficient (polynomial time) algorithm for a discrete optimization problem, there also may exist an explicit description of the convex hull of the feasible solutions, which in turn allows us to just solve a linear program to get an optimal solution to the problem [5]. For example, it is known that the minimum spanning tree problem can be solved in polynomial time, and the description of the convex hull of all spanning trees is also explicitly known. In a sense, the existence of a polynomial time algorithm for a discrete optimization problem is characterized by our ability to construct a linear program, with a polynomial number of variables and constraints, which gives an optimal solution to the discrete optimization problem. See chapter 3 of Wolsey [5] for more detailed theoretical discussion.

Since RLP can be solved in polynomial time but the description of the convex hull of all the feasible solutions to RLP is not known, it is natural to ask if there is a linear program whose number of variables and constraints is bounded by a polynomial function of the size of a given instance of RLP, and gives an optimal solution to RLP. In this paper, we give an answer to an interesting question which is closely related to the above question. The results presented in this paper are still valid in a more generalized case where nontrivial integer-valued upper and lower bounds are imposed on the amount of traffic routed in one direction of the given ring network for some selected node pairs, which is often the case in practical telecommunication network planning process where planners want to evaluate the capacity requirement of a ring network under various traffic scenarios and partial traffic routing plan to simulate various what-if scenarios.

We first present an integer programming formulation of RLP in section 2. Let z_{LP} and z_{RLP} be the optimal objective value of the LP relaxation of the formulation and that of RLP, respectively. By characterizing extreme points of the LP relaxation of the formulation, we show that z_{LP} is equal to p or $p + 0.5$, where p is a nonnegative integer. We also show that the set of feasible solutions to the LP relaxation is an integral polyhedron in some special cases. Further, we show that $z_{RLP} - z_{LP} \leq 1$ by constructing a feasible solution to RLP, whose corresponding objective value is less than or equal to $\lfloor z_{LP} \rfloor + 1$, from an optimal extreme point solution to the LP relaxation which has fractional coordinates. Therefore, it is clear that $z_{RLP} = \lceil z_{LP} \rceil$ if z_{LP} is equal to $p + 0.5$, where p is a nonnegative integer. On the other hand, if z_{LP} is equal to p, then either $z_{RLP} = z_{LP}$ or $z_{RLP} = z_{LP} + 1$. In this case, does there exist an LP which can tell if $z_{RLP} = z_{LP}$ or not ? Does there exists such an LP whose size is bounded by a polynomial function of n and $|K|$? We give an affirmative answer to these questions in section 3. Finally, we give concluding remarks in section 4 together with an LP-based polynomial time algorithm for RLP.

2. Analysis of LP Relaxation

The Ring Loading Problem with integer demand splitting (RLP) is defined on an undirected ring $G = (V, E)$ with $V = \{1, 2, \dots, v\}$ and $E = \{(1, 2), (2, 3), \dots, (v-1, v), (v, 1)\}$. The following are additional notation and definitions to be used in the formulation of RLP.

- K : set of selected node pairs (commodities),
- o_k, d_k : two nodes of a commodity k, for each $k \in K$, where $o_k < d_k$,
- r_k : traffic requirement of commodity k, for each $k \in K$, assumed to be a positive integer,
- P_k^+ : set of links which are used by the clockwise path of k, for each $k \in K$, i.e., $\{(o_k, o_k+1), (o_k+1, o_k+2), \dots, (d_k-1, d_k)\}$,
- P_k^- : set of links which are used by the counter-clockwise path of k, for each $k \in K$, i.e., $E \setminus P_k^+$,
- x_k : decision variable that represents the amount of traffic requirement of commodity k which are routed in the clockwise direction, for each $k \in K$.

Note that telecommunication traffic is typically bidirectional. i.e., traffic requirement r_k refers to traffic from o_k to d_k as well as from d_k to o_k; the transmission technology also typically allow for this amount of traffic requirement to be carried in both directions using capacity r_k. Hence, we only need to consider traffic requirements r_k, for $o_k < d_k$ [4].

If we route x_k units of traffic requirement of commodity k in the clockwise direction, for each $k \in K$, $(r_k - x_k)$ units are routed in the counter-clockwise direction. Therefore, we can formulate an integer program for RLP as follows :

$$\text{(RLP)} \quad \min \quad z \tag{1}$$

$$\text{s.t.} \quad x_k \le r_k, \quad \text{for all } k \in K,$$

$$\sum_{\{k \in K | e \in P_k^+\}} x_k + \sum_{\{k \in K | e \in P_k^-\}} (r_k - x_k) \le z, \quad \text{for all } e \in E, \tag{2}$$

$$x_k, \ \text{nonnegative integer}, \quad \text{for all } k \in K.$$

Note that the objective function (1) is to minimize the maximum traffic loads imposed on each link of the network. For ease of later expositions, let us rewrite constraints (2) as follows:

$$I_e : -z + \sum_{\{k \in K | e \in P_k^+\}} x_k - \sum_{\{k \in K | e \in P_k^-\}} x_k \le R_e,$$

where $R_e = -\sum_{\{k \in K | e \in P_k^-\}} r_k$, and let $L_e(x, z)$ be the left-hand-side of I_e. For a feasible solution (x^*, z^*), the value of the left-hand-side of I_e is denoted by $L_e(x^*, z^*)$. If there exist nontrivial integer-valued lower and upper bounds (l_k and u_k, respectively) for some $k \in K$, then the corresponding bound constraints should be added in the above formulation.

Now, we characterize extreme point solutions of the linear programming relaxation of (RLP) and analyze the strength of it. Let (LP) be the linear programming relaxation of (RLP), that is, (RLP) without integrality restrictions. Let P be the set of feasible solutions to (LP) and $(\bar{x}, \bar{y})$ be an extreme point of P. Let $EQ(\bar{x}, \bar{y})$ be the set of defining inequalities of P which are satisfied at equalities by $(\bar{x}, \bar{y})$ and let $E(\bar{x}, \bar{y}) = \{e \in E | L_e(\bar{x}, \bar{z}) = R_e\}$. Further, define $\bar{K}(\bar{x}) = \{k \in K | 0 < \bar{x}_k < r_k\}$. Then by substituting the variables $\bar{x}_k$, $k \in K \setminus \bar{K}(\bar{x})$ into each inequality of $EQ(\bar{x}, \bar{y})$, we can obtain the following system of linear equations :

$$-z + \sum_{\{k \in \bar{K}(\bar{x}) | e \in P_k^+\}} x_k - \sum_{\{k \in \bar{K}(\bar{x}) | e \in P_k^-\}} x_k = \bar{R}_e, \text{ for all } e \in E(\bar{x}, \bar{y}), \tag{3}$$

where $\bar{R}_e = R_e - \sum_{\{k \in K \setminus \bar{K}(\bar{x}) | e \in P_k^+\}} \bar{x}_k + \sum_{\{k \in K \setminus \bar{K}(\bar{x}) | e \in P_k^-\}} \bar{x}_k$.

The following proposition characterizes every extreme point of P. We call an integer c is even(odd) if the absolute value of c is even(odd), from now on.

Proposition 1 Let $(\bar{x}, \bar{y}) \in P$ be an extreme point of P, then $\bar{x}_k = l_k/2$ for all $k \in K$, where l_k is a nonnegative integer which is less than or equal to $2r_k$.
Proof. See Appendix. □

The above proposition means that every extreme point of P is half-integral. That is, if a variable corresponding to an extreme point of P has a fractional value, its fractional part is 0.5. Myung [2] showed that there exists a half-integral optimal solution to (LP). This fact, however, does not necessarily mean that every extreme point of P is half-integral.

Proposition 2 P is an integral polyhedron, that is every extreme point is integral, if all R_e's are either even or odd.
Proof. See Appendix. □

If r_k is even for each $k \in K$, the associated polyhedron P is integral by the above proposition 2. Now, we will analyze the strength of the bound of (LP). First, we need the following definition.

Definition 1 Two commodities $i, j \in K$ are (mutually) independent if one of the following two conditions holds, otherwise, i and j are dependent. $i)$ $P_i^+ \subset P_j^+$ (or $P_i^+ \supset P_j^+$), $ii)$ $P_i^+ \subset P_j^-$.

Note that if two commodities are independent, then each of the two traffic requirements can be routed without sharing a common link, i.e., if the condition $i)$ holds then $P_i^+ \cap P_j^- = \emptyset$ (or $P_i^- \cap P_j^+ = \emptyset$), and if the condition $ii)$ holds then $P_i^+ \cap P_j^+ = \emptyset$.

We say that a set of commodities $N \subseteq K$ is dependent, if i and j are dependent, for every pair of $i, j \in N$. Now, we present some observations. First, note that $P_j^+ \cup P_j^- = E$, for every $j \in K$.

Remark 1 For any pair of commodities $i, j \in K$, the followings hold. $i)$ $P_i^+ \subset P_j^+ (P_i^+ \supset P_j^+)$ if and only if $P_i^- \supset P_j^- (P_i^- \subset P_j^-)$, $ii)$ $P_i^+ \subset P_j^-$ if and only if $P_i^- \supset P_j^+$.

Remark 2 If $i, j \in K$ are dependent, the followings hold. $i)$ $P_i^+ \cap P_j^+ \neq \emptyset$, $ii)$ $P_i^+ \cap P_j^- \neq \emptyset$, $iii)$ $P_i^- \cap P_j^+ \neq \emptyset$.

Remark 3 Let $N = \{1, 2, \ldots, n\} \subseteq K$, where $o_1 \leq o_2 \leq \cdots \leq o_n$. If N is dependent, then $o_1 < o_2 < \cdots < o_n < d_1 < d_2 < \cdots < d_n$.

Now, let (SP) be (RLP) with $K = N$ and $r_i = 1$, for all $i \in N$, where $N = \{1, 2, \ldots, n\}$ is dependent with $o_1 \leq o_2 \leq \cdots \leq o_n$. From remark 3, we have $o_1 < \cdots < o_n < d_1 < \cdots < d_n$. It can be easily shown that $\bar{x}_i = 0.5$, for all $i \in N$ with the corresponding objective value $\bar{z} = n/2$, where $(\bar{x}, \bar{z})$ is an optimal solution of the linear programming relaxation of (SP). Now, consider the following feasible solution (x', z') to (SP) :

$$x'_i = 1, \text{ if } i \text{ is odd, for all } i \in N, x'_i = 0, \text{ otherwise,} \tag{4}$$
$$z' = \max_{e \in E}\{\textstyle\sum_{\{i \in N | e \in P_i^+\}} x'_i + \sum_{\{i \in N | e \in P_i^-\}} (1 - x'_i)\}.$$

That is, if i is odd, the traffic requirement of commodity i is routed in the clockwise direction, otherwise, it is routed in the counter-clockwise direction. The following proposition characterizes the corresponding objective value of the solution (4).

Proposition 3 Let (x', z') be a feasible solution to (SP) defined as (4). Then, $z' = (n/2+1)$, if n is even, and $z' = (n/2 + 0.5)$, if n is odd.

Proof. See Appendix. □

The following proposition and proposition 3 are keys to the analysis of the strength of the bound of (LP).

Proposition 4 Let $(\bar{x}, \bar{z})$ be an optimal extreme point solution to (LP), where $F \subseteq K$ with $|F| \geq 2$ is the set of commodities which correspond to the variables whose values are fractional. Then, if $i, j \in F$ are independent, we can construct another optimal solution $(x', \bar{z})$ to (LP) in which x'_i, x'_j are integral.

Proof. By proposition 1, $\bar{x}_i = p + 0.5, \bar{x}_j = q + 0.5$, for some nonnegative integers $p \leq r_i - 1, q \leq r_j - 1$. First, suppose that the condition $i)$ of definition 1 holds. Without loss of generality, assume that $P_i^+ \subset P_j^+$. Then, by remark 1, $P_i^- \supset P_j^-$. Consider a solution x' in which $x'_i = p + 1, x'_j = q$ and all the other variables remain the same. It can be easily shown that (x', z') is a feasible solution to (LP) with $z' = \bar{z}$. Now, suppose that the condition $ii)$ of definition 1 holds, i.e., $P_i^+ \subset P_j^-$. Then, $P_i^+ \cap P_j^+ = \emptyset$. So, consider the following feasible solution x' in which $x'_i = p+1, x'_j = q+1$ and all the other variables remain the same. It can be also easily shown that (x', z') is a feasible solution to (LP) with $z' = \bar{z}$. □

Now, recall that z_{LP} denotes the optimal objective value of (LP) and z_{RLP} denotes that of (RLP).

Theorem 1 $z_{RLP} - z_{LP} \leq 1$ and the bound is tight.

Proof. Suppose that an optimal extreme point solution (x^*, z^*) to (LP) is given, where $z^* = z_{LP}$. If x^* is integral, $z_{RLP} - z_{LP} = 0$. So, assume that x^* has some fractional coordinates. From proposition 4, we can construct another optimal solution (x', z^*) to (LP) by sequentially choosing a pair of independent commodities and changing values to integers. Let F be the set of commodities which correspond to the variables whose values are fractional in x'. If $F = \emptyset$, then $z_{RLP} - z_{LP} = 0$. So, assume that $F \neq \emptyset$. By the construction of x', F is dependent or has a single commodity.

Consider a modified instance of the given instance of (RLP), where the traffic requirement for each $k \in F$ is set to $r_k - 1$ but all the others remain the same. By proposition 1, the following solution $(\tilde{x}, \tilde{z})$ is a feasible solution to this modified instance of (RLP) :

$$\begin{aligned} &\tilde{x}_k = x'_k, \text{for all } k \in K \setminus F, \\ &\tilde{x}_k = x'_k - 0.5, \text{for all } k \in F, \\ &\tilde{z} = z_{LP} - (|F|/2). \end{aligned}$$

Then, consider an instance of (SP) with $K = F$. By proposition 3, we can construct a feasible solution $(\hat{x}, \hat{z})$ to this instance of (SP) with $\hat{z} = (|F|/2 + 1)$ if $|F|$ is even, and $\hat{z} = (|F|/2 + 0.5)$ if $|F|$ is odd. Now, if we set $\bar{x}_k = \tilde{x}_k$ for all $k \in K \setminus F$, and $\bar{x}_k = \tilde{x}_k + \hat{x}_k$ for all $k \in F$, then $(\bar{x}, \bar{z})$ is a feasible solution to the given instance of (RLP) whose objective value $\bar{z}$ is less than or equal to :

$$\begin{aligned} &z_{LP} - (|F|/2) + (|F|/2 + 1) = z_{LP} + 1, \text{ if } |F| \text{ is even}, \\ &z_{LP} - (|F|/2) + (|F|/2 + 0.5) = z_{LP} + 0.5, \text{ if } |F| \text{ is odd}. \end{aligned}$$

Therefore, $z_{RLP} - z_{LP} \leq 1$. Consider an instance of (RLP) with only two dependent commodities whose demands are all equal to 1. In this case, $z_{LP} = 1, z_{RLP} = 2$. So, the bound is tight. □

The fact that $z_{RLP} - z_{LP} \leq 1$ is already known by the previous studies such as Myung [2]. However, we want to mention that the above theorem still holds in generalized cases where nontrivial integer-valued upper and lower bounds are imposed on the decision variable x_k, for each $k \in D$, where $D \subseteq K$.

From theorem 1, we can obtain an optimal solution of (RLP) if $z_{LP} = p + 0.5$, for some nonnegative integer p. On the other hand, if $z_{LP} = p$ for some nonnegative integer p, either $z_{RLP} = p$ or $z_{RLP} = p + 1$. Recall that $z_{LP} = 1$ and $z_{RLP} = 2$ in the example instance given in the proof of theorem 1. Let us consider an instance of (RLP) defined on a 6-node ring network with $E = \{(i, i+1) | i = 1, \ldots, 5\} \cup \{(1, 6)\}$ and four commodities whose demands are all equal to 1, where $\{(o_k, d_k) | k = 1, \ldots, 4\} = \{(1, 4), (2, 3), (2, 5), (3, 6)\}$. In this example, it can be easily shown that $z_{RLP} = 2$ and an optimal extreme point solution to the corresponding LP relaxation is $x_k = 0.5, k = 1, \ldots, 4$, with $z_{LP} = 2$. As in the proof of theorem 1, we can construct a feasible solution $(\bar{x}, \bar{z})$ to (RLP) with $\bar{z} = 3$ starting from the fractional solution.

Suppose that $z_{LP} = p$ for some nonnegative integer p. Then, does there exist a linear program whose number of variables and constraints is bounded by a polynomial function of $|E|$ and $|K|$, which can provide information to determine if $z_{RLP} = p$ or not ? In the next section, we give an affirmative answer to this question.

3. A Strengthened Linear Program

For a pair of inequalities I_e and I_f, if exactly one of R_e and R_f is an odd number, we can obtain the following valid inequality I_{ef} to (RLP) :

$$I_{ef} : -z + \sum_{\{k \in K | e, f \in P_k^+\}} x_k - \sum_{\{k \in K | e, f \in P_k^-\}} x_k \leq \lfloor (R_e + R_f)/2 \rfloor.$$

Note that the right-hand-side of I_{ef} is equal to $(R_e+R_f)/2-0.5$ and $I_{ef}=I_{fe}$. Let $L_{ef}(x,z)$ be the left-hand-side of I_{ef}. Then, $L_{ef}(x,z)=(L_e(x,z)+L_f(x,z))/2$.

For ease of exposition, let $Q=\{(e,f)|$exactly one of R_e and R_f is odd, for each pair of $e,f\in E\}$. Let (LP′) be the LP relaxation of (RLP) obtained by adding all I_{ef}, $(e,f)\in Q$ to (LP), which yields a stronger LP relaxation of (RLP) than (LP). We also use $EQ(\bar{x},\bar{z})$ to denote the set of inequalities of (LP′) which are satisfied at equalities by a feasible solution $(\bar{x},\bar{z})$ to (LP′).

Proposition 5 Let $(\bar{x},\bar{z})$ be a feasible solution to (LP′). If $I_e\in EQ(\bar{x},\bar{z})$ and $I_f\in EQ(\bar{x},\bar{z})$, then both R_e and R_f are either odd or even.

Proof. Suppose that $I_e\in EQ(\bar{x},\bar{z})$ and $I_f\in EQ(\bar{x},\bar{z})$, but exactly one of R_e and R_f is odd. Then, $L_{ef}(\bar{x},\bar{z})=(L_e(\bar{x},\bar{z})+L_f(\bar{x},\bar{z}))/2=(R_e+R_f)/2>\lfloor(R_e+R_f)/2\rfloor$, thus, $(\bar{x},\bar{z})$ violates I_{ef}. □

Proposition 6 Let (x^*,z^*) be an extreme point solution to (LP′) with $z^*=z_{LP}$. If $I_{ef}\in EQ(x^*,z^*)$, for some $(e,f)\in Q$, then exactly one of I_e and I_f is in $EQ(x^*,z^*)$.

Proof. Suppose that $I_{ef}\in EQ(x^*,z^*)$. Clearly, both I_e and I_f cannot be in $EQ(x^*,z^*)$. Now, suppose that $I_e\notin EQ(x^*,z^*)$ and $I_f\notin EQ(x^*,z^*)$. Then the following hold :

$$L_e(x^*,z^*)+s_e=R_e,\ L_f(x^*,z^*)+s_f=R_f \text{ and}$$
$$L_{ef}(x^*,z^*)=\lfloor(R_e+R_f)/2\rfloor,\ \text{where } s_e>0 \text{ and } s_f>0.$$

From the construction of I_{ef}, $L_e(x,z)=L_{ef}(x,z)+l(x)$ and $L_f(x,z)=L_{ef}(x,z)-l(x)$, where $l(x)=\sum_{\{k\in K|e\in P_k^+,f\in P_k^-\}}x_k-\sum_{\{k\in K|e\in P_k^-,f\in P_k^+\}}x_k$. That is,

$$\begin{aligned}
&L_e(x^*,z^*)+s_e=L_{ef}(x^*,z^*)+l(x^*)+s_e=R_e,\\
&L_f(x^*,z^*)+s_f=L_{ef}(x^*,z^*)-l(x^*)+s_f=R_f,\\
&L_{ef}(x^*,z^*)+(s_e+s_f)/2=(R_e+R_f)/2.
\end{aligned}$$

Therefore, $s_e+s_f=1$, where $s_e>0$ and $s_f>0$.

Now, we show that $I_g\notin EQ(x^*,z^*)$, for all $g\in E\setminus\{e,f\}$. Suppose $I_g\in EQ(x^*,z^*)$, for some $g\in E\setminus\{e,f\}$. Without loss of generality, assume that R_g is odd and R_e is even, then, since $0<s_e<1$,

$$\begin{aligned}
L_{eg}(x^*,z^*)\ &=(L_e(x^*,z^*)+L_g(x^*,z^*))/2=(R_e+R_g)/2-s_e/2\\
&>(R_e+R_g)/2-1/2=\lfloor(R_e+R_g)/2\rfloor.
\end{aligned}$$

It means (x^*,z^*) violates I_{eg}, which contradicts the supposition that $I_g\in EQ(x^*,z^*)$, for some $g\in E\setminus\{e,f\}$.

Therefore, if $I_{ef}\in EQ(x^*,z^*)$ but $I_e\notin EQ(x^*,z^*)$ and $I_f\notin EQ(x^*,z^*)$, then $L_g(x^*,z^*)<R_g$, for all $g\in E$. It means that there exists a feasible solution (x^*,z') to (LP) such that $z'<z^*$, which is impossible because (x^*,z^*) is an optimal solution to (LP). □

The following theorem characterizes the strength of bounds provided by (LP′).

Theorem 2 If there exists a feasible solution to (RLP) whose objective value is equal to z_{LP}, then an optimal extreme point solution $(\hat{x},\hat{z})$ to (LP′) is integral with $\hat{z}=z_{LP}$. Otherwise, $(\hat{x},\hat{z})$ has possibly fractional coordinates with $\hat{z}>z_{LP}$.

Proof. Suppose that there exists a feasible solution to (RLP) whose objective value is equal to z_{LP}. Then, there exists an optimal extreme point solution $(\hat{x}, \hat{z})$ to (LP′) with $\hat{z} = z_{LP}$. Therefore, we have only to prove that $(\hat{x}, \hat{z})$ is integral. By proposition 5, R_e's have the same parity, for all $I_e \in EQ(\hat{x}, \hat{z})$. Also by proposition 6, if $I_{ef} \in EQ(\hat{x}, \hat{z})$, exactly one of I_e and I_f is in $EQ(\hat{x}, \hat{z})$. Let us assume that $I_{ef} \in EQ(\hat{x}, \hat{z})$ and $I_e \in EQ(\hat{x}, \hat{z})$. Then, $L_f(\hat{x}, \hat{z}) = R_f - 1$. Also note that I_{ef} is equal to an inequality $(I_e + I'_f)/2$, where $I'_f : L_f(x, z) \leq R_f - 1$. Therefore, $(\hat{x}, \hat{z})$ should be the unique solution to the following system of linear equations :

$$x_k = r_k, \text{ for all } k \in K \text{ such that } \hat{x}_k = r_k,$$
$$x_k = 0, \text{ for all } k \in K \text{ such that } \hat{x}_k = 0,$$
$$L_e(x, z) = R_e, \text{ for all } I_e \in EQ(\hat{x}, \hat{z}), \tag{5}$$
$$L_f(x, z) = R_f - 1, \text{ for all } I_{ef} \in EQ(\hat{x}, \hat{z}) \text{ and } I_e \in EQ(\hat{x}, \hat{z}). \tag{6}$$

Note that the right-hand-side values of all equations in (5) and (6) have the same parity. Now, consider some $k \in K$ such that $\hat{x}_k = r_k$. Note that x_k appears in all equations in (5) and (6) with the coefficients 1 or -1. Therefore, the right-hand side values of all equations in (5) and (6) after substituting $\hat{x}_k = r_k$ into them also have the same parity. By repeating the same process, finally, we can obtain a system of linear equations similar to (3) with the right-hand sides of the same parity. Now, by proposition 2, $(\hat{x}, \hat{z})$ is integral. This completes the first part of this theorem.

Now, suppose that $z_{RLP} > z_{LP}$. Then $\hat{z} > z_{LP}$, otherwise, as in the first part of this proof, $(\hat{x}, \hat{z})$ is integral with $\hat{z} = z_{LP}$, which contradicts $z_{RLP} > z_{LP}$. □

From the above theorem, (LP′) either gives an optimal integral solution to (RLP) if $z_{RLP} = z_{LP}$ or proves $z_{RLP} > z_{LP}$ when $z_{LP} = p$ for some nonnegative integer p. Moreover, since $z_{RLP} - z_{LP} \leq 1$, it is clear that $z_{RLP} - z_{LP'} < 1$, thus, $z_{RLP} = \lceil z_{LP'} \rceil$, where $z_{LP'}$ is the optimal objective value of (LP′). The number of additional inequalities $I_{ef}, (e, f) \in Q$, is at most $|E|^2/4$. Therefore, (LP′) has $|K|$ variables and at most $|E| + |E|^2/4$ constraints.

4. Concluding Remarks

In this paper, we present a strengthened linear program of which size is bounded by a polynomial function of the number of nodes (links) and the number of selected node pairs, which provides enough information to determine the optimal value of (RLP). Based on the results, we have tried to find the complete inequality description of the convex hull of feasible solutions of (RLP). For now, we just have partial results for that, however, we expect the convex hull description can be found in the near future. One final remark is that an LP-based polynomial time algorithm for (RLP) can be devised as follows by using theorem 2 and the proof of theorem 1 :

Algorithm A :

Step 1 : Solve (LP), let (x^*, z^*) be the obtained solution to (LP). If (x^*, z^*) is integral, it is an optimal solution to (RLP) and stop.

Step 2 : Construct an integral solution $(\bar{x}, \bar{z})$ as in the proof of theorem 1. If $\bar{z} = z^*$ or $\bar{z} = z^* + 0.5$, $(\bar{x}, \bar{z})$ is optimal and stop.

Step 3 : Construct (LP′) and solve it. Let $(\hat{x}, \hat{z})$ be the obtained optimal extreme point solution of (LP′). If $\hat{z} = z^*$, $(\hat{x}, \hat{z})$ is integral by theorem 2 and it is optimal, stop. Otherwise, $\hat{z} > z^*$, $z_{RLP} = z^* + 1$ and $(\bar{x}, \bar{z})$ is an optimal solution to (RLP).

Note that a linear program can be solved in polynomial time and we can construct (LP) and (LP$'$) in polynomial time. Also note that the number of fractional coordinates of an optimal solution to (LP) without z is at most $|E| - 1$, because the maximum rank of the systems of linear equalities (3) is at most $|E|$. Hence, we can construct $(\bar{x}, \bar{z})$ within $O(|E|^2)$ time. Therefore, Algorithm A has a polynomial time complexity.

The above algorithm may be slower than the combinatorial algorithms [2–4] developed so far. Since the theorems and propositions presented in section 2 and 3 hold in generalized cases where nontrivial integer-valued upper and lower bounds are imposed on decision variables(x_k's), the above algorithm can be used in such cases. However, previously developed fast combinatorial algorithms in [2–4] may need to be modified to handle such cases.

Acknowledgements

This work was supported by Hankuk University of Foreign Studies Research Fund.

References

[1] C.Y. Lee and S.G. Chang: Balancing loads on SONET rings with integer demand splitting. *Computers & Operations Research*, **24** (1997), 221–229.

[2] Y.S. Myung: An efficient algorithm for the ring loading problem with integer demand splitting. *SIAM Journal on Discrete Mathematics*, **14** (2001), 291–298.

[3] R. Vachani, A. Shulman, P. Kubat, and J. Ward: Multicommodity flows in ring networks. *INFORMS Journal on Computing*, **8** (1996), 235–242.

[4] B.F. Wang: Linear time algorithms for the ring loading problem with demand splitting. *Journal of Algorithms*, **54** (2005), 45–57.

[5] L. Wolsey: *Integer Programming* (John Wiley and Sons, New York, 1998).

[6] T.-H. Wu: *Fiber Network Service Survivability* (Artech House, MA, 1992).

Appendix

Proof of Proposition 1. Let B be the left-hand-side coefficient matrix of (3). After eliminating redundant equations, we can assume that $B = [B_1, B_2, \ldots, B_m]$ is an m by m nonsingular integral matrix and b is an m by 1 integral vector, where the first column of B, B_1, corresponds to the variable z. Note that each column corresponding to the variable x_k of (LP), for all $k \in K$, has one of the following two patterns :

$$[1, \ldots, 1, -1, \ldots, -1]^T \text{ and } [-1, \ldots, -1, 1, \ldots, 1, -1, \ldots, -1]^T.$$

Hence, B_i, for all $2 \leq i \leq m$, has one of the following three patterns :

$$[1, \ldots, 1, -1, \ldots, -1]^T, [-1, \ldots, -1, 1, \ldots, 1]^T, \text{ and } [-1, \ldots, -1, 1, \ldots, 1, -1, \ldots, -1]^T.$$

Since $B_1 = [-1, \ldots, -1]^T$, B_i cannot have $[1, \ldots, 1]^T$ or $[-1, \ldots, -1]^T$, for all $2 \leq i \leq m$.

Let us apply Gaussian elimination procedure to $(B : b)$. After the first step, B is changed to $B^{(1)}$, where

$$B^{(1)} = \begin{bmatrix} -1 & a^{(1)} \\ d & A^{(1)} \end{bmatrix}$$

$d = [0,\ldots,0]^T$ is an $(m-1) \times 1$ vector, $a^{(1)}$ is an $1 \times (m-1)$ vector each element of which is either 1 or -1, and $A^{(1)}$ is an $(m-1) \times (m-1)$ matrix each column of which has one of the following three patterns in which leading 0's may be omitted :

$$[0,\ldots,0,-2,\ldots,-2]^T, [0,\ldots,0,2,\ldots,2]^T, \text{ and } [0,\ldots,0,2,\ldots,2,0,\ldots,0]^T.$$

Note that b is changed to an m by 1 integral vector. We now only to apply Gaussian elimination procedure to $A^{(1)}$. It can be easily shown that after completing Gaussian elimination procedure to $A^{(1)}$, each nonzero element of the resulting upper-triangular matrix is either 2 or -2. Also the resulting right-hand-side vector is an integral vector. Note that we can use only subtractions at each elimination step without row exchanges. But column exchanges are possibly used. This proves proposition 1. □

Proof of Proposition 2. Suppose that $b = [b_1,\ldots,b_m]^T$ such that all b_i's are either even or odd. After the first step of Gaussian elimination procedure as in the proof of proposition 1, b is changed to $b^{(1)} = [b_1, b_2 - b_1, \ldots, b_m - b_1]^T$, where every $b_i - b_1$ is even, for all $2 \le i \le m$. Therefore, the result follows. □

Proof of Proposition 3. Recall that (SP) is a special case of (RLP) with $K = N$ and $r_i = 1$, for all $i \in N$, where $N = \{1,2,\ldots,n\}$ is dependent with $o_1 < \cdots < o_n < d_1 < \cdots < d_n$. Suppose that (x', z') is a feasible solution to (SP) given in (4). z' is equal to the maximum number of commodities which pass through links of G. Let $(i-j)$ be the set of commodities which pass through the set of links between two nodes i and j, where the set of links between i and j is $\{(i,i+1),\ldots,(j-1,j)\}$ if $i<j$ or $\{(i,i+1),\ldots,(n,1),\ldots,(j-1,j)\}$ if $i>j$.

i) if n is even : for all $1 \le i \le n-1$,

$$
\begin{aligned}
&(o_i - o_{i+1}) = \{2l-1 | 1 \le l \le (i+1)/2\} \cup \{2l | (i+1)/2 \le l \le n/2\}, \text{ if } i \text{ odd},\\
&(o_i - o_{i+1}) = \{2l-1 | 1 \le l \le i/2\} \cup \{2l | (i+2)/2 \le l \le n/2\}, \text{i f } i \text{ even},\\
&(d_i - d_{i+1}) = \{2l | 1 \le l \le (i-1)/2\} \cup \{2l-1 | (i+3)/2 \le l \le n/2\}, \text{ if } i \text{ odd},\\
&(d_i - d_{i+1}) = \{2l | 1 \le l \le i/2\} \cup \{2l-1 | (i+2)/2 \le l \le n/2\}, \text{i f } i \text{ even},\\
&(o_n - d_1) = \{2l-1 | 1 \le l \le n/2\}, \text{ and}\\
&(d_n - o_1) = \{2l | 1 \le l \le n/2\}. \text{ So },\\
&|(o_i - o_{i+1})| = (n/2)+1, \text{ if } i \text{ odd, otherwise } n/2, \text{ for all } 1 \le i \le n-1,\\
&|(o_n - d_1)| = |(d_n - o_1)| = n/2, \text{ and}\\
&|(d_i - d_{i+1})| = (n/2)-1, \text{ if } i \text{ odd, otherwise } n/2, \text{ for all } 1 \le i \le n-1.
\end{aligned}
$$

Therefore $z' = (n/2)+1$.

ii) if n is odd, in the same manner, we can get :

$$
\begin{aligned}
&|(o_i - o_{i+1})| = (n-1)/2+1, \text{ if } i \text{ odd, otherwise } (n-1)/2, \text{ for all } 1 \le i \le n-1,\\
&|(o_n - d_1)| = (n-1)/2+1,\\
&|(d_n - o_1)| = (n-1)/2, \text{ and}\\
&|(d_i - d_{i+1})| = (n-1)/2, \text{ if } i \text{ odd, otherwise } (n-1)/2+1, \text{ for all } 1 \le i \le n-1.
\end{aligned}
$$

Therefore $z' = (n-1)/2+1 = n/2+0.5$. □

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