THE ORDER OF \( n \) ITEMS PROCESSED ON \( m \) MACHINES.

ICHIRO NABESHIMA

The Metropolitan Hiroo High School.
(Presented at the 8th meeting, November 6, 1960)

1. PROBLEM

In this paper we consider the problem of deciding the order in which \( n \) items should be processed by \( m \) machines in order to minimize the time required to complete all the operations.

Let \( m \) machines be named by \( M_1, M_2, \ldots, M_m \), and let \( m_{k,i} \) be the time required to process the \( i \)th item on the machine \( M_k \) where the processing requires that the machines be used by the same numerical order for any item.

2. SOLUTION

In the following, we use the functional-equation approach. [1]

Let us define

\[
f[i, j, \ldots; t] = \text{the time consumed in processing the remained items after the processing of some definite sequence } S \text{ of items, when the machine } M_k \text{ is committed } t_{k-1} \text{ hours ahead for the machine } M_{k-1},
\]

\( k=2, 3, \ldots, m \); and an optimal scheduling procedure is employed.

In this case, the last machine \( M_m \) is committed \( t = \sum_{k=1}^{m-1} t_k \) hours ahead for the first machine \( M_1 \).

Then, after the sequence \( S \) of items, if \( i \)th item is processed first, we have (see Fig. 1)

\[
f[i, j, \ldots; t] = m_{1,i} + f[0, j, \ldots; g(i, t)]
\]

\[
g(i, t) = \sum_{k=2}^{m} \{m_{k,i} + \max(t_{k-1}^{(i)} - m_{k-1,i}, 0)\}
\]

where

\[
t_{1}^{(i)} = t_1,
\]

\[
t_{k}^{(i)} = t_k - \max(m_{k-1,i} - i_{k-1}^{(i)}, 0), \quad k=2, 3, \ldots, m-1.
\]

If we choose the \( j \)th item to follow, we obtain, by putting

\[
A_{k}^{(i)} = m_{k+1,i} - m_{k,j} + \max(t_{k}^{(j)} - m_{k,j}, 0),
\]

\[
A_{k}^{(j)} = m_{k+1,j} - m_{k,i} + \max(t_{k}^{(i)} - m_{k,i}, 0),
\]

\[
A_{k}^{(i,j)} = m_{k+1,j} - m_{k,i} + \max(t_{k}^{(i)} - m_{k,i}, 0) + \max(t_{k}^{(j)} - m_{k,j}, 0).
\]
The Order of \( n \) Items Processed on \( m \) Machines

\[ f[i, j, \ldots; t] = m_{1,i}t + m_{1,j} + f[0, 0, \ldots; g(i, j, t)] \]
\[ g(i, j, t) = m_{2,i}t + \max(A_i^{(0)}, 0) \]
\[ + m_{3,i}t + \max[A_i^{(0)} - \max(-A_i^{(0)}, 0), 0] \]
\[ + m_{4,i}t + \max[A_i^{(0)} - \max[\max(-A_i^{(0)}, 0) - A_i^{(0)}, 0], 0] \]
\[ + m_{5,i}t + \max[A_i^{(0)} - \max[\max[\max(-A_i^{(0)}, 0) - A_i^{(0)}, 0] - A_i^{(0)}, 0], 0] \]
\[ + \cdots \cdots \]
\[
geq g(j, i, t)^{\sum_{k=2}^{m}} \{m_{k,i} + \max[A_{k,1}^{(j)} - \max[\cdots \max[\max[-A_{k,0}^{(j)}, 0] - A_{k,0}^{(j)}, 0] \cdots]] - A_{k,0}^{(j)}, 0], 0]\]

where
\[
t_{k}^{(j)} = t_{k}^{(j)} = t_{k} - \max(m_{k-1,j} - t_{k-1}^{(j)}, 0)
\]
\[
t_{k}^{(j)} = t_{k} - \max(m_{k-1,j} - t_{k-1}^{(j)}, 0)
\]

and
\[
A_{k}^{(j)} = m_{k+1,j} + m_{k,i} + \max(t_{k}^{(j)} - m_{k,j}, 0).
\]

So that, in the case when \(g(i, j, t) < g(j, i, t)\), after the \(i\)th and \(j\)th item of the above both cases, if new \(t\)-term \(g\) which follows from \(g(i, j, t)\) is smaller than the corresponding \(t\)-term for the \(g(j, i, t)\) for any of the following items, than the order of operations which minimizes new \(t\)-term is optimal. That is to say, if this condition follows we choose the order of the items which yields minimum of \(g(i, j, t)\) and \(g(j, i, t)\).

Hence, we obtain the next theorem.

**Theorem.** An optimal ordering is determined by the following rule:
When the above mentioned condition follows, item \(i\) precedes item \(j\) if \(g(i, j, t) < g(j, i, t)\)

If there is equality, either ordering is optimal, provided that it is consistent with all the definite preferences.

### 3. SPECIALIZATION

In the case when \(m=2\), the condition of the theorem holds and we have the same result as both S. Johnson and R. Bellman had shown. [1], [2].

For the case when \(m \geq 3\), for example, if
\[
\min_{t} m_{k,i} \leq \max_{t} m_{k+1,i},
\]
\[
k = 1, 2, \ldots, m-2;
\]

then, as we have
\[
t_{k}^{(j)} \leq t_{k} = m_{k+1,i},
\]
\[
k = 1, 2, \ldots, m-2;
\]

the condition of the theorem holds. (see Fig. 2)

In this special case, we obtain, in (1)
\[
A_{k}^{(j)} = m_{k+1,i} - m_{k,j}, \quad k = 1, 2, \ldots, m-2;
\]
The Order of n Items Processed on m Machines

<table>
<thead>
<tr>
<th>Machines</th>
<th>Gantt chart</th>
</tr>
</thead>
<tbody>
<tr>
<td>M_1</td>
<td>m_1, t</td>
</tr>
<tr>
<td>M_2</td>
<td>m_2, t</td>
</tr>
<tr>
<td>M_3</td>
<td>m_3, t</td>
</tr>
<tr>
<td>M_4</td>
<td>m_4, t</td>
</tr>
<tr>
<td>M_m-1</td>
<td>m_m-1, t</td>
</tr>
<tr>
<td>M_m</td>
<td>m_m, t</td>
</tr>
</tbody>
</table>

![Gantt chart](image)

Fig. 2.

and then

\[
\max(-A_1^{(t)}, 0) = m_{1, t} - m_{2, t},
\]

\[
\max[\max(-A_1^{(t)}, 0) - A_2^{(t)}, 0] = \sum_{l=1}^{2} m_{l, t} - \sum_{l=2}^{3} m_{l, t},
\]

\[
\max[\max[\max(-A_1^{(t)}, 0) - A_2^{(t)}, 0] - A_3^{(t)}, 0] = \sum_{l=1}^{3} m_{l, t} - \sum_{l=2}^{4} m_{l, t},
\]

Consequently, from (1) we have

\[
g(i, j, t) = \sum_{k=2}^{m} \left[ m_{k, j} + \max \left( m_{k, t} - m_{k-1, j} \right) \right.
\]

\[- \left( \sum_{l=1}^{k-2} m_{l, j} - \sum_{l=2}^{k-1} m_{l, t} \right), 0 \right) + m_{m, j}
\]

\[+ \max \left( A_{m-1}^{(t)} - \left( \sum_{l=2}^{m-2} m_{l, j} - \sum_{l=2}^{m-1} m_{l, t} \right), 0 \right) \]  \( (3) \)

where

\[
A_{m-1}^{(t)} = m_{m, t} - m_{m-1, t} + \max(t_{m-1}^{(t)} - m_{m-1, t}, 0)
\]

\[= m_{m, t} - m_{m-1, t} - m_{m-1, t} + \max(t_{m-1}^{(t)}, m_{m-1, t})
\]
\[ t^{(i)}_{m-1} = t_{m-1} - \max(m_{m-2,i} - t^{(i)}_{m-2}, 0) \]
\[ = -m_{m-2,i} + t_{m-1} + \min(t^{(i)}_{m-2}, m_{m-2,i}) \]
\[ = -m_{m-2,i} + t_{m-1} + t^{(i)}_{m-2} \]
\[ = -m_{m-2,i} + t_{m-1} + t_{m-2} - \max(m_{m-3,i} - t^{(i)}_{m-3}, 0) \]
\[ = -m_{m-2,i} - m_{m-3,i} + t_{m-1} + t_{m-2} + \min(t^{(i)}_{m-3}, m_{m-3,i}) \]
\[ = -(m_{m-2,i} + m_{m-3,i}) + t_{m-1} + t_{m-2} + t^{(i)}_{m-3} \]
\[ = \ldots \]
\[ = -\sum_{i=1}^{m-2} m_{i,i} + \sum_{i=1}^{m-1} t_i \]
\[ = -\sum_{i=1}^{m-2} m_{i,i} + t \]

Hence, (3) becomes
\[ g(i, j, t) = \sum_{k=2}^{m-1} \left( m_{m,i} + \max \left[ \sum_{i=2}^{k} m_{i,i}, -\sum_{i=1}^{k-1} m_{i,j}, 0 \right] \right) + m_{m,i} + \max \left[ \sum_{i=2}^{m} m_{i,i} - \sum_{i=1}^{m-1} m_{i,j} - \sum_{i=1}^{m-1} m_{i,t} \right] \]
\[ + \max \left( t, \sum_{i=1}^{m-1} m_{i,i} \right), 0 \]
\[ = \sum_{k=2}^{m} m_{k,j} + \max \left[ m_{m,i} - m_{1,i} - \sum_{i=1}^{m-1} m_{i,j} \right] \]
\[ + \max \left( t, \sum_{i=1}^{m-1} m_{i,i} \right), 0 \]
\[ = -m_{1,i} - m_{1,j} + m_{m,i} + m_{m,j} \]
\[ + \max \left[ t, \sum_{i=1}^{m-1} m_{i,i}, m_{1,i} - m_{m,i} + \sum_{i=1}^{m-1} m_{i,j} \right] \]

Similarly we have
\[ g(j, i, t) = -m_{1,j} - m_{1,i} + m_{m,j} + m_{m,i} \]
\[ + \max \left[ t, \sum_{i=1}^{m-1} m_{i,j}, m_{1,j} - m_{m,j} + \sum_{i=1}^{m-1} m_{i,j} \right] \]

Hence, from \( g(i, j, t) < g(j, i, t) \) we have
\[ \max \left[ t, \sum_{i=1}^{m-1} m_{i,i}, m_{1,i} - m_{m,i} + \sum_{i=1}^{m-1} m_{i,j} \right] \]
\[ < \max \left[ t, \sum_{i=1}^{m-1} m_{i,j}, m_{1,j} - m_{m,j} + \sum_{i=1}^{m-1} m_{i,i} \right] \] (4)
So that, if
\[
\max \left[ \sum_{l=1}^{m-1} m_{l,i}, \ m_{1,i} - m_{m,i} + \sum_{l=1}^{m-1} m_{l,j} \right] < \max \left[ \sum_{l=1}^{m-1} m_{l,j}, \ m_{1,j} - m_{m,j} + \sum_{l=1}^{m-1} m_{l,i} \right]
\] (5)
then the left hand of (4) is not larger than the right hand of (4).

From (5) we have
\[
\sum_{l=1}^{m-1} m_{l,i} + \sum_{l=1}^{m-1} m_{l,j} + \max \left[ -\sum_{l=1}^{m-1} m_{l,i}, \ -\sum_{l=2}^{m} m_{l,i} \right] < \sum_{l=1}^{m-1} m_{l,i} + \sum_{l=1}^{m-1} m_{l,j} + \max \left[ -\sum_{l=1}^{m-1} m_{l,i}, \ -\sum_{l=2}^{m} m_{l,j} \right]
\]
hence we easily obtain the criterion,
\[
\min \left[ \sum_{l=1}^{m-1} m_{l,j}, \ \sum_{l=2}^{m} m_{l,i} \right] > \min \left[ \sum_{l=1}^{m-1} m_{l,i}, \ \sum_{l=2}^{m} m_{l,j} \right].
\]
Consequently we obtain the next corollary.

**Corollary.**

When
\[
\min_{i} m_{k,i} \geq \max_{i} m_{k+1,i}
\]
\[k=1, 2, \ldots, m-2.
\]
hold, an optimal ordering is determined by the following rule: Item \(i\) precedes item \(j\) if
\[
\min \left[ \sum_{l=1}^{m-1} m_{l,i}, \ \sum_{l=2}^{m} m_{l,i} \right] < \min \left[ \sum_{l=1}^{m-1} m_{l,j}, \ \sum_{l=2}^{m} m_{l,j} \right].
\]
If there is equality, either ordering is optimal.

For the case \(m=3\), this corollary coincides with the Johnson’s criterion. [2]

**REFERENCES:**