ON A SCHEDULING PROBLEM

TOJI MAKINO

Takasaki City College of Economics
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In this paper we consider the problem of deciding the order of the two items which should be processed by \( n \) machines in order to minimize the time required to complete all the operations.

Up to the present, many studies on the scheduling problem have been published. However, those studies only attempted to provide some results concerning the system with constant handling time, —the time required for processing.

So, in this paper we are going to obtain an optimal processing order in the case where the handling time is treated as random variables.

1. GENERAL CONSIDERATIONS

At first, we shall treat the system with the structure shown in the figure (1).

Two items \( A \) and \( B \) should be processed by \( n \) machines, that is, \( M_1, M_2, \ldots, M_n \).

Let \( T_{a_i} \) be the handling time of the item \( A \) on the machine \( M_i \), and let \( T_{b_i} \) be the handling time of the item \( B \). \( i = 1, 2, \ldots, n \).

Then these times are mutually independent random variables.

Suppose that the probability density function of the distribution of \( T_{a_i} \) and \( T_{b_i} \) are \( f_i(t) \) and \( g_i(t) \), respectively.
If we choose

\[ A \rightarrow B \]

for the order of process, then the total time, the time required to complete all the operations, \( T(n) \) becomes

\[ T(n) = T_{a_1} + U_n + T_{b_n}. \] (1)

Where,

\[
\begin{align*}
U_1 & \equiv 0 \\
U_{n+1} & = \{U_n + T_{b_n}\} \vee \{T_{a_2} + T_{a_3} + \cdots + T_{a_{n+1}}\}. \quad (\text{for } n \geq 1) \quad (2)
\end{align*}
\]

(Note) The symbol \( x \vee y \) denotes the maximum of \( x \) and \( y \).

In the following paragraph, we shall discuss precisely in the cases where \( n = 2 \) and 3.

### 2. TWO MACHINES

#### 2.1. General Expressions

As regards to the formulas (1) ~ (2), we get

\[ T(2) = T_{a_1} + (T_{b_1} \vee T_{a_2}) + T_{b_2}. \] (3)

We must introduce the density function \( u_2(t) \) of the distribution of

\[ U_2 = (T_{b_1} \vee T_{a_2}) \]

in order to obtain the moment generating function of the distribution of \( T(2) \).

But we can easily find the density function of \( U_2 \), as follows.

\[ u_2(t) = g_1(t) \cdot \int_0^t f_2(x) \, dx + f_3(t) \cdot \int_0^t g_1(x) \, dx \] (4)

since

\[ P_r\{U_2 < x\} = P_r\{T_{b_1} < x, T_{a_2} < x\} = P_r\{T_{b_1} < x\} \cdot P_r\{T_{a_2} < x\}. \]

Let \( M_1(\theta), M_2(\theta), M_3(\theta) \) be the moment generating functions of the distribution of random variables \( T_{a_1}, T_{b_2}, U_2 \), respectively.

Then \( M(\theta) \), the moment generating function of the distribution of \( T(2) \), can be obtained by

\[ M(\theta) = M_1(\theta) \cdot M_2(\theta) \cdot M_3(\theta) \] (5)
2. 2. In the case of the Exponential Handling Times

Let
\[ f_i(t) = \mu_i e^{-\mu_i t}, \quad g_i(t) = \nu_i e^{-\nu_i t}. \] (for \( t > 0 \) and \( i = 1, 2 \)) \hspace{1cm} (6)

Then,
\[ M_1(\theta) = \int_0^\infty e^{\theta t} \cdot \mu_1 e^{-\nu_1 t} dt = \frac{\mu_1}{\mu_1 - \theta}, \]
\[ M_2(\theta) = \int_0^\infty e^{\theta t} \cdot \nu_2 e^{-\nu_2 t} dt = \frac{\nu_2}{\nu_2 - \theta}, \]
and
\[ M_u(\theta) = \int_0^\infty e^{\theta t} \cdot u_2(t) dt = \int_0^\infty e^{\theta t} \left\{ \nu_1 e^{-\nu_2 t} \int_0^t \mu_2 e^{-\nu_2 x} dx + \mu_2 e^{-\nu_2 t} \int_0^t \nu_1 e^{-\nu_1 x} dx \right\} dt \]
\[ = \frac{\mu_2}{\mu_2 - \theta} + \frac{\nu_1}{\nu_1 - \theta} - \frac{\mu_2 + \nu_1}{\mu_2 + \nu_1 - \theta}. \]

Therefore, the moment generating function \( M(\theta) \) of the distribution of total time \( T(2) \) becomes
\[ M(\theta) = \left( \frac{\mu_1}{\mu_1 - \theta} \right) \left( \frac{\nu_2}{\nu_2 - \theta} \right) \left( \frac{\mu_2}{\mu_2 - \theta} + \frac{\nu_1}{\nu_1 - \theta} - \frac{\mu_2 + \nu_1}{\mu_2 + \nu_1 - \theta} \right). \] \hspace{1cm} (7)

Using the expression (7), we have the expectation \( E_{AB}(T) \) and variance \( V_{AB}(T) \) of total time \( T(2) \).

(Note) Here, the subscript \( AB \) is used to specify the order of jobs.

\[ E_{AB}(T) = \frac{1}{\mu_1} + \frac{1}{\mu_2} + \frac{1}{\nu_1} + \frac{1}{\nu_2} - \frac{1}{\mu_2 + \nu_1}, \] \hspace{1cm} (8)
\[ V_{AB}(T) = \frac{1}{\mu_1^2} + \frac{1}{\mu_2^2} + \frac{1}{\nu_1^2} + \frac{1}{\nu_2^2} - \frac{3}{(\mu_2 + \nu_1)^2}. \] \hspace{1cm} (9)

On the other hand, if we assume the order of jobs as, \( B \rightarrow A \),
the expectation and variance are merely obtained by replacing \( \mu_i \) and \( \nu_i \) in (8) and (9).

That is,
\[ E_{BA}(T) = \frac{1}{\mu_1} + \frac{1}{\mu_2} + \frac{1}{\nu_1} + \frac{1}{\nu_2} - \frac{1}{\mu_1 + \nu_2}. \]
On a Scheduling Problem

\[ V_{BA}(T) = \frac{1}{\mu_1^2} + \frac{1}{\mu_2^2} + \frac{1}{\nu_1^2} + \frac{1}{\nu_2^2} - \frac{3}{(\mu_1 + \nu_1)^2}. \]

Since
\[ E_{AB}(T) - E_{BA}(T) = \frac{1}{\mu_1 + \nu_2} - \frac{1}{\mu_2 + \nu_1}, \]
we can see that, if
\[ \mu_2 + \nu_1 < \mu_1 + \nu_2 \]
then
\[ E_{AB}(T) < E_{BA}(T) \]
holds.

Moreover, as regards to the variance, noting
\[ V_{AB}(T) - V_{BA}(T) = \frac{3}{(\mu_1 + \nu_2)^2} - \frac{3}{(\mu_2 + \nu_1)^2} \]
we can see that, if
\[ E_{AB}(T) < E_{BA}(T) \]
then
\[ V_{AB}(T) < V_{BA}(T). \]

2.3. Comparison to the Results of the Usual Model

We are going to compare the ordering in our model which minimizes the expectation \( E(T) \) with an optimal ordering in usual model (that is, the model with constant time to process), by some simple examples.

(Example 1)

<table>
<thead>
<tr>
<th>Item</th>
<th>Machine</th>
<th>( M_1 )</th>
<th>( M_2 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>A</td>
<td></td>
<td>5</td>
<td>10</td>
</tr>
<tr>
<td>B</td>
<td></td>
<td>20</td>
<td>40</td>
</tr>
</tbody>
</table>

(Table 1)

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An optimal ordering for usual system is determined by Johnson's criterion, which is $A\rightarrow B$.

And the total time is 65 (min.).

In our consideration, numerical value 5, 10, 20, 40 in Table (1) will be replaced by $\mu_1=1/5$, $\mu_2=1/10$, $\nu_1=1/20$, $\nu_2=1/40$, since we assumed the system with exponential probability density functions $f_i(t)$ and $g_i(t)$ for the handling time, and these distributions have the expectations $1/\mu_i$ and $1/\nu_i$, respectively.

Thus we have

$$\mu_2+\nu_1<\mu_1+\nu_2.$$  

Hence an optimal ordering which minimizes $E(T)$ is $A\rightarrow B$.

That is, we find the same result as that of usual system, in this case and we have 68 $\frac{1}{2}$ (min.) for the value of $E_{AB}(T)$.

However, in the following example, we can see that the solutions are not always similar.

(Example 2)

<table>
<thead>
<tr>
<th>Handling Time (min.)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Machine</td>
</tr>
<tr>
<td>---------------------</td>
</tr>
<tr>
<td>Item</td>
</tr>
<tr>
<td>A</td>
</tr>
<tr>
<td>B</td>
</tr>
</tbody>
</table>

(Table 2)

An optimal ordering for usual system is $A\rightarrow B$. (Total time=68 min.)

But we have

$$E_{AB}(T)>E_{BA}(T),$$
since

\[ \mu_2 + \nu_1 = \frac{11}{30}, \quad \mu_1 + \nu_2 = \frac{7}{20}. \]

Therefore we assert an optimal ordering as

\[ B \rightarrow A. \]

(The value of \( E_{BA}(T) \) become \( 71\frac{1}{7} \) min.)

Now, if we put

\[ \frac{\mu_2}{\mu_1} = \mu'_2, \quad \frac{\nu_1}{\mu_1} = \nu'_1, \quad \frac{\nu_2}{\mu_1} = \nu'_2, \]

then we see that the values of \( \mu'_2 \) in the above examples are equal to \( \frac{1}{2} \).

Whereupon we can compare the optimal ordering by Johnson's criterion with our ordering, using the figure (2).

That is, in the case where the points \((\nu'_1, \nu'_2)\) are plotted inside the regions \(\circ\) or \(\circ\), the optimal ordering by Johnson's criterion are not identical with our ordering.

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In fact, the point
\[ P\left(\nu_1'=\frac{3}{5}, \quad \nu_2'=\frac{1}{20}\right) \]
in the case of the example (2), is plotted inside the region \( \Omega \).
Therefore we can see the both ordering are not identical.

2. 4. In the case of the Erlang Handling Times

Similarly to the paragraph (2. 2), we assume that
\[
\begin{align*}
\hat{f}_i(t) &= \frac{(\mu_i k_i)^{k_i-1} \cdot e^{-\mu_i t}}{(k_i-1)!}, \quad \text{(for } i=1, 2 \text{)} \\
g_i(t) &= \frac{(\nu_1 S_i^{S_1} \cdot e^{-\nu_1 S_1} \cdot S_1^{S_1-1})}{(S_1-1)!} \end{align*}
\]

At first, we have to obtain the moment generating function \( M_u(\theta) \) of the distribution of random variable
\[ U_2=(T_{b_1} \lor T_{a_2}), \]
noting the total time \( T(2) \) is
\[ T(2)=T_{a_1}+(T_{b_1} \lor T_{a_2})+T_{b_2}. \]

Now, we get
\[
M_u(\theta) = \sum_{r=0}^{\infty} \frac{(k_2 \mu_2)^{k_2-1} \cdot e^{-\nu_2 k_2}}{(k_2-1)!} \cdot \left\{ 1-e^{-\nu_1 S_1} \cdot \sum_{r=0}^{S_1-1} \frac{(\nu_1 S_1)^r}{r!} \right\} \cdot e^{\theta t} dt
\]
\[
+ \sum_{r=0}^{\infty} \frac{(\nu_1 S_1)^{S_1} \cdot e^{-\nu_1 S_1} \cdot S_1^{S_1-1}}{(S_1-1)!} \cdot \left\{ 1-e^{-\nu_2 k_2} \cdot \sum_{r=0}^{k_2-1} \frac{(\nu_2 k_2)^r}{r!} \right\} \cdot e^{\theta t} dt
\]
\[
= \left( k_2 \mu_2 - \theta \right)^{k_2} \cdot \frac{S_1-1!}{(S_1-1)!} \cdot \frac{(S_1^r)}{r!} \cdot \frac{(S_1^r)}{r!} \cdot \left( \frac{1}{S_1^r} \right)^{k_2+r} \times \left( S_1^r \right) + \left( \frac{S_1^r}{S_1^r} - \frac{(k_2 \mu_2)^{k_2}}{(k_2-1)!} \frac{S_1-1!}{(S_1-1)!} \cdot \frac{(S_1^r)}{r!} \cdot \frac{(S_1^r)}{r!} \cdot \left( \frac{1}{S_1^r} \right)^{k_2+r} \times \left( S_1^r \right) \right)
\]
\[
\times \left( \frac{1}{S_1^r} \right)^{k_2+r} \cdot \left( (S_1+r-1)! \right) \cdot \left( (S_1+r-1)! \right). \quad (12)
\]

Therefore, the moment generating function \( M(\theta) \) of total time \( T(2) \) is obtained by...
\[ M(\theta) = M_1(\theta) \cdot M_2(\theta) \cdot M_3(\theta) . \]

Where,
\[
M_1(\theta) = \left( \frac{k_1\mu_1}{k_1\mu_1 - \theta} \right)^{k_1} ,
\]
\[
M_2(\theta) = \left( \frac{S_2\nu_2}{S_2\nu_2 - \theta} \right)^{S_2} .
\]

**Special Case**

At first, let us obtain an optimal processing order in the case where
\[
k_i = S_i = 2 \quad (i = 1, 2) .
\]

In this case, we get
\[
M_1(\theta) = \left( \frac{2\mu_1}{2\mu_1 - \theta} \right)^{2} , \quad M_2(\theta) = \left( \frac{2\nu_2}{2\nu_2 - \theta} \right)^{2} ,
\]
\[
M_3(\theta) = \frac{4\mu_2^2}{(2\mu_2 - \theta)^2} - \frac{4\nu_2^2}{(2\nu_2 + 2\nu_1 - \theta)^2} - \frac{16\mu_2^2\nu_1}{(2\mu_2 + 2\nu_1 - \theta)^3} + \frac{4\nu_1^2}{(2\nu_1 - \theta)^2} - \frac{4\nu_1^2}{(2\mu_2 + 2\nu_1 - \theta)^3} - \frac{16\mu_2\nu_1^2}{(2\mu_2 + 2\nu_1 - \theta)^3} .
\]

Thus we have
\[
M(\theta) = \left( \frac{2\mu_1}{2\mu_1 - \theta} \right)^{2} \cdot \left( \frac{2\nu_2}{2\nu_2 - \theta} \right)^{2} \cdot \left\{ \frac{4\mu_2^2}{(2\mu_2 - \theta)^2} - \frac{4\nu_2^2}{(2\nu_2 + 2\nu_1 - \theta)^2} - \frac{16\mu_2\nu_1^2}{(2\mu_2 + 2\nu_1 - \theta)^3} + \frac{4\nu_1^2}{(2\nu_1 - \theta)^2} - \frac{4\nu_1^2}{(2\mu_2 + 2\nu_1 - \theta)^3} \right\}.
\]

Hence
\[
E_{AB}(T) = \frac{1}{\mu_1} + \frac{1}{\nu_2} + \frac{1}{\nu_1} - \frac{\mu_2^2 + 3\mu_2\nu_1 + \nu_1^2}{(\mu_2 + \nu_1)^3} , \quad (13)
\]
and
\[
E_{AB}(T) - E_{BA}(T) = \frac{\mu_1^2 + 3\mu_1\nu_2 + \nu_2^2 - \mu_2^2 + 3\mu_2\nu_1 + \nu_1^2}{(\mu_1 + \nu_2)^3} . \quad (14)
\]

Using the expression (14), we can determine an optimal ordering.

Now, we can obtain the ordering in the case where \( k_i = 1, 2; s_i = 1, 2 \) (for \( i = 1, 2 \)), by similar calculations.

Those results are shown in the following table (3).
Toji Marino

<table>
<thead>
<tr>
<th>Handling Time</th>
<th>$M(\theta)$</th>
<th>$E_{AB}(T)$</th>
<th>Optimal Ordering</th>
</tr>
</thead>
</table>
| **Type [1];** | $\begin{align*} k_i &= 1, \quad S_i = 2, \\
|               | \quad \text{(for } i = 1, 2) \end{align*}$ | $T_0 - T_1$ | if $T_1 > T_1'$  |
|               | $M_{\alpha_1}(\theta) \cdot M_{\mu_2}(\theta) \cdot M_{\mu_2, \nu_1}(\theta)$ |             | $A \rightarrow B$ |
| **Type [2];** | $\begin{align*} k_1 &= S_1 = 1, \\
|               | \quad k_2 = S_2 = 2 \end{align*}$ | $T_0 - T_2$ | if $T_2 > T_2'$  |
|               | $M_{\alpha_1}(\theta) \cdot M_{\mu_2}(\theta) \cdot M_{\mu_2, \nu_1}(\theta)$ |             | $A \rightarrow B$ |
| **Type [3];** | $\begin{align*} k_1 &= 2, \quad S_i = 1, \\
|               | \quad \text{(for } i = 1, 2) \end{align*}$ | $T_0 - T_2$ | if $T_2 > T_2'$  |
|               | $M_{\alpha_1}(\theta) \cdot M_{\mu_2}(\theta) \cdot M_{\mu_2, \nu_1}(\theta)$ |             | $A \rightarrow B$ |
| **Type [4];** | $\begin{align*} k_i &= S_i = 2, \\
|               | \quad k_2 = S_2 = 1. \end{align*}$ | $T_0 - T_1$ | if $T_1 > T_1'$  |
|               | $M_{\alpha_1}(\theta) \cdot M_{\mu_2}(\theta) \cdot M_{\mu_2, \nu_1}(\theta)$ |             | $A \rightarrow B$ |

**Table (3)**

The symbols in the table (3) are defined as follows:

$$M_{\alpha_1}(\theta) = \frac{\mu_1}{\mu_1 - \theta}, \quad M_{\mu_2}(\theta) = \frac{\nu_2}{\nu_2 - \theta},$$

$$M_{\mu_2, \nu_1}(\theta) = \left( \frac{2\mu_2}{2\mu_1 - \theta} \right)^2, \quad M_{\mu_2, \nu_1}(\theta) = \left( \frac{2\nu_2}{2\nu_2 - \theta} \right)^2,$$

$$M_{\mu_2, \nu_1}(\theta) = \left\{ \frac{\mu_2}{\mu_2 - \theta} - \frac{\mu_2}{\mu_2 + 2\nu_1 - \theta} - \frac{2\mu_2 \nu_1}{(\mu_2 + 2\nu_1 - \theta)^2} \right\} + \frac{4\nu_1^2}{(2\nu_1 - \theta)^2} - \frac{4\nu_1^2}{(\mu_2 + 2\nu_1 - \theta)^2}.$$
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\[ M_{2\nu_2, \nu_1}(\theta) = \begin{cases} \frac{\nu_1}{\nu_1 - \theta} - \frac{\nu_1}{2\mu_2 + \nu_1 - \theta} - \frac{2\mu_2\nu_1}{(2\mu_2 + \nu_1 - \theta)^2} \\ + \frac{4\mu_2^2}{(2\mu_2 - \theta)^2} - \frac{4\mu_2^2}{(2\mu_2 + \nu_1 - \theta)^2} \end{cases} \]

\[ T_0 = \frac{1}{\mu_1} + \frac{1}{\mu_2} + \frac{1}{\nu_1} + \frac{1}{\nu_2}, \]

\[ T_1 = \frac{\mu_2 + 4\nu_1}{(\mu_2 + 2\nu_1)^2}, \quad T'_1 = \frac{\mu_1 + 4\nu_2}{(\mu_1 + 2\nu_2)^2}, \]

\[ T_2 = \frac{4\mu_2 + \nu_1}{(2\mu_2 + \nu_1)^2}, \quad T'_2 = \frac{4\mu_1 + \nu_2}{(2\mu_1 + \nu_2)^2}. \]

(Note) In the case where

\[ k_i = 1, \quad S_i = 1, \quad (\text{for } i = 1, 2), \]

of course, we have the same results with the exponential case.

(Paragraph 2.2)

In the case where

\[ k_i \to \infty, \quad S_i \to \infty, \quad (i = 1, 2) \]

we have the same results as an optimal ordering by Johnson's criterion.

Now, let us show the figure (3) after the model of figure (2), assuming the same values of \( \mu_1, \mu_2, \nu_1 \) and \( \nu_2 \), that is,

\[ \frac{\mu_2}{\mu_1} = \frac{\nu_1}{\nu_1} = \frac{\nu_2}{\nu_2} = \frac{1}{2}. \]

Then we can said if the point \( (\nu_1', \nu_2') \) are plotted upside the curve, then an optimal ordering is \( A \to B \), using the figure (3).
3. THREE MACHINES

3.1. General Expressions

As regards to the formulas (1) (2), we have

\[ T(3) = T_{a1} + \{(T_{b1} \lor T_{a2}) + T_{b2}\} \lor \{ T_{b2} + T_{a3}\} + T_{b3}. \]

At first, let us obtain the moment generating function \( M_u(\theta) \) of

\[ U_3 = \{(T_{b1} \lor T_{a2}) + T_{b2}\} \lor \{ T_{a2} + T_{a3}\}. \]

Now,

\[ P_r\{U_3 < x\} = P_r\{T_{b1} + T_{b2} < x, T_{a2} + T_{b2} < x, T_{a2} + T_{a3} < x\}. \]

Here we consider two cases of partitioned

Case [1] \( \cdots T_{b1} + T_{b2} > T_{a2} + T_{a3} \),
Case [2] \( \cdots T_{b1} + T_{b2} < T_{a2} + T_{a3} \).
For Case [1], we have

\[ P_r\{U_3 < x\} = P_r\{(T_{b_1} \vee T_{a_2}) + T_{b_2} < x\}. \]

On the other hand, for Case [2], we have

\[ P_r\{U_3 < x\} = P_r\{(T_{b_2} \vee T_{a_2}) + T_{a_2} < x\}. \]

Let

- the probability that Case [1] occurs be \( P_{[1]} \),
- the probability that Case [2] occurs be \( P_{[2]} \),
- the moment generating function of \( (T_{b_1} \vee T_{a_2}) + T_{b_2} \) be \( M_{[1]}(\theta) \),
- the moment generating function of \( (T_{b_2} \vee T_{a_2}) + T_{a_2} \) be \( M_{[2]}(\theta) \).

Then we get

\[ M_n(\theta) = P_{[1]} \cdot M_{[1]}(\theta) + P_{[2]} \cdot M_{[2]}(\theta). \]

Using above expressions, we can find the moment generating function \( M(\theta) \) of total time.

### 3.2. In the case of the Exponential Handling Time

The probability density functions \( g_{12}(t) \) and \( f_{23}(t) \) of the distributions of random variables \((T_{b_1} + T_{b_2})\) and \((T_{a_2} + T_{a_3})\) are

\[ g_{12}(t) = \frac{\nu_1^2}{\nu_2 - \nu_1} \cdot (e^{-\nu_1 t} - e^{-\nu_2 t}) \]

\[ f_{23}(t) = \frac{\mu_3^2}{\mu_3 - \mu_2} \cdot (e^{-\mu_2 t} - e^{-\mu_3 t}), \]

respectively.

Accordingly, the probability \( P_{[1]} \) of \( T_{b_1} + T_{b_2} > T_{a_2} + T_{a_3} \) becomes

\[
P_{[1]} = \iint_{t_1 + t_2} g_{12}(t_1) \cdot f_{23}(t_2) dt_1 dt_2 \]

\[
= \mu_2 \mu_3 \cdot \frac{\nu_1^2 \cdot \nu_1 \cdot \nu_2^2 + \mu_2 \mu_3 + (\nu_1 + \nu_2)(\mu_2 + \mu_3)}{(\mu_2 + \nu_1)(\mu_3 + \nu_1)(\mu_2 + \nu_2)(\mu_3 + \nu_2)}.
\]

And the probability \( P_{[2]} \) of \( T_{b_1} + T_{b_2} < T_{a_2} + T_{a_3} \) becomes

\[
P_{[2]} = \iint_{t_1 + t_2} g_{12}(t_1) \cdot f_{23}(t_2) dt_1 dt_2 \]

\[
= \nu_1^2 \cdot \frac{\mu_2^2 + \mu_3^2 + \mu_2 + \nu_1 \cdot \nu_2^2 + (\nu_1 + \nu_2)(\mu_2 + \mu_3)}{(\mu_2 + \nu_1)(\mu_3 + \nu_1)(\mu_2 + \nu_2)(\mu_3 + \nu_2)}.
\]
The moment generating functions \( M_{[1]}(\theta) \) and \( M_{[2]}(\theta) \) of the distributions of \( (T_{b_1} \vee T_{a_2}) + T_{b_2} \) and \( (T_{b_2} \vee T_{a_2}) + T_{b_2} \) are

\[
M_{[1]}(\theta) = \left( \frac{\mu_2}{\mu_2 - \theta} + \frac{\nu_1}{\nu_1 - \theta} - \frac{\mu_2 + \nu_1}{\mu_2 + \nu_1 - \theta} \right) \cdot \left( \frac{\nu_2}{\nu_2 - \theta} \right),
\]

\[
M_{[2]}(\theta) = \left( \frac{\mu_3}{\mu_3 - \theta} + \frac{\nu_2}{\nu_2 - \theta} - \frac{\mu_3 + \nu_2}{\mu_3 + \nu_2 - \theta} \right) \cdot \left( \frac{\mu_2}{\mu_2 - \theta} \right).
\]

Therefore we can obtain the moment generating function \( M(\theta) \) of the total time \( T(3) \), using above expressions

\[
M(\nu) = \left( \frac{\mu_1}{\mu_1 - \nu} \right) \cdot \left( \frac{\nu_3}{\nu_3 - \nu} \right) \cdot \left\{ P_{[1]} \cdot M_{[1]}(\nu) + P_{[2]} \cdot M_{[2]}(\nu) \right\}. \quad (15)
\]

Thus we get the expectation \( E_{AB}(T) \) of the total time.

That is,

\[
E_{AB}(T) = \frac{1}{\mu_1} + \frac{1}{\nu_3} + \left\{ P_{[1]} \left( \frac{1}{\mu_2} + \frac{1}{\nu_1} - \frac{1}{\mu_2 + \nu_1} + \frac{1}{\nu_2} \right) \right. \\
\left. + P_{[2]} \left( \frac{1}{\mu_3} + \frac{1}{\nu_2} - \frac{1}{\mu_3 + \nu_2} + \frac{1}{\mu_2} \right) \right\}.
\]

\[
= \frac{1}{\mu_1} + \frac{1}{\mu_2} + \frac{1}{\nu_3} + \frac{\mu_2}{\nu_3 (\mu_2 + \nu_1)} \cdot P_{[1]} + \frac{\nu_2}{\mu_3 (\mu_3 + \nu_2)} \cdot P_{[2]}.
\quad (16)
\]

On the other hand, the expectation \( E_{BA}(T) \) is obtained simply by replacing the symbols \( \mu_i \) and \( \nu_i \).

Using these values, we can determine an optimal ordering.

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