SOME RESULTS ON ROAD TRAFFIC DISTRIBUTION

TAKEJI SUZUKI

Defense Academy, Yokosuka
(Received June 11, 1966)

§ 1. Introduction

In a paper [1] by Leo Breiman it is shown that, under weak assumptions, the number of cars in an arbitrary interval on the space will be asymptotically Poisson distributed as the time tends to infinity. After that, under the same assumptions as those of Breiman, the above result is generalized by T. Théden [2] as follows: the cars on the space as the time tends to infinity will be distributed according to a Poisson process.

In this paper it will be shown that under the similar assumptions as above the number of cars on the time-axis will be distributed according to a Poisson process as the time tends to infinity (Theorem 1).

Furthermore, it is shown that that the number of cars in the time-interval (0, t] is asymptotically linear in t as $t \to \infty$ (Theorem 2). Also the relation between the spatial and time density of the velocity is shown in the theorem 3.

§ 2. Assumptions and Notations

At $t=0$, let there be a set of starting points $X_1, X_2, \ldots$ on the negative spatial axis which are obtained as observations of a stochastic point process. Concerning this process $\{X_n, n=1, 2, 3, \ldots\}$ we assume the following three conditions.

(a) A spatial density $\sigma$ exists with probability one,
Some Results on Road Traffic Distribution

Fig. 1.

i.e.,

\[ \lim_{x \to \infty} \{ \text{no. of } X_k \text{ in } (0, -x)]/x = \sigma, \]

with probability one.

(b) For any finite interval \( I \) on the spatial axis, the expected number of \( X_k \) in \( I \) can be bounded above by \( M \), where \( M \) depends only on the length of \( I \).

Associated with \( X_k \) is \( V_k \), its velocity. The \( V_k \) are assumed independent each other and of the \( X_1, X_2, \ldots \), with common distribution

\[ G(v) = P(V_k < v), \quad G(v) = 0 \quad (v \leq 0). \]

Further,

\[ G(v) = \int_0^v g(u) du, \]

where \( g(u) \) is almost everywhere continuous (with respect to Lebesgue measure), bounded on every finite interval and \( g(v) = 0 \) \( (v > v_0) \) for some positive number \( v_0 \). Then the expectation of \( V_k \) exists and finite, say \( E(V) \).

Denote by \( X_k(t) \) the position of the \( k \)th car after time \( t \), then we have \( X_k(t) = X_k + tV_k \). Let \( T_k \) be the time at that \( X_k(t) = 0 \) and \( N(t_1, t_2) \) be the number of \( T_k \) in the time interval \( (t_1, t_2) \), where \( t_1 < t_2 \) and define
$N_k(t) =$ no. of $k$ such that $v \leq V_k < v + \Delta v$ and $T_k \in (0, t]$.

### 3. Theorems

**Theorem 1.** Under (a), (b), (c) above for fixed $T$, $j$,

\[
\lim_{t \to \infty} P(N(t, t + T) = j) = \frac{\lambda^j}{j!} e^{-\lambda}
\]

where $\lambda = \sigma T E(V)$. Further, let $(t_1, t_1 + T_1], (t_2, t_2 + T_2], \ldots, (t_n, t_n + T_n]$ be $n$ disjoint but otherwise arbitrary time-intervals, then for fixed $T_1, T_2, \ldots, T_n; j_1, j_2, \ldots, j_n$,

\[
\lim_{t_1, \ldots, t_n \to \infty} P(N(t, t + T) = j, \nu = 1, 2, \ldots, n) = \prod_{\nu=1}^{n} \left( \frac{\lambda_\nu^{j_\nu}}{j_\nu!} \right) e^{-\lambda_\nu}
\]

where $\lambda_\nu = \sigma T E(V)$.

**Proof.** The proof of the theorem can be done in exactly the same way as in [1] and [2].

We will note that, for fixed $X_1, X_2, \ldots$, the $T_k$ are independent. For sake of completeness of the proof it will be sufficient only to show that

\[
\lim_{t \to \infty} \sum_{k=1}^{\infty} P(t \leq T_k \leq t + T | X_1, X_2, \ldots) = \lambda
\]

and

\[
\lim_{t \to \infty} \sup_k P(t \leq T_k \leq t + T | X_1, X_2, \ldots) = 0
\]

for fixed $T$.

**Proof of (3).**

\[
S_t = \sum_{k=1}^{\infty} P(t \leq T_k \leq t + T | X_1, X_2, \ldots) = \sum_k P\left(-\frac{X_k}{t+T} \leq V_k \leq -\frac{X_k}{t} \mid X_1, X_2, \ldots\right) = \sum_k \left[ G\left(-\frac{X_k}{t}\right) - G\left(-\frac{X_k}{t+T}\right) \right].
\]
Now let $M_t(y) = \text{no. of } X_k \text{ in } (0, -ty] \ (y \geq 0)$ and place $m_t(y) = M_t(y)/t$. Then using the assumption (a),
\[ m_t(y) \to a y \ (t \to \infty) \]
with probability one (and the same exceptional set for all $y$). Then $S_t$ may be rewritten as follows.

\[
S_t = \int_0^\infty \left[ G(y) - G\left( \frac{t}{t+T}, y \right) \right] dM_t(y)
\]
\[
= \int_0^\infty \frac{G(y) - G\left( \frac{t}{t+T}, y \right)}{y/t} \cdot y dm_t(y)
\]
\[
= \int_0^\infty \left( \frac{1}{y/t} \int_{t/T}^y g(u) du \right) y dm_t(y)
\]
\[
= \int_0^\infty \left( t \int_{t/T}^1 g(yz) dz \right) y dm_t(y).
\]

If $g(y)$ is continuous, then as $t \to \infty$,
\[
t \int_{t/T}^1 g(yz) dz \to \begin{cases} g(y)T & \text{unif. in } y \\ 0 & (y \geq y_0 + \varepsilon), \end{cases}
\]
where $\varepsilon$ is any small positive number.

Because, for any $\varepsilon > 0$ there exists a number $\delta$ independent of $y$ such that $|g(yz) - g(y)| < \varepsilon$ for $|z-1| < \delta$. For $t$ sufficiently large,
\[
\left| t \int_{t/T}^1 g(yz) dz - g(y)T \right|
\]
\[
= \left| t \int_{t/T}^1 g(yz) - (t+T) \int_{t/T}^1 g(y)dz \right|
\]
\[
\leq t \int_{t/T}^1 |g(yz)-|dz + \frac{T^2}{t+T} \cdot g(y)
\]
\[
\leq \frac{tT}{t+T} \varepsilon + \frac{T^2}{t+T} \max_{y} g(y) < \varepsilon'.
\]
Therefore,

\[ \lim_{t \to \infty} S_t = \sigma T f_0 g(y) dy = \sigma TE(V) = \lambda \text{ (say)}. \]

If \( g(v) \) is not continuous, then using the assumption (c) and the same method as in [1] we have the result (5).

**Proof of (4).** Using (c),

\[
P(t \leq t_k \leq t + T | X_1, X_2, \cdots) = P\left( t \leq -\frac{X_k}{V_k} \leq t + T | X_1, X_2, \cdots \right)
\]

\[
= P\left( -\frac{X_k}{t + T} \leq -\frac{X_k}{t} \right) \left| X_1, X_2, \cdots \right)
\]

\[
= \int_{-\frac{X_k}{t} + T}^{\frac{X_k}{t}} g(u) du \leq \frac{T}{t + T} v_0 \max g(u).
\]

Therefore, for fixed \( T \),

\[
\lim_{t \to \infty} \sup_k P(t \leq T_k \leq t + T | X_1, X_2, \cdots) \leq \lim_{t \to \infty} \frac{T}{t + T} v_0 \max g(v) = 0.
\]

In the following we denote \( N(0, t) \) by \( N(t) \).

**Theorem 2.** Under (a), (b) and (c),

\[ \lim_{t \to \infty} \frac{N(t)}{t} = \sigma E(V) \]

in probability.

**Proof.** First we consider the case where \( X_1, X_2, \cdots \) are fixed. Define the following random variables,

\[ Y_k(t) = \varphi\left( V_k + \frac{X_k}{t} \right), \]

where \( \varphi(x) = 1 \ (x \geq 0) \), \( = 0 \ (x < 0) \).

Then
Some Results on Road Traffic Distribution

(7) \[ N(t) = \sum_{k=1}^{\infty} Y_k(t) \]

and for fixed \( t \) the \( Y_k(t) \) are independent random variables, however, they have not a common distribution. In fact,

\[
P(Y_k(t) = 0 | X_1, X_2, \ldots) = P \left( V_k < -\frac{X_k}{t} \mid X_1, X_2, \ldots \right) = G \left( -\frac{X_k}{t} \right) = q_k \to 1 \quad (k \to \infty)
\]

and put \( p_k = 1 - q_k \).

According to the assumption (c), the summation on the right-hand side of (7) is finite.

Now if we define

\[ k(t) = \min \left\{ k \mid -\frac{X_k}{t} \leq v_0, \quad -\frac{X_{k+1}}{t} > v_0 \right\}, \]

then

\[
N(t) = \sum_{k=1}^{k(t)} Y_k(t) = \frac{k(t)}{t} \cdot \frac{\sum_{k=1}^{k(t)} Y_k(t)}{k(t)}.
\]

(8)

For the application of the strong law of large numbers we need to examine some properties.

\[ k(t) = \text{no. of cars in the interval } (0, -tv_0] \text{ on the spatial axis.} \]

\[ = \text{no. of } \frac{X_k}{t} \text{ in } (0, -v_0] \]

\[ = \text{no. of } -\frac{X_k}{t} \text{ in } (0, v_0] \]

\[ = \int_0^{v_0} dM(t) = \int_0^{v_0} tdm(t). \]

Therefore,
Furthermore,

\[
E(Y_k(t)|X_1, X_2, \cdots) \leq 1 \quad \text{and} \quad Var(Y_k(t)|X_1, X_2, \cdots) = \rho \cdot q < 1,
\]

then

\[
\sum_{k=1}^{\infty} \frac{Var(Y_k(t)|X_1, X_2, \cdots)}{k^2} \leq \sum_{k=1}^{\infty} \frac{1}{k^2} < \infty.
\]

And

\[
E(N(t)|X_1, X_2, \cdots) = E(\sum_{k=1}^{k(t)} Y_k(t)|X_1, X_2, \cdots)
\]

\[
= \sum_{k=1}^{k(t)} P(Y_k(t) = 1|X_1, X_2, \cdots)
\]

\[
= \sum_{k=1}^{k(t)} \left[ 1 - G\left(\frac{X_k}{t}\right) \right]
\]

\[
= \int_{0}^{\infty} [1 - G(y)]dM_t(y) + t \int_{0}^{\infty} [1 - G(y)]dM_t(y),
\]

then from (9),

\[
\lim_{t \to \infty} E\left( \frac{N(t)}{k(t)} \bigg| X_1, X_2, \cdots \right) = \lim_{t \to \infty} \frac{t}{k(t)} \int_{0}^{\infty} [1 - G(y)]dM_t(y)
\]

\[
= \frac{1}{\sigma \nu_0} \int_{0}^{\infty} [1 - G(y)]dy = \frac{1}{\nu_0} E(V).
\]

By (8), (10), (11) and the strong law of large numbers, for fixed \(X_1, X_2, \cdots\),

\[
\lim_{t \to \infty} \frac{N(t)}{t} = \sigma E(V)
\]

with probability one. According to the assumption (a) \(P(N(t)/t \leq x) \to 0 (x < \sigma E(V)) \) or 1 (elsewhere), then \(N(t)/t \to \sigma E(V)\) in probability, i.e., the theorem is true.

**Corollary.** Under (a), (b) and (c),
Some Results on Road Traffic Distribution

\[ \lim_{t \to \infty} E\left( \frac{N(t)}{t} \right) = \sigma E(V). \]

**Proof.** With the proof of the above and the Lebesgue bounded convergence theorem,

\[
\lim_{t \to \infty} E\left( \frac{N(t)}{t} \right) = \lim_{t \to \infty} E\left( \frac{N(t)}{t} \left| X_1, X_2, \ldots \right. \right) \\
= \lim_{t \to \infty} E\left( \int_0^{t_0} [1 - G(y)] \, dm_t(y) \right) \\
= E\left( \int_0^{t_0} [1 - G(y)] \, dy \right) = \sigma E(V).
\]

**Theorem 3.** Under (a), (b) and (c),

\[ \lim_{t \to \infty} \frac{N_c(t)}{N(t)} = \frac{1}{E(V)} \int_{v}^{v+\Delta v} u \varphi(u) \, du \]

in probability.

**Proof.** First we consider the case where \( X_1, X_2, \ldots \) are fixed.

Define the following random variables,

\[ Z_k(t) = \varphi\left( V_k + \frac{X_k}{t} \right) \varphi(v - \varphi(V_k - v)) \varphi(v + \Delta v - V_k). \]

Then

\[ N_c(t) = \sum_{k=1}^{k(t)} Z_c(t), \]

where \( k(t) \) is defined in the proof of theorem 2.

For the application of the strong law of large numbers to the following expression,

\[ \frac{N_c(t)}{t} = \frac{k(t)}{t} \sum_{k=1}^{k(t)} Z_c(t), \]

we will examine some properties as follows.

Clearly,
\( (15) \quad \sum_{k=1}^{\infty} \frac{V_{ax}(Z_k(t)|X_1^t, X_2^t, \cdots)}{k^2} \leq \sum_{k=1}^{\infty} \frac{1}{k^2} < \infty. \)

since \( 0 \leq Z_k(t) \leq 1 \) for all \( k \).

And also

\[
E(N_v(t)|X_1, X_2, \cdots) = \sum_k P(Z_k(t) = 1|X_1, X_2, \cdots)
\]

\[
= \int_v^{v+dv} g(u)du \int_0^u dM_i(y)
\]

\[
= t \int_v^{v+dv} g(u)du \int_0^u dm_i(y).
\]

Therefore, using (9),

\[
\lim_{t \to \infty} E \left( \frac{N_v(t)}{N(t)} \right| X_1, X_2, \cdots)
\]

\[
= \lim_{t \to \infty} \frac{t}{k(t)} \cdot \lim_{t \to \infty} \int_v^{v+dv} g(u)du \int_0^u dm_i(y)
\]

\[
= \frac{1}{v_0} \int_v^{v+dv} ug(u)du.
\]

From (14), (15), (16) and the strong law of large numbers,

\[
\lim_{t \to \infty} \frac{N_v(t)}{t} = \sigma \int_v^{v+dv} ug(u)du.
\]

(17)

The above result (17) is true for each fixed values of \( X_1, X_2, \cdots \) again with the possible exception of a set of probability zero, i.e., it is true with probability one.

With the result of the preceding theorem and (17), we have

\[
\lim_{t \to \infty} \frac{N_v(t)}{N(t)} = \frac{1}{E(V)} \int_v^{v+dv} ug(u)du
\]

in probability.

Copyright © by ORSJ. Unauthorized reproduction of this article is prohibited.
REFERENCES
