NOTE ON THE CRITICAL PATH ANALYSIS FOR A PROJECT WITH A DIVISIBLE ACTIVITY

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In a paper published in JORSA Vol. 13 (1965), W. S. Jewell [2] presented an algorithm for minimizing the total duration of a project with a divisible activity. The mathematical formulation of the problem is as follows:

Minimize \( t \),
subject to
\[
\begin{align*}
    v_j - v_i & \geq T_{ij}, & \text{for } (i, j) & \in \overline{D}, \\
    v_j - v_i - t_{ij} & \geq 0, & \text{for } (i, j) & \in D, \\
    t_{ij} & \geq 0, \\
    \sum_{(i, j) \in D} t_{ij} & \geq U, \\
    v_N - v_1 - t & = 0,
\end{align*}
\]

where \( \overline{D}, D, U, T \)'s are given.

He considered the following parametric programming problem \( P|Q \) and its dual one \( D|Q \).

\( P|Q: \) Minimize \( (Qt - \sum_{D} t_{ij}) \),
subject to
\[
\begin{align*}
    v_j - v_i & \geq T_{ij}, & \text{for } (i, j) & \in \overline{D}, \\
    v_j - v_i - t_{ij} & \geq 0, & \text{for } (i, j) & \in D, \\
    t_{ij} & \geq 0, \\
    v_N - v_1 - t & = 0.
\end{align*}
\]
His algorithm consists of two parts, that is, the starting procedure for finding an initial solution of $D|Q$ for a sufficiently large positive $Q$, and the minimal-flow subroutine for decreasing $Q$ so as to allocate more time to the divisible activity. The latter is one of the primal-dual algorithms. The procedure is terminated when $\sum_{D} t_{ij}$ reaches $U$, since it is proved in [2] that if an optimal solution of $D|Q$ for some $Q$, $(v_{i}, t_{ij}, t)$, satisfies that $\sum_{D} t_{ij}=U$, then it is also optimal to the original problem.

Here, the author suggests that the usual CPM (critical path method) is directly applicable to the problem without introducing a modified algorithm. We consider the following parametric programming problem $D^{*}|T$ and its dual one $P^{*}|T$.

\[D^{*}|T: \text{Maximize } \sum_{D} t_{ij},\]

subject to

\[v_{j} - v_{i} \geq T_{ij}, \quad \text{for } (i,j) \in \overline{D},\]

\[v_{j} - v_{i} - t_{ij} \geq 0, \quad M \geq t_{ij} \geq 0, \quad M \geq t_{ij} \geq 0, \quad v_{N} - v_{1} = T,\]

where $M$ is a sufficiently large positive number.

\[P^{*}|T: \text{Minimize } [T q + M \sum_{D} y_{ij} - \sum_{D} T_{ij} x_{ij}],\]
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subject to \[ x_{ij} \geq 0, \quad \text{for } (i, j) \in DU \overline{D}, \]
\[ y_{ij} \geq 0, \quad \text{for } (i, j) \in D, \]
\[ \sum_{(i, j) \in DU \overline{D}} x_{ij} - q = 0, \]
\[ \sum_{(i, j) \in DU \overline{D}} x_{ij} - \sum_{j \in DU \overline{D}} x_{ji} = 0 \quad (i = 2, 3, \ldots, N-1), \]
\[ \sum_{(i, j) \in DU \overline{D}} x_{jN-q} = 0, \]
\[ x_{ij} + y_{ij} \geq 1, \quad \text{for } (i, j) \in D. \]

If \( T < M \), the condition that \( M \geq t_{ij} \geq 0 \) for \((i, j) \in D\), may be replaced by the condition that \( t_{ij} \geq 0 \) for \((i, j) \in D\), in \( D^*|T \) and \( y \)'s may be neglected in \( P^*|T \). Hence, from Proposition 2.1. of Kurata [4], it is proved that if \((v_i, t_{ij})\) resp. \((x_{ij}, y_{ij}, q)\) is the optimal solution of \( D^*|T \) resp. \( P^*|T \) for \( T < M \), \((v_i, t_{ij}, t = T)\) resp. \((x_{ij})\) is the optimal solution of \( P|Q = q \) resp. \( D|Q = q \). Furthermore, \( D^*|T \) is equivalent to one of the usual CPM problems:

Maximize \( \sum_P c_{ij} t_{ij}, \)

subject to \[ v_j - v_i - t_{ij} \geq 0, \quad \text{for } (i, j) \in P, \]
\[ D_{ij} \geq t_{ij} \geq d_{ij}, \quad \text{for } (i, j) \in P, \]
\[ v_N - v_1 = T, \]

when \( P = DU \overline{D}, \)
\[ D_{ij} = \begin{cases} M, & \text{for } (i, j) \in D, \\ T_{ij}, & \text{for } (i, j) \in \overline{D}, \end{cases} \]
\[ d_{ij} = \begin{cases} 0, & \text{for } (i, j) \in D, \\ T_{ij}, & \text{for } (i, j) \in \overline{D}, \end{cases} \]
\[ c_{ij} = \begin{cases} 1, & \text{for } (i, j) \in D, \\ 0, & \text{for } (i, j) \in \overline{D}. \end{cases} \]
So, we can obtain an optimal solution of the original problem by applying the usual CPM to $D^*|T$. In this case we can easily find a solution of $P^*|T=\infty$, as an initial solution, which is as follows:

$$x_{ij}=0 \quad \text{for} \quad (i,j) \in DU\bar{D},$$

$$y_{ij}=1 \quad \text{for} \quad (i,j) \in D,$$

$$q=0.$$

And by decreasing $T$, less time will be allocated to the divisible activity in the reverse order of Jewell’s, and the procedure is terminated when $\sum_{D} t_{ij}$ reaches $U$.

**References**


