AN ALGORITHM FOR FINDING ALL EXTREMAL RAYS
OF POLYHEDRAL CONVEX CONES
WITH SOME COMPLEMENTARITY CONDITIONS

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Abstract

In this paper, we show a method for finding all extremal rays of polyhedral convex cones with some complementarity conditions. The polyhedral convex cone is defined as the intersection of half-spaces expressed by linear inequalities. By a complementarity extremal ray, we mean an extremal ray vector that satisfies some complementarity conditions among its elements. Our method is iterative in the sense that, knowing all sub-complementary extremal rays of the intersection of several half-spaces, we add repeatedly a new half-space to the half-spaces on the foregoing stage and determine all sub-complementary extremal rays of the new polyhedral convex cone thus formed, until all half-spaces are taken into consideration. Since, in the process of computation, we deal only with sub-complementary extremal rays, we could avoid the exceeding growth of the number of extremal rays. And it is of interest to note that the more complementarities there are, the less amount of computations we need. In the latter part, we apply this method to the general linear complementarity problem, to the non-convex quadratic programming and to a mathematical programming with control variables.

1. Problem

We define a polyhedral convex cone $P_m$ by a set of nonnegative vector $x \in \mathbb{R}^n$ in the intersection of $m$ half-spaces. That is,
where \( x, a_1, \ldots, a_m \) are vectors in \( \mathbb{R}^n \) and the symbol ' denotes the transposition of matrices or vectors. We introduce the slack variables \( \lambda = (\lambda_1, \ldots, \lambda_m)' \) to transfer the inequalities in (1.1) into equalities. Thus, we have

\[
(1.2) \quad P_m = \{x \mid x \geq 0, a_1'x_1 = 0, \ldots, a_m'x_m = 0\}.
\]

Now, we put some complementarity conditions on the elements of \( x \) and \( \lambda \). For example, \( x_1 \lambda_1 = 0, x_2 \lambda_2 = 0, \lambda_1 \lambda_2 = 0, x_3 \lambda_3 = 0 \) etc. We call them the complementarity conditions.

The problem is to find out all extremal rays of \( P_m \) that satisfy these conditions. We call an extremal ray in such conditions a complementary extremal ray and a vertex of a polyhedral set in such conditions a complementary vertex.

Our method is iterative in the sense that knowing all complementary extremal rays of the polyhedral convex cone

\[
(1.3) \quad P_{k-1} = \{x \mid x \geq 0, a_1'x_1 = 0, \ldots, a_{k-1}'x_{k-1} = 0\},
\]

we add a constraint \( a_k'x_k = 0 \) to it to determine all complementary extremal rays of the polyhedral convex cone

\[
(1.4) \quad P_k = \{x \mid x \in P_{k-1}, a_k'x_k = 0\}.
\]

Here, when we mention of the complementary extremal rays of \( P_k \), we only consider the complementarity conditions among \( x_1, \ldots, x_n, \lambda_1, \ldots, \lambda_k \).

We take no account of the complementarity conditions related to \( \lambda_{k+1}, \ldots, \lambda_m \).

The latter conditions are taken into consideration step by step as we proceed our algorithm and when \( k \) attains \( m \), the complementarity conditions among all variables \( x_1, \ldots, x_n, \lambda_1, \ldots, \lambda_m \) will be taken into consideration. As, at step \( k \) of the algorithm, we only consider a subset of the given complementarity conditions, we call it the sub-complementarity conditions. Similarly, we mean by a sub-complementary extremal ray or a sub-complementary vertex that satisfies

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these sub-complementarity conditions among its elements and corresponding slack variables. In regard to $P_k$, let

$$C_k = \{ x \in P_k, \ l'x = 1 \},$$

where $l' = (1, \ldots, 1)$. $C_k$ is a convex polyhedron. As is well known, there is a one to one correspondence between the vertices of $C_k$ and the extremal rays of $P_k$.

Indeed, the correspondence between $x \in P_k$ ($x \neq 0$) and $y = x/(l'x) \in C_k$ is one to one and the vertices of $C_k$ and the extremal rays of $P_k$ correspond with each other.

Also, by this correspondence, the sub-complementary extremal rays of $P_k$ correspond to the sub-complementary vertices of $C_k$ and vice versa. So, we hereafter deal with the set of sub-complementary vertices of $C_k$ which we denote $V_k$.

For our problem, if we could find all vertices of $C_m$, we could choose among them the complementary extremal rays. M.L. Balinski [1] showed an algorithm for finding all vertices of convex polyhedral sets by means of the simplex method. But we wish to find only vertices in the complementarity conditions. As far as such a purpose is aimed, Balinski's method will not be said to be very effective. On the other hand, Motzkin, Raiffa, Thompson and Thrall [3] presented "the Double Description Method" for linear programming problems, in which they tried to find out all extremal rays of a polyhedral convex cone. But, this method will not be so effective as the simplex method for linear programs, because the number of extremal rays grows exceedingly as the number of variables and constraints increases.

Our method is basically along Motzkin's method to which we add care of degeneracy and complementarity. And strong complementarity conditions will avoid the exceeding growth of the number of extremal rays.

2. Algorithm

Step 1. Initialization.
For

\[ C_0 = \{ x | x > 0, \lambda'x = 1 \} \]

the sub-complementary vertex set is

\[ V_0 = \{ e_i | e_i: \text{the } i-\text{th unit vector}, i=1,\ldots,n \} \]

Repeat the following steps for \( k=1,\ldots,m \).

**Step 2. Adding a constraint.**

Suppose all sub-complementary vertices of \( C_{k-1} \) are known. Let it be

\[ V_{k-1} = \{ v_i \} \]

and let

\[ \lambda_{ik} = \alpha_k' v_i \]  

**Step 2.1.** If \( \lambda_{ik} > 0 \) for all \( v_i \in V_{k-1} \), then the adding constraint \( \alpha_k' x > 0 \) is not binding. That is

\[ C_k = C_{k-1} \]

Let

\[ V_k = V_{k-1} \]

( Go to step 2.5.)

**Step 2.2.** If \( \lambda_{ik} < 0 \) for all \( v_i \in V_{k-1} \), then \( C_k \) is null.

( The end.)

**Step 2.3.** If \( \lambda_{ik} \leq 0 \) for all \( i \), then let

\[ V_k = \{ v_i | v_i \in V_{k-1}, \lambda_{ik} = 0 \} \]

( Go to the beginning of step 2. Increase \( k \) by one.)

**Step 2.4.** If, for some \( i \) and some \( j \), \( \lambda_{ik} > 0 \) and \( \lambda_{jk} < 0 \), then try the following [Common Zero Test] for \( v_i \) and \( v_j \). If they pass the test, then compose a vector \( w_{ij} \) by

\[ w_{ij} = -\{ \lambda_{jk} / (\lambda_{ik} - \lambda_{jk}) \} v_i + \{ \lambda_{ik} / (\lambda_{ik} - \lambda_{jk}) \} v_j \]  

(2.5)
The \( w_{ij} \) is on the line segment joining \( v_i \) and \( v_j \) and on the hyperplane \( H_k : a_k'x = 0 \). Try this process for all pairs of \( v_i \) (with \( \lambda_{ik} > 0 \)) and \( v_j \) (with \( \lambda_{jk} < 0 \)).

Then let

\[
\bar{V}_k = \{ v_i \mid v_j \in V_{k-1}, a_k'v_i > 0; w_{ij} \text{ by } (2.5) \}.
\]

( Go to step 2.5.)

**Step 2.5.** Try the following [Sub-Complementarity Test] to the elements of \( \bar{V}_k \) to remove all non sub-complementary vertices of \( C_k \) and let the remaining set be \( \bar{V}_k' \).

( Go to step 2.6.)

**Step 2.6.** Try the following [Degeneracy Test] to \( \bar{V}_k \) to remove all non-vertex points of \( C_k \) from \( \bar{V}_k \) and let the remaining set be \( V_k \).

( Go to the beginning of step 2. Increase \( k \) by one.)

[Common Zero Test]

For \( v_i \) and \( v_j \), let

\[
(2.6) \quad \lambda_{is} = a_s'v_i \quad (s=1, \ldots, k-1),
\]

\[
(2.7) \quad \lambda_{js} = a_s'v_j \quad (s=1, \ldots, k-1)
\]

and let the extended vectors \( v_i^o \) and \( v_j^o \) of \( v_i \) and \( v_j \) be

\[
(2.8) \quad v_i^o = (v_{i1}, \ldots, v_{in}, \lambda_{i1}, \ldots, \lambda_{ik-1}, \lambda_{ik})',
\]

\[
(2.9) \quad v_j^o = (v_{j1}, \ldots, v_{jn}, \lambda_{j1}, \ldots, \lambda_{jk-1}, \lambda_{jk})'
\]

respectively. They are \((n+k)\)-vectors. If \( v_i^o \) and \( v_j^o \) have no less than \((n-2)\) common zeros in their corresponding elements, then they pass the test.

Otherwise, they fail.

[Sub-Complementarity Test]

For each \( v_i \) of \( \bar{V}_k \), check the sub-complementarity among the elements of its extended vector \( v_i^o \). If it does not satisfy the conditions, then remove \( v_i \) from \( \bar{V}_k \).
[Degeneracy Test]

\( \mathcal{V}_k \) consists of \( v_i \in \mathcal{V}_{k-1} \) and \( w_{ij} \) composed by (2.5). Let \( \mathcal{V}_k \) be the subset of \( \mathcal{V}_k \) composed of the points on the hyperplane \( H_k : a_k'x = 0 \). Of course \( w_{ij} \in \mathcal{V}_k \).

If \( w_{ij} \) can be expressed by a convex combination of other points of \( \mathcal{V}_k \), \( w_{ij} \) is not a vertex of \( \mathcal{C}_k \). To see this test the following.

If there exist \( w_{ij} \in \mathcal{V}_k \) and \( y_t \in \mathcal{V}_k \) whose extended vectors we denote \( w_{ij}^0 \) and \( y_t^0 \) respectively, such that for every positive elements of \( y_t^0 \), the corresponding elements of \( w_{ij}^0 \) are also positive and there is at least one positive element of \( w_{ij}^0 \) whose corresponding element of \( y_t^0 \) is zero, then \( w_{ij} \) is not a vertex of \( \mathcal{C}_k \). And we remove it from \( \mathcal{V}_k \).

3. Validity of the method

[Lemma 1] Let \( \mathcal{W}_{k-1} \) be the set of all vertex of \( \mathcal{C}_{k-1} \) and let \( \mathcal{W}_k \) be the set of points obtained from \( \mathcal{W}_{k-1} \) by applying step 2 of the preceding section to \( \mathcal{W}_{k-1} \) instead of \( \mathcal{V}_{k-1} \) except [Sub-Complementarity Test]. Then \( \mathcal{W}_k \) is the vertex set of \( \mathcal{C}_k \).

Proof: If all vertices of \( \mathcal{C}_k \) are non-degenerate, the lemma is true even if \( \mathcal{W}_k \) is obtained from \( \mathcal{W}_{k-1} \) by applying step 2 of the preceding section except [Degeneracy Test], (see [3]). But when some vertices of \( \mathcal{C}_k \) are degenerate, it may be happen that some non-vertex points of \( \mathcal{C}_k \) are contained in \( \mathcal{W}_k \) (corresponding to \( \mathcal{V}_k \) in step 2.4.). [Degeneracy Test] prevents such troubles. We need to try the test only to the newcomers on the hyperplane \( H_k : a_k'x = 0 \). Let \( w_i \) and \( w_j \in \mathcal{W}_{k-1} \) be positioned on the opposite sides of \( H_k \) and

\[
\begin{align*}
\lambda_{ik} &= a_k'w_i > 0, \\
\lambda_{jk} &= a_k'w_j < 0.
\end{align*}
\]

If they pass [Common Zero Test], we define a newcomer \( w_{ij} \) by

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Now, let \( \mathbb{W}_k \) be the subset of \( \mathbb{W}_k \) composed of the points on the hyperplane \( H_k \).

If, for \( w_{ij} \), there exists \( y_t \in \mathbb{W}_k \) such that for every positive elements of \( y_t^o \) (the extended vector of \( y_t \)), the corresponding elements of \( w_{ij}^o \) are also positive and there is at least one positive element of \( w_{ij} \) for which the corresponding element of \( y_t^o \) is zero, then \( w_{ij} \) is not a vertex of \( C_k \). This can be seen as follows.

Let

\[
(3.4) \quad \xi_1 = (w_{ij} + cy_t)/(1+\epsilon)
\]

and

\[
(3.5) \quad \xi_2 = (w_{ij} - cy_t)/(1+\epsilon),
\]

where \( \epsilon \) is a sufficiently small positive number. Then \( \xi_1, \xi_2 \in C_k \cap H_k \) and \( \xi_1 \neq \xi_2 \).

And we have,

\[
(3.6) \quad w_{ij} = ((1+\epsilon)/2)\xi_1 + ((1-\epsilon)/2)\xi_2.
\]

Thus, \( w_{ij} \) is not a vertex of \( C_k \).

Conversely, as \( \mathbb{W}_k \) contains all vertices of \( C_k \) on \( H_k \) and every non-vertex point of \( C_k \) on \( H_k \) can be expressed by a convex combination of vertices on \( H_k \), all non-vertex point of \( \mathbb{W}_k \) can be removed by [Degeneracy Test]. This can be seen as follows. First, let \( w_i \) and \( w_j \) be any two different vertices of \( C_k \) on \( H_k \), then their extended vectors \( w_i^o \) and \( w_j^o \) have their positive elements at, at least, one different position. For, if they have their positive elements wholly at common positions, let \( \xi_1 \) and \( \xi_2 \) be the vectors defined by (3.4) and (3.5) respectively, after replacing \( w_{ij} \) by \( w_i \) and \( y_t \) by \( w_j \) on the right hand sides. Then, \( \xi_1, \xi_2 \in C_k \cap H_k \) and \( \xi_1 \neq \xi_2 \). And we have,

\[
(3.7) \quad w_i = ((1+\epsilon)/2)\xi_1 + ((1-\epsilon)/2)\xi_2.
\]

Thus, \( w_i \) is not a vertex of \( C_k \) on \( H_k \) and also \( w_j \). Using this fact and using
the representation of a non-vertex point of $C_k$ on $H_k$ by at least two different vertices of $C_k$ on $H_k$. we can conclude that all non-vertex points of $W_k$ can be removed by [Degeneracy Test]. (Q.E.D.)

**[Lemma 2]** Let $V_k$ be the sub-complementary vertex of $C_k$. Then we can get $V_k$ from $V_{k-1}$ by step 2.

**Proof:** Assume the vertex set of $C_{k-1}$ is known, which has the sub-complementary vertex set $V_{k-1}$ and the non-sub-complementary vertex set $U_{k-1}$. Similarly, the vertex set $W_k$ of $C_k$ consists of $V_k$ and non-sub-complementary $U_k$. By lemma 1, $W_k$ is composed of some vertices of $C_{k-1}$ and some of the newcomers defined by (3.3). The former we denote $\{w_i\}$, the latter $\{w_{ij}\}$. Then we have,

1. If $w_i \in U_{k-1}$, then also $w_i \in U_k$.
2. If $w_i \in V_{k-1}$, then $w_i$ may belong to $U_k$ or to $V_k$.
3. If $w_{ij}$ is defined by (3.3) and
   a. if $w_i \in V_{k-1}$ and $w_j \in V_{k-1}$, then $w_{ij}$ may belong to $U_k$ or to $V_k$,
   b. otherwise, $w_{ij}$ belongs to $U_k$.

Thus, we have shown that $V_k$ can only be obtained from $V_{k-1}$. (Q.E.D.)

As $V_o$ has such property, we demonstrated the validity of the method.

Now, we have the following theorem:

**[Theorem 1]** $V_k$ defined in the section 2 is the sub-complementary vertex set of $C_k$.

**[Corollary]** Any sub-complementary point of $C_k$ can be expressed by a convex combination of vectors in $V_k$.

**Remark:** We could also replace the step 2 of the preceding algorithm by the dual simplex method, because the added constraint is a cutting plane.

4. **Example and computational remark**

Example. Solve the system,
Extremal Rays of Polyhedral Convex Cones

\begin{align*}
(4.1a) & \quad \lambda_1 = x_1 + x_2 + x_3 + x_4 - x_5 + 2x_6 \geq 0 \\
(4.1b) & \quad \lambda_2 = x_1 - x_2 + x_3 - x_4 + x_5 - 3x_6 \geq 0 \\
(4.1c) & \quad \lambda_3 = -x_1 - x_2 + 2x_6 \geq 0 \\
(4.1d) & \quad \lambda_4 = -x_1 + x_2 + x_6 \geq 0 \\
(4.1e) & \quad \lambda_5 = x_1 - x_2 + x_6 \geq 0 \\
(4.1f) & \quad x_1, x_2, x_3, x_4, x_5, x_6, \lambda_1, \lambda_2, \lambda_3, \lambda_4, \lambda_5 \geq 0
\end{align*}

with the complementarity conditions

\begin{align*}
(4.1g) & \quad x_1\lambda_1 = x_2\lambda_2 = x_3\lambda_3 = x_4\lambda_4 = x_5\lambda_5 = 0.
\end{align*}

We show the process of solution in Table 4.1. To avoid decimal numbers, we were not restricted to the condition \( l'x = l \). In this table, a row means a point and its extended form. The points \( V_1 \) to \( V_6 \) are unit vectors corresponding to \( C_0 \). And \( V_k = V(i,j) \) means that the point \( V_k \) is obtained from \( V_i \) and \( V_j \) by the formula (2.5). \( _\) means non-complementary elements. Final solutions are \( V_3, V_9, V_{13} \) and \( V_{18} \).

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<td>_</td>
<td>_</td>
<td>_</td>
<td>_</td>
<td>_</td>
<td>4</td>
<td>1</td>
</tr>
<tr>
<td>( \lambda_4 )</td>
<td>_</td>
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<td>_</td>
<td>_</td>
<td>_</td>
<td>_</td>
<td>_</td>
<td>2</td>
<td>5</td>
</tr>
<tr>
<td>( \lambda_5 )</td>
<td>_</td>
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<td>_</td>
<td>_</td>
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</tr>
</tbody>
</table>

Table 4.1.

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Remark: Degeneracy may happen rarely. And we need not to try [Degeneracy Test] at every step. We may try it at the final stage to final candidates.

5. Applications

Recently, many papers have published on the linear complementarity problem [2]. But we can apply so-called principal pivoting method or complementary pivot method only for matrices with special structures. There would be many other complementarity problems whose matrices are not of such structures, for example, the Kuhn-Tucker conditions for nonconvex quadratic programms.

In this section, we apply our method to the general linear complementarity problem, to the nonconvex quadratic programming and to a mathematical programming with control variables.

(a) The general linear complementarity problem

We define a generalized linear complementarity problem as the problem to find $x \in \mathbb{R}^n$ which satisfies the following system:

\begin{align}
\text{(5.1)} & \quad Ax = b, \\
\text{(5.2)} & \quad x \geq 0 \\
\text{(5.3)} & \quad \text{the given complementarity conditions among the elements of } x,
\end{align}

where $A$ is an $(m,n)$ matrix, $m \leq n$, $\text{rank}(A) = m$, and $b$ is an $m$-vector.

As $\text{rank}(A)$ is $m$, there is a regular submatrix $M$ of order $m$ of $A$. By multiplying $M^{-1}$ to (5.1) from the left and by rearranging the result, we have the canonical form:

\begin{align}
\text{(5.4)} & \quad \lambda = By + d,
\end{align}

where $\lambda \in \mathbb{R}^m$ is the vector of the basic variables corresponding to $M$, $y \in \mathbb{R}^{n-m}$ is the vector of the nonbasic variables, $d = M^{-1}b \in \mathbb{R}^m$ and $B$ is an $(m, n-m)$ matrix.
Then we have the following theorem:

[Theorem 2] The general linear complementarity problem which satisfies (5.1) to (5.3), can be reduced to the complementary extremal ray problem of the polyhedral convex cone \( F \) defined by

\[
F = \{(y,t) | \lambda = By + dt \geq 0, y \geq 0, t \geq 0, t \in \mathbb{R}^1\}.
\]

And when \( d \neq 0 \), any complementary ray of \( F \) with a positive \( t \) can be normalized so as to become a solution of the original problem and any solution \( x \) of the original problem can be represented as the sum of the convex linear combination \( x_1 \) of the normalized complementary extremal rays of \( F \) and of the nonnegative combination \( x_2 \) of the complementary extremal rays of \( F \) with \( t = 0 \).

Proof: The relationship between the solutions of the inhomogeneous system (5.1) to (5.2) and the homogeneous system \( \lambda = By + dt \) is well known, (see, for example, [4]). We can get the theorem by adding the complementarity conditions to the relationship. (Q.E.D.)

(b) The nonconvex quadratic programming

It is well known that the Kuhn-Tucker conditions for a quadratic programming is a linear complementarity problem. When the coefficient matrix of the quadratic form in the objective function to be minimized is positive semidefinite (i.e. the convex quadratic programming), we have an efficient algorithm to solve it, due to P. Wolfe [6]. For nonconvex case, we have not such a good algorithm. But as the minimizing point satisfies the Kuhn-Tucker conditions, we can solve the corresponding linear complementarity problem by our method to choose the global optimum point among the solutions. Indeed, the example in the preceding section is the Kuhn-Tucker conditions for the following nonconvex quadratic programming.

Minimize \( 2x_1 - 3x_2^2 + (x_1^2 + 2x_1x_2 - x_2^2)/2 \),

subject to \( x_1 + x_2 \leq 2 \),
Thus, we have the global optimum point \((x_1=1/2, x_2=3/2)\).

As to the detail of the algorithm which includes several devices to reduce the amount of computations, see [5].

(c) A mathematical programming with control variables

We consider the following two linear programming problems including control variables \(\lambda\).

[Problem I]

\[
\text{Maximize } \quad (c+K\lambda)'x, \\
\text{subject to } \quad Ax \leq b + F\lambda, \ x \geq 0, \ \lambda \geq 0.
\]

[Problem II]

\[
\text{Maximize } \quad (d+L\lambda)'x, \\
\text{subject to the same constraints with [Problem I], where } x, a, c \in \mathbb{R}^n, \ \lambda \in \mathbb{R}^k, \ b \in \mathbb{R}^m \text{ and matrices } A, F, K \text{ and } L \text{ are of order } (m,n), (m,k), (n,k), \text{ respectively.}
\]

For a given \(\lambda\), there may be two optimum points of the two problems. Our problem is how to determine the control variables \(\lambda\) to let the two optimum points coincide with each other. By applying the Kuhn-Tucker conditions for this problem, we get the following theorem:

[Theorem 3] In order that the optimum points of the two problems coincide with each other, it is necessary and sufficient to find \(\bar{x}, \bar{y}, \bar{z}, \bar{\lambda}, \bar{\xi}, \bar{\eta} \) and \(\bar{\zeta}\) which satisfy

\[
\begin{align*}
\xi &= -Ax + F\lambda + b, \\
\eta &= A'y - K\lambda - c,
\end{align*}
\]
(5.9c) $\zeta = A'z - L\lambda - d,$

(5.9d) $\eta ' x = \xi ' x = \xi ' y = \xi ' z = 0,$

where $\zeta \in \mathbb{R}^m$, $\eta$ and $\zeta \in \mathbb{R}^n$ and all variables must be nonnegative.

And the $\bar{x}$ is the common optimum point and the $\bar{\lambda}$ is the corresponding value of the control vector.

Proof: Obvious. (Q.E.D.)

Because (5.9) is a generalized linear complementarity problem, we can apply our method to get the solutions.

References


