TWO-AND THREE-MACHINE FLOW-SHOP MAKESPAN SCHEDULING PROBLEMS WITH ADDITIONAL TIMES SEPARATED FROM PROCESSING TIMES

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Abstract  This paper considers two- and three-machine flow-shop makespan scheduling problems with additional times separated from processing times.

For two-machine case we investigate the optimality of the passing of jobs and give an optimal algorithm in permutation case with five independent times, that is, setup and removal times, start and stop lags and transportation times. It generalizes known algorithms. And for a three-machine case with independent setup times alone, we present several sufficient optimality conditions for a specified sequence $\omega_1$ and clarify not only their mutual relation but also the superiority of $\omega_1$ as an approximate sequence. Those results generalize the results in usual three-machine flow-shop case.

1. Introduction


For a three-machine flow-shop makespan scheduling problem with sequence-independent setup times, Yoshida and Hitomi [20] has shown that the passing of jobs may produce a sequence having minimum makespan, and Maruyama and Nabeshima [11] proposed two solvable cases under some restraints on removal.
times for similar problem with sequence-independent setup times and removal times.

Practical significances of those additional times are obvious and are described in detail in the papers [12], [20], [10] and [18].

Then in the first sections (§§ 2.1-2.3) in this paper we investigate a general two-machine flow-shop makespan scheduling problem with such five additional times under the same assumptions as in the usual two-machine flow-shop case [4] [2] and show by simple examples that the passing of jobs may produce an optimal schedule. Next, we give, in permutation scheduling case, an optimal algorithm, that is, a criterion for constructing an optimal sequence in our general problem. Our algorithm includes known algorithms as special cases.

In sections §§ 3.1-3.7, this paper deals with a three-machine permutation makespan scheduling problem with sequence-independent setup times and, in order to clarify not only the relation among polynomially solvable cases [6] [5] but also the superiority of a specified sequence \( \omega_1 \) as an approximate sequence, we identify several new sufficient optimality conditions for \( \omega_1 \).

In this paper, the following notations are used. Those are deterministically given where \( i=1,2,\ldots,n \) and \( t=1,2,3 \) in § 2, \( t=1,2,3 \) in § 3 (\( P_{i,t}, S_{i,t}^{c} \)).

\( P_{i,t} \): Sequence-independent processing time of job \( i \) on the \( t \)-th machine \( M_{t} \).

\( S_{i,t}^{c} \): Sequence-independent setup times of \( M_{t} \) for job \( i \). Setup can start prior to the processing of the job \( i \) if machine is idle.

\( R_{i,t}^{c} \): Sequence-independent removal times of \( M_{t} \) for job \( i \). Removal follows directly after the processing of the job \( i \).

\( D_{i} \): Start lag of job \( i \) which requires that the processing of job \( i \) on \( M_{2} \) must not start sooner than \( D_{i} \) time-units after the starting of its processing on \( M_{1} \).

\( E_{i} \): Stop lag of job \( i \) which requires that the processing of job \( i \) on \( M_{2} \) must not complete sooner than \( E_{i} \) time-units after the completion of its processing on \( M_{1} \).

\( t_{i} \): Transportation time of job \( i \) which is required to transport job \( i \) directly after the completion of its processing on \( M_{1} \) to \( M_{2} \) for its processing on \( M_{2} \).

\( \omega_{1} \): A specified sequence \( \omega_{1} \) for three-machine case with sequence-independent setup time alone is constructed by a single criterion: if

\[
\min \left[ S_{1}^{1}+P_{i,1}+P_{i,2}-S_{1}^{3}, P_{j,2}+P_{j,3} \right] \leq \min \left[ S_{2}^{1}+P_{j,1}+P_{j,3}+S_{2}^{3}, P_{i,2}+P_{i,3} \right]
\]
holds, then job $i$ precedes job $j$ where either ordering is optimal provided
the equality holds.

   Case with Five Separated Independent Times

2.1 On the Passing of Jobs

   When additional times are sequence-independent setup time alone, it is
   sufficient to consider only permutation schedules as demonstrated in Yoshida
   and Hitomi's paper [20].

   Also, if additional times are sequence-independent removal times alone,

   On the other hand, we can illustrate by examples that the passing of a
   job may produce an optimal schedule in the cases where at least one additional
   times among five such times exist, except the above $S_i^t$ case and $R_i^t$ case. For
   example, next example shows the above fact for the $(S_i^t, R_i^t)$ case:

   Example. $(S_i^t, R_i^t)$ case. We consider two jobs 1 and 2 with related data
given by Table 1.

   Table 1

<table>
<thead>
<tr>
<th>job</th>
<th>$S_i^1$</th>
<th>$P_{i,1}$</th>
<th>$R_i^1$</th>
<th>$S_i^2$</th>
<th>$P_{i,2}$</th>
<th>$R_i^2$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>2</td>
<td>2</td>
<td>2</td>
<td>3</td>
<td>2</td>
<td></td>
</tr>
<tr>
<td>2</td>
<td>2</td>
<td>2</td>
<td>12</td>
<td>11</td>
<td>2</td>
<td></td>
</tr>
</tbody>
</table>

   Permutation schedules 1 2 and 2 1 have makespan $M(1 2)=24$ and $M(2 1)=25$
respectively, but a schedule (1 2 on $M_1$, 2 1 on $M_2$) gives smaller makespan
$M(1^2 2^1)=22$. In fact it is optimal since $M(1^2 2^1)=40$.

   Similarly we can easily give the examples showing that $M(1^2 2^1)$ is the
smallest, for the other cases.

   Then, in order to identify a simple optimality criterion, in the next
sections we seek for an optimal schedule among the set of permutation schedules,
that is, sequences in our general problem.

2.2 The Optimal Algorithm for General Permutation Case

   The optimal algorithm for general problem restricted to permutation
schedules consists of the following steps:

   Step 1. Let $T_{ij}$ denote the revised start lag, defined by
Step 2. Let $G$ and $H$ be two fictitious machines having respective processing times for job $i$ as $G_i$ and $H_i$, where $G_i$ and $H_i$ are defined by

\[ G_i = S_i^1 - S_i^2 + T_i, \quad H_i = T_i - P_{i,1} + P_{i,2} - R_i^1 + R_i^2. \]


Step 4. An optimal sequence constructed in step 3 also gives an optimal sequence for the original general permutation scheduling problem.

By this algorithm we have an optimal sequence in polynomial time $O(n \log n)$.

Particular Studies
As special cases of our algorithm, we have the following:

For every job $i$.

1. If $D_i = P_{i,1}$, $E_i = P_{i,2}$ and $t_i = 0$, then the algorithm reduces to the algorithm presented separately by Sule [18], and Maruyama and Nabeshima [11].

2. If $S_i^t = 0$ and $R_i^t = 0$, then the algorithm reduces to the algorithm by Maggu et al [10] where $t_i = T_i - P_{i,1}$.

3. If $S_i^t = 0$, $R_i^t = 0$, $D_i = P_{i,1}$, and $E_i = P_{i,2}$, then the algorithm reduces to the algorithm by Maggu and Das [9].

4. If $S_i^t = 0$, $R_i^t$ and (a) $t_i = 0$, or (b) $D_i \geq P_{i,1} + t_i$ and $E_i > t_i + P_{i,2}$, then the algorithm reduces to the algorithm by Mitten [12] and Johnson [8].

5. If $S_i^t = 0$, $R_i^t = 0$, $D_i = P_{i,1}$, $E_i = P_{i,2}$ and $t_i = 0$, then the algorithm reduces to the algorithm by Johnson [7] and Bellman [1].

2.3 Proof of Optimal Algorithm
We compare the makespans of two sequences $S = \pi \{i j\} \pi'$ and $S' = \pi' \{i j\} \pi$ where $S'$ is a sequence obtained only by interchanging two adjacent jobs $i$ and $j$ in $S$, and $\pi$ and $\pi'$ are subsequences or may be empty.

Let $C_t(i)$ be the finishing time (here finish means end of removal) of job $i$ on $M_t$.

Then, we have for $S$
(2) \[ C_1(i) = C_1(\ell) + S_1^1 + p_1 + R_1^1 \]
\[ C_2(i) = \max \{ \max(C_2(\ell) + S_2^1, C_1(\ell) + S_1^1 + D_1, C_1(\ell) + S_1^1 + \delta(t_{ij})) \} \]
\[ + p_{i,2} + C_1(\ell) + S_2^1 + p_{1,1} + R_1^2 \]
\[ = S_1^2 + p_{i,2} + R_1^2 + \max \{ C_2(\ell), C_1(\ell) + S_1^1 - S_2^2 + T_1 \} \]

where \( C_1(\ell) = 0 \) (t=1,2) when \( i, j \) are the first two jobs of \( S \).

Using relations (2) and (3), finally we obtain

(4) \[ C_1(ij) = C_1(j) = C_1(\ell) + S_1^1 + p_{j,1} + R_1^1 + S_j^1 + p_j + 1 + R_j^1 \]
and \[ C_2(ij) = C_2(j) = S_j^2 + p_{j,2} + R_j^2 + S_j^2 + p_{j,1} + R_j^2 \]

(5) \[ + \max \{ C_2(\ell), C_1(\ell) + S_1^1 - S_2^1 + T_1, C_1(\ell) + S_1^1 + p_{1,1} + R_1^1 \]
\[ + S_j^1 + T_j - S_2^2 - p_{j,2} - R_j^2 \} \]

Similarly we obtain for \( S' \)

(6) \[ C_1(ji) = C_1(i) = C_1(\ell) + S_j^1 + p_{1,1} + R_j^1 + S_j^1 + p_{j,1} + R_j^1 \]
and \[ C_2(ji) = C_2(i) = S_j^2 + p_{j,2} + R_j^2 + S_j^2 + p_{j,1} + R_j^2 \]

(7) \[ + \max \{ C_2(\ell), C_1(\ell) + S_j^1 - S_2^1 + T_j, C_1(\ell) + S_j^1 + p_{j,1} + R_j^1 \]
\[ + S_1^1 + T_1 - S_2^2 - p_{j,2} - R_j^2 \} \]

Since \( C_1(ij) = C_1(ji) \) holds from (4) and (6), if \( C_2(ij) \leq C_2(ji) \) holds, then makespan of \( S \), \( M(S) \), is not larger than \( M(S') \) because generally \( C_2(i) \) is a nondecreasing function of \( C_2(\ell) \) and \( C_1(\ell) \) where job \( i \) is a next job of job \( \ell \), as is clear from (3).

Substituting (5) and (7) into \( C_2(ij) \leq C_2(ji) \), if an inequality:

\[ \max \{ S_1^1 - S_2^1 + T_1, S_1^1 + p_{1,1} + R_1^1 + S_1^1 + T_{ij} - S_2^2 - p_{j,2} - R_j^2 \} \]
\[ \leq \max \{ S_j^1 - S_2^1 + T_j, S_j^1 + p_{j,1} + R_j^1 + S_j^1 + T_{ij} - S_2^2 - p_{j,2} - R_j^2 \} \]

holds, then \( C_2(ij) \leq C_2(ji) \) follows.

Subtracting \( S_1^1 - S_2^1 + T_1 + S_j^1 - S_2^1 + T_j \) from both sides of (8), it follows.

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max \{ -S^1_j + S^2_j - T_{j1}, -T_{j1} + P_{j1,1} - P_{j1,2} + R^1_j - R^2_j \}
\leq \max \{ -S^1_i + S^2_i - T_{i1}, -T_{i1} + P_{i1,1} - P_{i1,2} + R^1_i - R^2_i \},

that is, our criterion:

$$\min (G_j, H_j) \leq \min (G_j, H_j).$$

Q.E.D.

3. Solvable Cases and An Approximate Sequence \( \omega_1 \) in Three-Machine Case with Sequence-Independent Setup Times Alone

3.1 An Expression of Makespan

Let every job be available at time zero and be processed on three machines \( M_1, M_2, M_3 \) in that order and no passing of jobs be allowed.

For any sequence \( \omega = (1, 2, \ldots, n-1, n) \) in which number \( i \) means \( i \)-th job in \( \omega \), completion time \( C_t(i) \) of the processing of job \( i \) on \( M_t \) is determined by recurrence relation:

$$C_t(i) = \max \{ C_{t-1}(i), C_{t-1}(i)+S^t_i \} + P_{i,t}, (i=1, \ldots, n, t=1, 2, 3)$$

where \( C_{t-1}(i)+S^t_i \) denotes completion time of the setup for job \( i \) on \( M_t \), and \( C_0(i)=0, C_0(i)=0 \).

Then, makespan of \( \omega, M(\omega) \), can be expressed by critical path approach.

$$M(\omega) = \max_{0 \leq r \leq s \leq n} \left\{ \sum_{i=1}^{r} (S^1_i + P_{i,1}) + P_{r,2} + \sum_{i=r+1}^{s} (S^2_i + P_{i,2}) + P_{s,3} + \sum_{i=s+1}^{n} (S^3_i + P_{i,3}) \right\}.$$

Notice that this is a machine-base expression of the makespan.

3.2 A Relation Between \( M(\omega) \) and \( m(\omega) - \Sigma \{ (P_{i,2} - S^2_i) \} \)

As preliminary for deriving our results, we need this relation.

From (9) it follows

$$M(\omega) = \sum_{i=1}^{r^o} (S^1_i + P_{i,1}) + P_{r^o,2} + \sum_{i=r^o+1}^{s^o} (S^2_i + P_{i,2}) + P_{s^o,3} + \sum_{i=s^o+1}^{n} (S^3_i + P_{i,3}),$$

where, in a case \( r^o \leq 1, r^o \) and \( s^o \) denote the corner jobs on the first machine \( M_1 \) and the second machine \( M_2 \) respectively along any critical path for \( M(\omega) \).

Therefore, we can express \( M(\omega) \) as

$$M(\omega) = \max_{r^o \leq r \leq s \leq n} \left\{ \sum_{i=1}^{r} (S^1_i + P_{i,1}) + P_{r,2} + \sum_{i=r+1}^{s} (S^2_i + P_{i,2}) + P_{s,3} + \sum_{i=s+1}^{n} (S^3_i + P_{i,3}) \right\} \equiv A(\omega; r^o, s),$$

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and

\[
M(\omega) = \max_{0 \leq r \leq s^o} \left[ \sum_{i=1}^{r} (S^1_i + P_i) + P_i,2 + \sum_{i=r+1}^{s^o} (S^2_i + P_i,2) + P_i,3 + s^o \sum_{i=s^o+1}^{n} (S^3_i + P_i,3) \right] \\
\equiv A(\omega; r, s^o)
\]

respectively.

Next, we consider a related two-machine problem without setup times in which every job \(i\) has its processing times \(\sum_{t=1}^{2} (S^t_i - S^t_{i-1}) = S^1_i + P_i,1 + P_i,2 - S^3_i \) and \(\sum_{t=2}^{i} P_i,t\) on the first and second machines respectively. This problem corresponds to the two-machine problem with setup times in which every job \(i\) has its setup times \(S^1_i\) separated from processing times \(P_i,1 + P_i,2\) on the first machine and its setup times \(S^3_i\) separated from processing times \(P_i,2 + P_i,3\) on the second machine. cf. [20]

Then, for any sequence \(\omega=(1, 2, \ldots, n-1, n)\), the makespan of \(\omega\), \(m(\omega)\), in the related two-machine problem is expressed by

\[
m(\omega) = \max_{0 \leq u \leq n} \left[ \sum_{i=1}^{u} (S^1_i + P_i,1 + P_i,2 - S^3_i) + \sum_{i=u}^{n} (P_i,2 + P_i,3) \right] \\
= \sum_{i=1}^{u} (S^1_i + P_i,1 + P_i,2 - S^3_i) + \sum_{i=u}^{n} (P_i,2 + P_i,3),
\]

where, in a case \(u^o \geq 1\), \(u^o\) is a corner job on the first machine along any critical path for \(m(\omega)\) and in a case \(u^o=0\) which can happen when \(S^1_i + P_i,1 + P_i,2 - S^3_i \leq 0\), critical path for \(m(\omega)\) exists on the second machine alone.

From (12), we have

\[
m(\omega) - \sum_{i=1}^{n} (P_i,1 - S^3_i) = u^o \sum_{i=1}^{u^o} (S^1_i + P_i,1 + P_i,2 - S^3_i) + \sum_{i=u^o+1}^{n} (S^3_i + P_i,3) \\
\equiv B(\omega; u^o).
\]

Then, from (10) and (13), it follows

\[
\forall \{r^o, s^o\}, \forall u^o \geq 0; \\
M(\omega) = \{m(\omega) - \sum_{i=1}^{n} (P_i,2 - S^3_i) \}
\]

\[
\geq A(\omega; r^o, s) - B(\omega; u^o) \\
\geq A(\omega; u^o, s) - B(\omega; u^o) \\
= \max_{u^o \leq s \leq n} \left[ \sum_{i=u^o+1}^{s} (S^2_i + P_i,1) - \sum_{i=u^o+1}^{s-1} (S^3_i + P_i,3) \right] \\
\equiv D(\omega; u^o, s).
\]
Similarly we have from (11) and (13)

\[ M(\omega) = \{m(\omega) - \sum_{i=1}^{n} (P_i \cdot S_i^3)\} \]
\[ \geq A(\omega; r, u^o) - B(\omega; u^o) \]
\[ = \max_{0 \leq r \leq u^o} \left[ \sum_{i=r}^{u^o-1} (S_{i+1}^2 + P_i \cdot S_i^3) - \sum_{i=r+1}^{u^o} (S_i^1 + P_i \cdot S_i^3) \right] \]
\[ \equiv E(\omega; r, u^o). \]

Here, if we put \( s = u^o \) and \( r = u^o \) to the expressions in the brackets of (15) and (16) respectively, those expressions become zero. Therefore, \( D(\omega; u^o, s) \geq 0 \) and \( E(\omega; r, u^o) \geq 0 \) follow.

From (15) and (16) we obtain

\[ M(\omega) \geq m(\omega) - \sum_{i=1}^{n} (P_i \cdot S_i^3) + \max [D(\omega; u^o, s), E(\omega; r, u^o)] \]
\[ \geq m(\omega) - \sum_{i=1}^{n} (P_i \cdot S_i^3). \]

On the other hand, it follows from (14)

\[ M(\omega) = \{m(\omega) - \sum_{i=1}^{n} (P_i \cdot S_i^3)\} \]
\[ \leq A(\omega; r^o, s) - B(\omega; r^o) = D(\omega; r^o, s). \]

Similarly we obtain from (16)

\[ M(\omega) - \{m(\omega) - \sum_{i=1}^{n} (P_i \cdot S_i^3)\} \]
\[ \leq A(\omega; r, s^o) - B(\omega; s^o) = E(\omega; r, s^o). \]

From the above it follows

\[ M(\omega) \leq m(\omega) - \sum_{i=1}^{n} (P_i \cdot S_i^3) + \min [D(\omega; r^o, s), E(\omega; r, s^o)]. \]

Finally, let us define \( M(\omega; r=s=u) \) \((0 \leq u \leq n)\) as

\[ M(\omega; r=s=u) = \sum_{i=1}^{u} (S_i^1 + P_i, 1) + P_{u, 2} + P_{u, 3} + \sum_{i=u+1}^{n} (S_i^3 + P_i, 3), \]

that is, the makespan of \( \omega \) in original three-machine problem when in case \( u \geq 1 \) both corner jobs \( r \) and \( s \) on the respective machines \( M_1 \) and \( M_2 \) are assumed.
to be identical and equal to \( u \). If \( u=0 \), it corresponds to the critical path on the last machine \( M_3 \).

Then, we have from (13)

\[
(19) \quad Vu(0 \leq u \leq n), \quad Vu^o; \quad M(\omega; \; r=s=u) \leq m(\omega) - \sum_{i=1}^{n} (P_i, 2^{-S_i}) = \max_{0 \leq u \leq n} M(\omega; \; r=s=u) = M(\omega; \; r=s=0) \leq M(\omega).
\]

So that from (17), (18) and (19) we have the following inequalities (20) in Theorem 1:

**Theorem 1.** For any sequence \( \omega=(1,2,...,n-1,n) \) in our three-machine problem, we have

\[
Vu(0 \leq u \leq n) \; \text{for} \; M(\omega), \quad Vu^o(0 \leq u \leq n) \; \text{for} \; m(\omega), \quad Vu(0 \leq u \leq n);
\]

\[
(20) \quad M(\omega; \; r=s=u) \leq m(\omega) - \sum_{i=1}^{n} (P_i, 2^{-S_i}) = \max_{0 \leq u \leq n} M(\omega; \; r=s=u)
\]

\[
= M(\omega; \; r=s=0) \leq m(\omega) - \sum_{i=1}^{n} (P_i, 2^{-S_i}) + \max_{0 \leq u \leq n} \left\{ \sum_{i=1}^{u} (S_{i,1}^2 + P_{i,1}) \right\}
\]

\[
= \max_{0 \leq u \leq n} \left\{ \sum_{i=1}^{u} (S_{i,1}^2 + P_{i,1}) \right\}
\]

\[
\leq M(\omega) \leq m(\omega) - \sum_{i=1}^{n} (P_i, 2^{-S_i}) + \max_{0 \leq u \leq n} \left\{ \sum_{i=1}^{u} (S_{i,1}^2 + P_{i,1}) \right\}
\]

\[
\leq M(\omega) \leq m(\omega) - \sum_{i=1}^{n} (P_i, 2^{-S_i}) + \max_{0 \leq u \leq n} \left\{ \sum_{i=1}^{u} (S_{i,1}^2 + P_{i,1}) \right\}
\]

\[
\leq M(\omega) \leq m(\omega) - \sum_{i=1}^{n} (P_i, 2^{-S_i}) = M(\omega).
\]

### 3.3 Fundamental Result derived from Theorem 1

From Theorem 1 we obtain the following lemma which shows equivalent conditions for any sequence.

In the next section those equivalent conditions are shown to become sufficient optimality conditions for a specified sequence \( \omega_1 \) which is constructed by using a single criterion (1).

**Lemma.** For any sequence \( \omega=(1,2,...,n-1,n) \) in our three-machine problem, the following conditions (21), (1), (2), (3) and (4) where \( r^0 \) and \( s^0 \) are for \( M(\omega) \) and \( u^o \) is for \( m(\omega) \) are equivalent to each other.

\[
(21) \quad m(\omega) - \sum_{i=1}^{n} (P_i, 2^{-S_i}) = M(\omega).
\]
Flow-Shop Makespan Scheduling Problems

1). \( \exists s^o; \max_{0 \leq r \leq s^o} \frac{\sum (S^2_{i+1} + P^i_{1,2}) - \sum (S^1_{i+1} + P^i_{1,1})}{i=r} = 0, \)

or \( \exists r^o; \max_{r^o \leq s \leq n} \frac{\sum (S^2_{i+1} + P^i_{1,2}) - \sum (S^3_{i+1} + P^i_{1,3})}{i=r^o} = 0. \)

2). \( \exists (r^o, s^o); \sum_{i=r^o}^{s^o-1} (S^2_{i+1} + P^i_{1,2}) \geq \sum (S^1_{i+1} + P^i_{1,1}), \)

or \( \sum_{i=r^o}^{s^o-1} (S^2_{i+1} + P^i_{1,2}) \geq \sum (S^3_{i+1} + P^i_{1,3}), \)

3). \( \exists (r^o, s^o), \exists u^o; r^o = s^o = u^o. \)

4). \( \exists u(0 \leq u \leq n); M(\omega; r=s=u) = M(\omega). \)

In this case, it follows \( u = u^o \) for some \( u^o. \)

Proof: 1). Condition 1) yields (21) from (20).

2). Max-terms in 1) take their maxima when \( r=r^o \) and \( s=s^o \) respectively. Since max-terms in 1) are not negative in itself and 2) yields their non-positivities, 1) follows from 2).

3). If 3) holds, 2) follows. Next if (21) holds, we have a critical path expression of \( M(\omega) \) from (13)

\[
M(\omega) = \sum_{i=1}^{u^o} (S^1_{i+1} + P^i_{1,1}) + P^u^o_{2} + P^u^o_{3} + \sum_{i=u^o+1}^{n} (S^3_{i+1} + P^i_{1,3}).
\]

This shows that there exists some critical path for \( M(\omega) \) which has corner job set \( \{r^o, s^o\} \) satisfying \( r^o = s^o = u^o \), that is, 3) holds.

From the above, we have 3) \( \Rightarrow \) 2) \( \Rightarrow \) 1) \( \Rightarrow \) (21) \( \Rightarrow \) 3).

4). 4) is equivalent to (21) from (20).

So that the conclusion of the lemma follows. Q.E.D.

3.4 Sufficient Optimality Conditions for A Specified Sequence \( \omega_1 \)

Using Lemma, we obtain equivalent fundamental (weakest) sufficient optimality conditions for a specified sequence \( \omega_1 \) as shown in the next theorem:

Theorem 2. If any one of the following equivalent fundamental conditions (F); 1), i.e. (F'); 2), i.e. (F''); 3), i.e. (F'''), holds for a specified sequence \( \omega_1 = (1, 2, ..., n-1, n) \) where \( r^o \) and \( s^o \) are for \( M(\omega_1) \) and \( u^o \) is for \( m(\omega_1) \), then \( \omega_1 \) becomes an optimal sequence in the problem:

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In this case \( u = u^* \) holds for some \( u^* \).

**Proof:** Since the equivalence of (F), (F'), (F'') and (F''') follows from that of conditions (21), 2), 3) and 4) in Lemma, it is sufficient only to prove that (F) yields the optimality of \( \omega_1 \).

Then, from \( m(\omega_1) \leq m(\omega_0) \) and \( M(\omega_0) \leq M(\omega_1) \) for an optimal sequence \( \omega_0 \) and also from (20) for \( \omega = \omega_0 \), we have

\[
m(\omega_1) - \sum_{i=1}^{n} (P_{i,2} - S_{i,2}^3) \leq m(\omega_0) - \sum_{i=1}^{n} (P_{i,2} - S_{i,2}^3) \leq M(\omega_0) - M(\omega_1).
\]

Therefore, (F) yields \( M(\omega_0) = M(\omega_1) \), that is, \( \omega_1 \) is an optimal sequence.

**Q.E.D.**

**Remark. Bound of \( M(\omega_1) - M(\omega_0) \).** Directly from the above proof, a bound of the deviation of an approximate value \( M(\omega_1) \) by an approximate sequence \( \omega_1 \) from an unknown optimal value \( M(\omega_0) \) is given by

\[
M(\omega_1) - M(\omega_0) \leq m(\omega_1) - \sum_{i=1}^{n} (P_{i,2} - S_{i,2}^3).
\]

Next, applying a fundamental condition (F'), we obtain the following concrete sufficient optimality conditions for \( \omega_1 \), that are stronger than the fundamental conditions proposed in Theorem 2:

**Theorem 3.** If any one of the following conditions \((A_2), (A), (A_3)\) and \((A_4)\) including corner jobs \( \omega_1 = (1, 2, \ldots, n-1, n) \) in our three-machine problem, \( \omega_1 \) becomes an optimal sequence in the problem:

1). \( \exists s^o, \forall i(0 \leq i \leq s^o - 1); \quad S_{i+1,1}^2 + P_{i,2} \leq S_{i+1,1}^2 + P_{i+1,1} \), \((A_2)\)
or \( \exists \omega, \forall i (r^0 \leq i \leq \omega - 1): S^2_{i+1} + P_{i+1, 2} \leq S^3_{i+1} + P_{i, 3}. \)

2). \( \exists \omega; \max_{0 \leq i \leq S^0 - 1} (S^2_{i+1} + P_{i, 2}) \leq \min_{1 \leq i \leq S^0} (S^1_{i} + P_{i, 1}), \) (A)

or \( \exists \omega; \max_{r^0 + 1 \leq i \leq \omega} (S^2_{i} + P_{i, 2}) \leq \min_{r^0 \leq i \leq \omega - 1} (S^3_{i+1} + P_{i, 3}). \)

3). \( \exists \{r^0, s^0\}; S^2_{r^0 + 1} + P_{r^0, 2} \leq S^3_{r^0 + 1} + P_{r^0, 3}, \)

\( \forall i (r^0 \leq i \leq s^0 - 1), \max_{r^0 \leq i \leq s^0 - 1} (S^2_{i} + P_{i, 2}) \leq \min_{r^0 \leq i \leq s^0} (S^1_{i} + P_{i, 1}, S^3_{i+1} + P_{i, 3}), \) (A3)

\( S^2_{s^0} + P_{s^0, 2} \leq S^1_{s^0} + P_{s^0, 2}. \)

4). \( \exists \{r^0, s^0\}; \forall i (r^0 \leq i \leq s^0); \)

\( \max_{r^0 \leq i \leq s^0} (S^2_{i} + P_{i, 2}) \leq \min_{r^0 \leq i \leq s^0} (S^1_{i} + P_{i, 1}, S^3_{i+1} + P_{i, 3}). \) (A4)

Proof: 1). (A2) yields (F'). 2). (A) yields (A2).

3). Assume that (F') in Theorem 2 does not follow from (A3), then we have for \( \omega_1 = (1, 2, \ldots, n-1, n) \)

\( \forall \{r^0, s^0\}; \)

\( \sum_{i=r^0}^{s^0-1} (S^2_{i} + P_{i, 2}) > \sum_{i=r^0+1}^{s^0} (S^1_{i} + P_{i, 1}), \)

and

\( \sum_{i=r^0+1}^{s^0} (S^2_{i} + P_{i, 2}) > \sum_{i=r^0}^{s^0-1} (S^3_{i+1} + P_{i, 3}). \)

In (22) and (23) we have from (A3)

\( \sum_{i=r^0+1}^{s^0} (S^2_{i} + P_{i, 2}) \leq \sum_{i=r^0+1}^{s^0-1} (S^1_{i} + P_{i, 1}) \) and \( \sum_{i=r^0}^{s^0-1} (S^2_{i+1} + P_{i, 2}) \leq \sum_{i=r^0}^{s^0-1} (S^3_{i+1} + P_{i, 3}) \)

respectively. Therefore, from (22) and (23) they follow

\( P_{x^0, 2} - P_{s^0, 2} > 0 \) and \( P_{s^0, 2} - P_{x^0, 2} > 0 \)

respectively. This is a contradiction. So that (A3) must yield (F').

4). (A4) yields (A3). Q.E.D.

Remark. (A2) and (A3) are weaker than (A) and (A4) respectively and are stronger than (F') \( \Leftrightarrow \) (F) from the above proof.

Next, we give concrete conditions (A2), (A), (A3) and (A4) for \( \omega_1 \) which
do not contain corner jobs $r^o$ and $s^o$ for $M(w_1)$ and are stronger than the respective conditions $(A_2)$, $(A)$, $(A_3)$ and $(A_4)$ presented in Theorem 3. The results are obvious from Theorem 3:

Theorem 4. The followings are sufficient optimality conditions for $w_1 = (1,2,\ldots,n-1,n)$:

1. $\forall i(0 \leq i \leq n-1)$: $S_{i+1}^2 + P_{i,2} \leq S_{i+1}^1 + P_{i+1,1}$, \hspace{1cm} (A_2)

$S_{i+1}^2 + P_{i,2} \leq S_{i+1}^3 + P_{i,3}$

2. $\max \{S_{i+1}^2 + P_{i,1}, 2\} \leq \min \{S_{i+1}^1 + P_{i,1}, 1\}$, \hspace{1cm} (A)

or $\max \{S_{i+1}^2 + P_{i,1}, 2\} \leq \min \{S_{i+1}^3 + P_{i,1}, 3\}$.

3. $S_1^2 \leq S_1^3$.

$\forall i(1 \leq i \leq n-1)$; $\max \{S_i^2, S_i+1^2\} + P_{i,2} \leq \min \{S_i+1^1 + P_{i,1}, S_i+1^3 + P_{i,3}\}$, \hspace{1cm} (A_3)

4. $\forall i(0 \leq i \leq n)$; $\max \{S_i^2, S_i+1^2\} + P_{i,2} \leq \min \{S_i+1^1 + P_{i,1}, S_i+1^3 + P_{i,3}\}$.

(A_4)

Remark. $(A_2)$ and $(A_4)$ are weaker than $(A)$ and $(A_4)$ respectively.

3.5 A Condition (G)

We show here an additional condition (G) which has similar inequalities to a fundamental condition $(F')$, but is slightly stronger than $(F')$.

A condition (G) is

$\forall r, \forall s(0 \leq r \leq s \leq n)$; $\frac{S_s}{s} \sum_{i=r}^{s} (S_{i+1}^2 + P_{i,2}) \leq \frac{S_s}{s} \sum_{i=r+1}^{s} (S_{i+1}^1 + P_{i,1})$,

or $\frac{S_s}{s} \sum_{i=r+1}^{s} (S_{i+1}^2 + P_{i,2}) \leq \frac{S_s}{s} \sum_{i=r+1}^{s} (S_{i+1}^3 + P_{i,3})$.

Then next theorem holds:

Theorem 5. (G) is stronger than $(F')$ and is equivalent to a condition $(A_2)$ and is weaker than a condition $(A_3)$.

Proof: Obviously (G) yields $(F')$. Next, $(A_2)$ yields (G) and inversely (G) for every $r=s-1 (s=1,2,\ldots,n)$ yields $(A_2)$. Therefore, (G) is equivalent

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to $(A_2)$. Finally, by following similar reasoning to the proof of 3) in Theorem 3, we have the result that $(A_3)$ yields $(G)$. Q.E.D.

3.6 Numerical Examples Illustrating Solvable Cases

We present three simple examples for illustrating some solvable cases, in which the first two are for our problem with setup times and the last one is for the usual flow-shop case.

Example 1. (Condition $A_2$)

Consider a problem $(n=6, m=3)$ given in Table 2. Although condition $(A)$ does not hold, observing $w_1=(1,2,3,4,5,6)$ and $(1,2,4,3,5,6)$, we find that $w_1=(1,2,3,4,5,6)$ satisfies the second inequalities of weaker condition $(A_2)$. cf. Table 2. So that this $w_1$ is an optimal sequence. Here another $w_1$ is not optimal $(M(w_1)=49)$.

Table 2. Example 1.

<table>
<thead>
<tr>
<th>$i$</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
</tr>
</thead>
<tbody>
<tr>
<td>$S_i$</td>
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<td>3</td>
<td>5</td>
<td>3</td>
<td>1</td>
<td>4</td>
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<td>$P_{i,1}$</td>
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<tr>
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<td>2</td>
<td>3</td>
<td>4</td>
<td>1</td>
<td>2</td>
</tr>
<tr>
<td>$P_{i,2}$</td>
<td>2</td>
<td>5</td>
<td>3</td>
<td>5</td>
<td>4</td>
<td>1</td>
</tr>
<tr>
<td>$S_i$</td>
<td>6</td>
<td>4</td>
<td>4</td>
<td>3</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>$P_{i,3}$</td>
<td>5</td>
<td>3</td>
<td>6</td>
<td>4</td>
<td>3</td>
<td>1</td>
</tr>
<tr>
<td>$S_i^1+P_{i,1}+P_{i,2}-S_i^3$</td>
<td>5</td>
<td>7</td>
<td>8</td>
<td>8</td>
<td>8</td>
<td>5</td>
</tr>
<tr>
<td>$P_{i,2}+P_{i,3}$</td>
<td>7</td>
<td>8</td>
<td>9</td>
<td>9</td>
<td>7</td>
<td>2</td>
</tr>
</tbody>
</table>

For an optimal $w_1$, obviously the weakest conditions $(F)$: $M(w_1)=46=47-(20-19)=m(w_1)-\sum_{i=1}^{6} (P_{i,2}-S_i)$ and $(F')$: $\exists (r^o, s^o)$, $\exists u^o$; $r^o=s^o=1$ or 2 or 3, are satisfied. cf. Table 3 where $C_t$ $(t=1,2,3)$ denotes completion times of setup in parentheses and processing of every job and $C_{12}$ and $C_{23}$ mean completion times of every job on the first and second machines in related two-machine problem respectively, and also each sequence of arrows shows related critical path for $M(w_1)$ and $m(w_1)$.

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Table 3. Example 1.

<table>
<thead>
<tr>
<th>$\omega_1$</th>
<th>$r^s, s^s$</th>
<th>$r^s, s^s$</th>
<th>$r^s, s^s$</th>
<th>$s^s$</th>
<th>$s^s$</th>
</tr>
</thead>
<tbody>
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<td>2</td>
<td>3</td>
<td>4</td>
<td>5</td>
<td>6</td>
</tr>
<tr>
<td>$C_1$</td>
<td>9(12)</td>
<td>15(20)</td>
<td>24(27)</td>
<td>30(31)</td>
<td>35(39)</td>
</tr>
<tr>
<td>$C_2$</td>
<td>11(13)</td>
<td>20(23)</td>
<td>27(31)</td>
<td>36(37)</td>
<td>41(43)</td>
</tr>
<tr>
<td>$C_3$</td>
<td>16(20)</td>
<td>23(27)</td>
<td>33(36)</td>
<td>40(41)</td>
<td>44(45)</td>
</tr>
</tbody>
</table>

Example 2. (Case where $r^s=s^s=u^s=0$ holds)

If we put $S_2^2=14$ in Example 1 (Table 2), then under $S_1^3 \geq 16$ second inequalities of ($A_2$) is also satisfied by $\omega_1=(1,2,3,4,5,6)$ and so this $\omega_1$ remains optimal.

In this case we have $S_1^2 > S_1^1 + P_{1,1}^1$ and $S_1^3 \geq \max(S_1^1 + P_{1,1}^1, S_1^2 + P_{1,2}^1)$ for $M(\omega_1)$ and $S_1^1 + P_{1,1}^1 + P_{1,2}^1 - S_1^3 \leq -5$ for $M(\omega_1)$, which means that critical paths for $M(\omega_1)$ and $m(\omega_1)$ exist on $M_2$ and $M_3$ ($S_1^3=16$) where $r^s=s^s=0$, or on $M_2$ ($S_1^3 \geq 16$) where $r^s=s^s=0$, and on the second machine where $u^s=0$ respectively.

Specifically in a case $S_1^2=14$ and $S_1^3=16$ in Table 2, we have (F):

$M(\omega_1)=51=42-(20-29)=m(\omega_1)-\sum_{i=1}^{6} P_{i,1}^1-S_1^3$ and (F''): $\exists(x^s, s^s)$; $r^s=s^s=u^s=0$.

Example 3. (Usual flow-shop case [14])

For a problem ($n=6$, $m=3$) given in Table 4, Johnson condition (A) does not hold, but for $\omega_1=(1,2,3,4,5,6)$ first inequalities of weaker condition ($A_2$):

$\forall i (1 \leq i \leq n-1); P_{i,2}^1 \leq P_{i+1,1}^1$, is satisfied. cf. Table 4. So $\omega_1$ is optimal.

Obviously, they hold (F): $M(\omega_1)=18=31-13=m(\omega_1)-\sum_{i=1}^{6} P_{i,1}^1$, and (F''): $\exists(x^s, s^s)$; $r^s=s^s=3$ cf. Table 5.
Table 4. Example 3.

<table>
<thead>
<tr>
<th>i</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
</tr>
</thead>
<tbody>
<tr>
<td>pij</td>
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<td>2</td>
<td>3</td>
<td>4</td>
<td>3</td>
<td>2</td>
</tr>
<tr>
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<td>3</td>
<td>4</td>
<td>2</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>pij</td>
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<td>3</td>
<td>3</td>
<td>2</td>
<td>2</td>
<td>1</td>
</tr>
</tbody>
</table>

Table 5. Example 3.

<table>
<thead>
<tr>
<th>ω̊</th>
<th>ω̊</th>
<th>ω̊</th>
<th>ω̊</th>
<th>ω̊</th>
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<td>3</td>
<td>4</td>
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<td>6</td>
<td></td>
</tr>
<tr>
<td>C1</td>
<td>1</td>
<td>3</td>
<td>6</td>
<td>10</td>
<td>13</td>
<td>15</td>
</tr>
<tr>
<td>C2</td>
<td>3</td>
<td>6</td>
<td>10</td>
<td>12</td>
<td>14</td>
<td>16</td>
</tr>
<tr>
<td>C3</td>
<td>5</td>
<td>9</td>
<td>13</td>
<td>15</td>
<td>17</td>
<td>18</td>
</tr>
</tbody>
</table>

3.7 Inclusion Relation among Proposed Conditions

Figure 1 depicts an inclusion relation among the proposed conditions for a specified sequence ω̊ in our NP-complete (hard) three-machine flow-shop problem with independent setup times.

![Inclusion Relation](image)

Fig. 1. An Inclusion Relation (m=3).
The weakest condition which may be most useful is \((F'')\) and that which does not contain corner jobs \(r^o\) and \(s^o\) is \((G)\) or equivalently \((A_2)\).

It setup times are assumed to be included in the processing times, Figure 1 reduces to an inclusion relation among conditions for a Johnson approximation sequence \(\omega_1\) in the usual flow-shop case \((m=3)\) where \((A)\), \((A_4)\), \((F)\) and \((F'')\) are the conditions separately found by S.M. Johnson [7], Burns and Rooker [3], Szwarc [19] and Raghavachari [16] respectively, whereas our results show their mutual relation including other conditions.

4. Conclusion

First we investigated the optimality of the passing of jobs and proposed optimal permutation algorithm in general two-machine flow-shop makespan scheduling problem with five independent times, that is, setup and removal times, start and stop lags and transportation times, separated from processing times.

Proposed algorithm includes known algorithm as special cases and constructs an optimal sequence in polynomial-time \(0(n \log n)\).

Next we considered NP-complete three-machine flow-shop makespan scheduling problem with separated sequence-independent setup times alone in order to identify solvable cases and an approximate sequence.

Proposed sufficient optimality conditions for a specified sequence \(\omega_1\) identify the optimality of \(\omega_1\) in polynomial time \(0(n \log n)\).

Therefore, several types of polynomially solvable cases and their inclusion relation and also the superiority of that sequence \(\omega_1\) as an approximate sequence were clarified in our NP-complete problem.

Note that our conditions always yield an optimal sequence \(\omega_1\) having a critical path with corner jobs \(r^o\) and \(s^o\) such that \((F'')\): \(r^o=s^o=u^o\) \((0\leq u^o\leq n)\) holds, that is, generally \((2su^o(sn-1))\) having a critical path with two horizontal and one vertical segments as exemplified in Table 3 \((r^o=s^o=2, \text{ or } 3)\) and Table 5 \((r^o=s^o=3)\).

As already noted, our results are reduced to the results in the usual NP-complete three-machine permutation flow-shop makespan problem and extends known solvable cases when we put every setup times zero. cf. [13] [14] [17]

As further extensions of the results for the present three-machine case, we remark that 1. our results can be further generalized for general NP-complete \(m\)-machine \((m\geq 3)\) permutation case, having, for \(m=4\), additional conditions derived by applying the transformation method [2] first proposed in [15], and 2. including further the transportation times, time lags and/or
suitably restricted removal times in our problem, we can obtain more general complicated results.

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