CONFIDENCE REGION METHOD FOR A STOCHASTIC PROGRAMMING PROBLEM

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Abstract We propose a minimax model with a "quadratic" recourse. In stochastic linear programming models, a decision maker has been assumed to know the probability distribution of random variables. Here we consider the case that the parameters of distribution are unknown. We impose the restrictions on the unknown parameters from the viewpoint of a confidence region, and then seek a minimax solution that minimizes the worst case of the parameters. This model reflects the situation minimizing the maximal possible damage. Especially, the independent normal distribution model is discussed in detail. The analysis for a sufficiently large sample size and a numerical result are given.

1. Introduction

The elements of coefficient matrices of linear programming problems have been assumed to be exactly known. However, in a real life, we never know these values with the full conviction, so we have to include the uncertainty in the formulation of the practical problems. In a usual stochastic programming problem, this uncertainty is viewed as a random model characterized by a probability distribution, and in order to solve the problems under uncertainty, several approaches have been proposed, e.g., two-stage stochastic programs with recourse [6], chance-constrained stochastic programs [1] and so on. In most of these approaches it is assumed that the probability distributions of the random variables are perfectly known. But when the parameters of distributions are unknown, we cannot apply these approaches as they are. In this situation we need to obtain some knowledge about the parameters of distributions in statistical ways. Jagannathan [3] proposed an approach...
with sample informations for a given prior distribution of unknown parameters using a Bayesian approach.

Here we propose a so-called game theoretic minimax approach in section 2. First we impose the restrictions on the unknown parameters which are estimated by a confidence region of them based on random samples drawn from a parent population, and then we seek a minimax solution that minimizes the worst case of the parameters among those restrictions, that is, we minimize the maximal possible damage for a given significance level. In section 3, we investigate the independent normal distribution model in detail. The minimax model is solved numerically using some properties, and the discussion of limiting property is added. Finally we show a numerical example in section 4.

2. Formulation of a Minimax Model

A linear stochastic programming problem is given as follows:

\[ \text{LP: } \text{Minimize } c^T x \]
\[ \text{subject to } Ax = b, \]
where \( c \) and \( x \) are \( n \) real vectors, \( b \) is an \( m \) real vector and \( A \) is an \( m \times n \) real matrix with a full rank. Here we assume that \( A \) and \( c \) are exactly known and \( b \) is a random vector having a distribution function \( F(\cdot;\theta) \) where \( \theta \) is an unknown parameter vector. We consider the following two-stage stochastic programming problem \( P_0 \) with a "quadratic" recourse:

\[ \text{P}_0: \text{Minimize } c^T x + E\left[ \sum_{i=1}^{m} d_i y_i^2 \right] \]
\[ \text{subject to } Ax + y = b, \]
where \( d=(d_1,d_2,\ldots,d_m) \), weight of each constraint infeasibility, is an \( m \)-vector with positive elements, and \( A_i \) is an \( i \)-th row of matrix \( A \). The quadratic recourse is more tractable than a simple recourse in this model and reflects the situation that the infeasibility of constraint has a critical meaning. \( P_0 \) is rewritten to the following problem \( P'_0 \):

\[ \text{P}'_0: \text{Minimize } c^T x + E\left[ \sum_{i=1}^{m} d_i (A_i x - b_i)^2 \right] \]
\[ = c^T x + \int \cdots \int dF(t_1,t_2,\ldots,t_m;\theta) \]

The unknown parameter \( \theta \) is imposed the restrictions estimated by the confidence region \( S \) with a certain significance level. We consider the worst case of the parameter \( \theta \) among \( S \), and then minimize the objective function of
That is, we propose a following minimax model \( P \):

\[
P: \quad \text{Minimize } L = c^T x + \max_{\theta \in S} \int_{i=1}^{m} d_{i}(A_i x - t_i)^2 \, dx
\]

This model reflects on the situation that we should make a decision minimizing the maximal possible damage if the correct value of the parameter \( \theta \) is not known perfectly.

### 3. Normal Distribution Model

Let \( b_i, \ i = 1, \ldots, m \), be an independently, normally distributed random variable, respectively. First we construct the confidence region \( S \) of parameters (i.e., mean and variance). For this purpose, we define the following notations.

- \( \mu_i \) : mean of \( b_i, \ i = 1, \ldots, m \)
- \( \sigma_i^2 \) : variance of \( b_i, \ i = 1, \ldots, m \)
- \( \bar{\mu}_i \) : sample mean of \( b_i, \ i = 1, \ldots, m \)
- \( s_i^2 \) : sample variance of \( b_i, \ i = 1, \ldots, m \)
- \( N \) : sample size
- \( \alpha \) : significance level (\%)
- \( \phi(\cdot) \) : probability density function of independent multivariate standard normal distribution

#### 3.1. Confidence region of parameters

(i) Confidence region of mean \( \mu_i, \ i = 1, \ldots, m \)

Note that the distribution of Hotelling statistics, \( F \),

\[
F = \frac{N(N-m)}{m(N-1)} \sum_{i=1}^{m} \frac{(\mu_i - \bar{\mu}_i)^2}{s_i^2}
\]

is the \( F \)-distribution with \((m,N-m)\) degrees of freedom. Using this statistics \( F \), the confidence region of \( \mu_i, \ i = 1, \ldots, m \), is given by

\[
\sum_{i=1}^{m} \frac{(\mu_i - \bar{\mu}_i)^2}{s_i^2} \leq m(N-1) \frac{F_{\alpha}(m,N-m)}{N(N-m)},
\]

where \( F_{\alpha}(\cdot, \cdot) \) is an \( \alpha \) percentile of \( F \)-distribution.

(ii) Confidence region of variance \( \sigma_i^2, \ i = 1, \ldots, m \)

Since, in general, it is difficult to find the confidence region of the
variance-covariance matrix, we find it approximately by that of each variance for an independent case. The distribution of \((N-1)s_i^2/\sigma^2_i\) is the chi-square distribution with \((N-1)\) degrees of freedom. Therefore the confidence region of \(\sigma^2_i, i=1,\ldots,m\), is given by

\[
\frac{(N-1)s_i^2}{\chi^2_{\beta}(N-1)} \leq \sigma^2_i \leq \frac{(N-1)s_i^2}{\chi^2_{1-\beta}(N-1)}, \quad i=1,\ldots,m,
\]

where \(\beta = \frac{1}{2}a^1/m\) and \(\chi^2_{\beta}(\cdot)\) is a \(\beta\) percentile of chi-square distribution. Then this region is a rectangular region.

From (i) and (ii), the confidence region \(S\) is

\[
S = \left\{ (u_i, \sigma^2_i), i=1,\ldots,m \mid \sum_{i=1}^{m} \frac{(u_i - \bar{u})^2}{s^2_i} \leq \frac{m(N-1)}{N(N-m)} F_{\alpha}(m,N-m), \right. \\
\left. \frac{(N-1)s_i^2}{\chi^2_{\beta}(N-1)} \leq \sigma^2_i \leq \frac{(N-1)s_i^2}{\chi^2_{1-\beta}(N-1)}, \quad i=1,\ldots,m \right\}
\]

### 3.2. Minimax model

We consider the maximizing part of minimax model \(P\) under the above setting, i.e.,

\[
P': \text{Maximize} \quad \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} \sum_{i=1}^{m} d_t(A_i x - t_i)^2 \phi(t_1,\ldots,t_m; u_i, \sigma^2_i) dt_1 \cdots dt_m \\
= \sum_{i=1}^{m} \{ d_i(A_i x - u_i)^2 + d_i \sigma^2_i \}.
\]

Moreover this problem \(P'\) is divided into the following two problems:

\[
P_1: \text{Maximize} \quad L_1 = \sum_{i=1}^{m} d_i \sigma^2_i \\
\text{subject to} \quad \frac{(N-1)s_i^2}{\chi^2_{\beta}(N-1)} \leq \sigma^2_i \leq \frac{(N-1)s_i^2}{\chi^2_{1-\beta}(N-1)}, \quad i=1,\ldots,m,
\]

\[
P_2: \text{Maximize} \quad L_2 = \sum_{i=1}^{m} d_i (A_i x - u_i)^2 \\
\text{subject to} \quad \sum_{i=1}^{m} \frac{(u_i - \bar{u})^2}{s^2_i} \leq \frac{m(N-1)}{N(N-m)} F_{\alpha}(m,N-m) \leq K
\]

Then \(P_1\) corresponds to the variances and \(P_2\) corresponds to the means. Since maximum of \(P_1\) is independent of \(x\), we may regard it as a constant in a minimizing of \(P\) with respect to \(x\). Hence we consider only \(P_2\).

In order to solve \(P\), we show the several properties.
Property 1. $L_2$ is maximized on the boundary of the feasible region, and further the sign of $(\mu^*_i - \bar{\nu}_i)$ is opposite to that of $(A_i \bar{x} - \bar{\nu}_i)$, where $\mu^*_i$ is an optimum of $\nu_i$, $i=1, \ldots, m$.

Proof: It is easily shown that $L_2$ is a convex function in every $\nu_i$, $i=1, \ldots, m$, and so the first part of property 1 is proved directly by the theory of convexity. Actually, $L_2$ is maximized when

$$
\frac{(\mu^*_i - \bar{\nu}_i)^2}{s_i^2} = K_i^2, \quad i=1, \ldots, m,
$$

that is,

$$
\mu^*_i = \bar{\nu}_i \pm s_i K_i, \quad i=1, \ldots, m,
$$

where $K_i \geq 0$ and $\sum_{i=1}^{m} K_i^2 = K$. Since $(A_i \bar{x} - \bar{\nu}_i)^2 = (A_i \bar{x} - \bar{\nu}_i + s_i K_i)^2$, $\mu^*_i$ which maximizes $L_2$ is given as follows:

$$
\begin{align*}
\mu^*_i &= \bar{\nu}_i - s_i K_i & \text{if } A_i \bar{x} - \bar{\nu}_i \geq 0, \\
\mu^*_i &= \bar{\nu}_i + s_i K_i & \text{if } A_i \bar{x} - \bar{\nu}_i < 0.
\end{align*}
$$

Property 2. $P_2$ is transformed into a concave programming problem $P'_2$ by making a variable transformation:

$$
\begin{align*}
z_i &= \frac{(\mu^*_i - \bar{\nu}_i)^2}{s_i^2}, \quad i=1, \ldots, m.
\end{align*}
$$

Proof: From

$$
\frac{z_i^2}{s_i^2} = -\frac{d_i s_i |A_i \bar{x} - \bar{\nu}_i|}{2 z_i^{3/2}} < 0,
$$

$L'_2$ is a concave function. Therefore property 2 follows from property 1.

Solving $P'_2$, the optimum $z_i^*$, $i=1, \ldots, m$, is expressed as follows:

$$
\begin{align*}
z_i^* &= \frac{d_i^2 s_i^2 (A_i \bar{x} - \bar{\nu}_i)^2}{(\lambda - d_i s_i^2)^2},
\end{align*}
$$

where $\lambda$ is the Lagrange multiplier of $L'_2$.

Property 3. $\lambda$ is the largest solution of the equation

$$
\begin{align*}
\sum_{i=1}^{m} \frac{d_i^2 s_i^2 (A_i \bar{x} - \bar{\nu}_i)^2}{(\lambda - d_i s_i^2)^2} = K.
\end{align*}
$$

Proof: Denoting the maximum value of $L'_2$ with $L'_2^*(\lambda)$,
Let $\lambda_1$, $\lambda_2$ ($\lambda_1 < \lambda_2$) be solutions of eq.(3.15), then

\[
\sum_{i=1}^{m} d_i^2 \left( A_i x - \bar{\mu}_i \right)^2 \left( \frac{\lambda}{\lambda - d_i s_i^2} \right)^2 = \sum_{i=1}^{m} d_i^2 \left( A_i x - \bar{\mu}_i \right)^2 \left( \frac{\lambda_2}{\lambda_2 - d_i s_i^2} \right)^2 .
\]

Therefore, from eq.(3.17),

\[
\left( \lambda_1 + \lambda_2 \right) \sum_{i=1}^{m} d_i^2 \left( A_i x - \bar{\mu}_i \right)^2 \left( \frac{\lambda_2 - \lambda_1}{\lambda_1 - d_i s_i^2} \right)^2 = 2 \sum_{i=1}^{m} d_i^2 \left( A_i x - \bar{\mu}_i \right)^2 \left( \frac{\lambda_2 - \lambda_1}{\lambda_2 - d_i s_i^2} \right)^2 .
\]

Using eq.(3.16) and (3.18),

\[
L_2^*(\lambda_2) - L_2^*(\lambda_1) = \frac{\left( \lambda_2 - \lambda_1 \right)^3}{2} \sum_{i=1}^{m} d_i^2 \left( A_i x - \bar{\mu}_i \right)^2 \left( \frac{\lambda_2 - \lambda_1}{\lambda_2 - d_i s_i^2} \right)^2 > 0 .
\]

Hence $\lambda$ is a largest solution of eq.(3.15).

**Property 4.** $\lambda > \lambda_0 \iff \max_{i \in I} d_i s_i^2$ , where $I$ is the set of indices such that $A_i x - \bar{\mu}_i \neq 0$.

**Proof:** From eq.(3.15), we define

\[
G(\lambda) \triangleq \sum_{i=1}^{m} d_i^2 \left( A_i x - \bar{\mu}_i \right)^2 \left( \frac{\lambda}{\lambda - d_i s_i^2} \right)^2 - K .
\]

Since

\[
\lim_{\lambda \to 0} G(\lambda) = +\infty > 0 ,
\]

\[
\lim_{\lambda \to +\infty} G(\lambda) = -K < 0 ,
\]

and

\[
G'(\lambda) = -2 \sum_{i \in I} \frac{d_i^2 \left( A_i x - \bar{\mu}_i \right)^2}{\left( \lambda - d_i s_i^2 \right)^3} > 0 ,
\]

the unique solution exists in $(\lambda_0, +\infty )$. □

Consequently, the problem $P$ is expressed as follows:

\[
P' : \text{Minimize } \quad L = c \ x + \sum_{i=1}^{m} d_i^2 \left( A_i x - \bar{\mu}_i \right)^2 \left( \frac{\lambda}{\lambda - d_i s_i^2} \right)^2
\]

\[
\text{subject to } \quad \sum_{i=1}^{m} \frac{d_i^2 \left( A_i x - \bar{\mu}_i \right)^2}{\left( \lambda - d_i s_i^2 \right)^2} = K, \quad \lambda > \lambda_0 .
\]
where we neglect the constant associated with $P_1$. In order to solve this problem $P'$, we utilize the following property.

**Property 5.** For $\lambda \geq \frac{3}{2} \lambda_0$, $P'$ is a convex programming problem.

**Proof:** It is sufficient to show that the Hessian matrix of the objective function is positive semi-definite for $\lambda \geq \frac{3}{2} \lambda_0$. The elements of the Hessian matrix, $H=(h_{ij})$, $i,j=1,\ldots,n+1$, are given as follows:

$$h_{ij} = \frac{\partial^2 L}{\partial x_i \partial x_j} = 2 \sum_{k=1}^{m} d_k a_{ki} a_{kj} \frac{\lambda^2}{(\lambda-d_k s_k^2)^2}, \quad i,j=1,\ldots,n,$$

(3.25) \hspace{1cm} $h_{n+1,i} = h_{i,n+1} = \frac{\partial^2 L}{\partial x_i \partial \lambda} = -4 \sum_{k=1}^{m} d_k a_{ki} (A_k x - \mu_k) \frac{\lambda d_k s_k^2}{(\lambda-d_k s_k^2)^2}, \quad k=1,\ldots,n$,

$$h_{n+1,n+1} = \frac{\partial^2 L}{\partial \lambda^2} = 2 \sum_{k=1}^{m} d_k (A_k x - \mu_k)^2 \frac{d_k s_k^2 (2\lambda+d_k s_k^2)}{(\lambda-d_k s_k^2)^2},$$

where $a_{ij}$ denotes the $(i,j)$-th element of matrix $A$. Note that $H$ is divided into two matrices, that is,

(3.26) \hspace{1cm} $H = \begin{bmatrix} u_1 & \cdots & u_n & v_1 & \cdots & v_n \end{bmatrix} + \begin{bmatrix} 0 \\ \vdots \\ w \end{bmatrix},$

where $u_i$, $i=1,\ldots,n$, and $v$ are m-vectors with $u_i = \frac{\sqrt{2d_i} \lambda}{\lambda-d_i s_i^2} a_{ji}$, and $v_j = \frac{2\sqrt{2d_i}(A_k x - \mu_k) d_i s_i^2}{(\lambda-d_i s_i^2)^2}$ as $j$-th element, $j=1,\ldots,m$, respectively, and

(3.27) \hspace{1cm} $w = 2 \sum_{k=1}^{m} \frac{(A_k x - \mu_k)^2 d_k s_k^2}{(\lambda-d_k s_k^2)^2} (2\lambda-3d_k s_k^2).$

Since for $\lambda \geq \frac{3}{2} \lambda_0$, $w$ is non-negative, $H$ is expressed as the sum of two positive semi-definite matrices. Therefore $H$ is positive semi-definite. \qed

In order to solve this problem $P'$, we should consider several cases according to the value of $\lambda$ because of the dependency on $x$. Define

(3.28) \hspace{1cm} $\lambda_{\text{max}} = \max_{1 \leq i \leq n} d_i s_i^2.$

**Case 1.** $\lambda \geq \frac{3}{2} \lambda_{\text{max}}$.

Since it is clear that $\lambda_0 \leq \lambda_{\text{max}}$, from property 5, the problem $P'$ is a convex program. Then we use some approaches for a constrained non-linear programming problem, e.g., the steepest descent method with a penalty func-

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tion.

Case 2. \( \lambda_0 < \lambda < \frac{3}{2} \lambda_{\text{max}} \):

For fixed \( \lambda \), \( L \) is convex with respect to \( x \). Then, for several fixed \( \lambda \) between \( \lambda_0 \) and \( \frac{3}{2} \lambda_{\text{max}} \), we seek optimal solutions and choose the best ones among them. But we must divide \( \lambda \) into the following some intervals because the constraints differ on each interval. For a sake of convenience, we rearrange \( d_i s_i^2 \), \( i=1, \ldots, m \), in an increasing order with its constraint. By the definition of \( \lambda_0 \),

\[
(3.29) \quad \lambda_0 = d_m s_m^2
\]

means

\[
(3.30) \quad A_{i,x} \neq \mu_i, \quad i=1, \ldots, m,
\]

and

\[
(3.31) \quad \lambda_0 = d_p s_p^2, \quad \text{for } 1 \leq p < m,
\]

means

\[
(3.32) \quad A_{i,x} \neq \mu_i, \quad i=1, \ldots, p,
\]

\[
A_{i,x} = \mu_i, \quad i=p+1, \ldots, m.
\]

Thus \( p \) denotes the largest index such that \( A_{i,x} \neq \mu_i \). For \( \lambda_0 = d_p s_p^2, \ 1 \leq p < m \), the first constraint of \( P' \) is relaxed as follows by a way of compensation that the equality constraints \( A_{i,x} = \mu_i, \ i=p+1, \ldots, m \), are imposed. Moreover, since \( \text{rank } A = n \), the constraints eq. (3.32) should be infeasible for \( p=m-n \). That is, the constraints of problem \( P' \) are partitioned into \( n \) cases as follows.

\[
(3.33) \quad \sum_{i=1}^{p} \frac{d_i s_i^2 (A_{i,x} - \mu_i)^2}{\lambda_i (\lambda - d_i s_i^2)^2} = K,
\]

\[
(3.34) \quad \sum_{i=1}^{m} \frac{d_i s_i^2 (A_{i,x} - \mu_i)^2}{\lambda_i (\lambda - d_i s_i^2)^2} = K,
\]

\[
\text{for } m-n \leq p < m,
\]

Hence we need to obtain an optimal solution for every \( p \).

Choosing the best optimal solution among optimal ones of above two cases, we obtain the global optimal solutions. Next we show the existence of this best optimal solution.

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Lemma 1. The objective function $L$ is bounded subject to eq.(3.33) or eq.(3.34).

Proof: Assume that the optimal solution is unbounded, then since the denominator of the first constraint is bounded, the left hand side of it must be unbounded. This contradicts boundness of $K$. 

Lemma 2. The solution of $P'$ is continuous with respect to $\lambda$.

Proof: Let $x_1$ and $x_2$ be optimal solutions when $\lambda = \lambda_1$ and $\lambda = \lambda_2$, respectively. It is easily shown that $L(x_1) \rightarrow L(x_2)$ as $\lambda_2 \rightarrow \lambda_1$ and the solution is unique because this problem has a strictly convex objective function and a convex feasible region for fixed $\lambda$. From this convexity the continuity of the solution is shown.

We can find the solution of this problem approximately with discretizing value of $\lambda$.

Theorem. We can obtain the approximately global optimal solution by means of choosing the best optimal solution among optimal ones for each discretized $\lambda$.

3.3. Limiting property

We discuss the problem with sufficiently large sample size. We see, from a consistency of sample mean and variance, that the correct value of parameters can be known as $N \rightarrow \infty$. With these parameters, $P_0$ is rewritten to a problem $PL$ with perfect information as follows:

\begin{equation}
PL: \text{Minimize} \quad c^T x + \sum_{i=1}^{m} \left\{ d_i (A_i x - u_i)^2 + d_i \sigma_i^2 \right\}.
\end{equation}

Here we show that our proposed model with finite sampling with size $N$, say $P$, tends to $PL$ as $N \rightarrow \infty$.

Lemma 3. Let $M$ and $N$ be positive integers, then

\begin{equation}
\lim_{N \rightarrow \infty} \frac{\chi^2_{M}(N)}{N} = 1
\end{equation}

and

\begin{equation}
\lim_{N \rightarrow \infty} \frac{F_{\alpha}(M,N)}{N} = 0,
\end{equation}

where $0 < \alpha < 1$.

Proof: As in [4], with the Cornish-Fisher expansion,

\begin{equation}
\sqrt{2\chi^2_{M}(N)} - \sqrt{2N-1} = u_{\alpha} + O\left(\frac{1}{\sqrt{N}}\right),
\end{equation}

where $u_{\alpha}$ is a percentile of a standard normal distribution. Since $|u_{\alpha}| < \infty$,
for $0 < a < 1$, eq. (3.36) is proved.

Denoting the probability density function of $F$-distribution with $(M, \infty)$ degrees of freedom with $f_{M, \infty}(t)$,

\[
(3.39) \quad f_{M, \infty}(t) = \frac{1}{\Gamma\left(M/2\right)} \left(M/2\right)^{-M/2} t^{M/2 - 1} e^{-M/2 t},
\]

it is clear that an $a$ percentile of this distribution, $F_a(M, \infty)$, finitely exists.

Remark 1. From lemma 3, the problem $P$ with unknown parameters tends to the problem $P_L$ with known parameters as a sample size $N$ tends to infinity.

Remark 2. In the problem $P'$, $\lambda$ tends to infinity as $N \to \infty$.

Remark 3. Note that sample variances are not concerned with a decision making for a sufficiently large sample size. They are concerned with only the magnitude of the recourse.

Remark 4. This proposed minimax model would be useful to the other stochastic programming problems which the parameters of their random variables are unknown.

4. Numerical Example

Consider a following problem:

\[
(4.1) \quad \text{PE: Minimize} \quad 2x_1 + x_2
\]

subject to $x_1 + x_2 = b_1$

$2x_1 - x_2 = b_2$

$x_2 = b_3$

where $b_1$, $b_2$, and $b_3$ are independently, normally distributed random variables with unknown parameters, and let a weight vector $d = (10, 5, 10)$. Using an artificial result of sampling of a computer simulation, we give the sample means and variances in Table 1.

Table 1. Sample means and variances ($N=11$)

<table>
<thead>
<tr>
<th></th>
<th>$\bar{b}$</th>
<th>$s^2$</th>
<th>$d \cdot s^2$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$b_1$</td>
<td>2.979</td>
<td>0.007</td>
<td>0.070</td>
</tr>
<tr>
<td>$b_2$</td>
<td>0.056</td>
<td>0.360</td>
<td>1.800</td>
</tr>
<tr>
<td>$b_3$</td>
<td>1.020</td>
<td>0.043</td>
<td>0.430</td>
</tr>
</tbody>
</table>

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In this case, the problem $P_E$ is transformed into the following our model:

\[(4.2) \quad P_E': \text{Minimize } \quad 2x_1 + x_2 + 10(x_1 + x_2 - 2.979)^2/(\lambda - 0.070)^2
\] \[+ 5(2x_1 - x_2 - 0.056)^2/(\lambda - 1.800)^2
\] \[+ 10(x_2 - 1.020)^2/(\lambda - 0.430)^2, \]

subject to \[10(x_1 + x_2 - 2.979)^2 \cdot 0.070/(\lambda - 0.070)^2
\] \[+ 5(2x_1 - x_2 - 0.056)^2 \cdot 1.800/(\lambda - 1.800)^2
\] \[+ 10(x_2 - 1.020)^2 \cdot 0.430/(\lambda - 0.430)^2 = 1.388, \]

where a significance level is set to 0.05. The optimal solutions for each case are given in Table 2.

Table 2. The optimal solutions (1)

<table>
<thead>
<tr>
<th>Case</th>
<th>range of $\lambda$</th>
<th>$\lambda^*$</th>
<th>$x_1^*$</th>
<th>$x_2^*$</th>
<th>$L^*$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>$\lambda \geq 2.70$</td>
<td>2.700</td>
<td>0.736</td>
<td>1.710</td>
<td>16.809</td>
</tr>
<tr>
<td>2</td>
<td>$1.80 &lt; \lambda \leq 2.70$</td>
<td>1.802</td>
<td>0.803</td>
<td>1.551</td>
<td>12.244</td>
</tr>
<tr>
<td></td>
<td>$p=3$</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>2</td>
<td>$0.43 &lt; \lambda \leq 1.80$</td>
<td>1.471</td>
<td>0.821</td>
<td>1.587</td>
<td>13.232</td>
</tr>
<tr>
<td></td>
<td>$p=2$</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>2</td>
<td>$0.07 &lt; \lambda \leq 0.43$</td>
<td>0.430</td>
<td>0.538</td>
<td>1.020</td>
<td>31.386</td>
</tr>
<tr>
<td></td>
<td>$p=1$</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

The best optimal solution among these ones is $(\lambda^*, x_1^*, x_2^*) = (1.802, 0.803, 1.551)$ and $L^* = 12.244$. Furthermore, in Table 3, the best optimal solutions for some sample sizes are given.

Table 3. The optimal solutions (2)

<table>
<thead>
<tr>
<th>$N$</th>
<th>$\lambda^*$</th>
<th>$x_1^*$</th>
<th>$x_2^*$</th>
<th>$L^*$</th>
</tr>
</thead>
<tbody>
<tr>
<td>11</td>
<td>1.802</td>
<td>0.803</td>
<td>1.551</td>
<td>12.244</td>
</tr>
<tr>
<td>100</td>
<td>5.870</td>
<td>0.825</td>
<td>1.710</td>
<td>11.355</td>
</tr>
<tr>
<td>300</td>
<td>8.642</td>
<td>0.820</td>
<td>1.649</td>
<td>10.640</td>
</tr>
<tr>
<td>1000</td>
<td>16.270</td>
<td>0.840</td>
<td>1.651</td>
<td>10.271</td>
</tr>
<tr>
<td>$\infty$</td>
<td>-</td>
<td>0.967</td>
<td>1.695</td>
<td>9.887</td>
</tr>
</tbody>
</table>

5. Conclusion

We have proposed a minimax model. When the distribution of parameters of random variables are assumed to be unknown, our approach is seemed to be not only reasonable but useful. It notes that the maximizing process with respect to variances does not affect upon an optimal decision of our model.

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though they are unknown. Furthermore we show that, for a sufficiently large sample size, our model tends to one with a perfect information. Hereafter the better solving algorithm should be considered. But it is ascertained the non-convexity of the objective function by a simulation, so it is remained a difficulty on solving our model.

Appendix. Derivation of Problem $P'$ by a Duality Theorem

The problem (3.34) $P'$ which is equivalent to the problem (2.4) $P$ for a normal model is also derived by a duality theorem \[2\]. To solve the maximizing process of minimax problem $P$, it is sufficient to consider only the problem (3.37) $P_2$.

From \[2\], Theorem 2.1, 

\[
(A.1) \quad \sup_{\mu} \left\{ L_2 = \sum_{i=1}^{m} d_i (A_2 x - \mu_i^2)^2 \right\} \leq K
\]

\[
= \inf_{\lambda} \left\{ \sup_{\mu} \{ L(\mu, \lambda) = \sum_{i=1}^{m} d_i (A_2 x - \mu_i^2)^2 + \lambda \left( \sum_{i=1}^{m} \frac{(\mu_i - \bar{u}_i)^2}{s_i^2} \right) \} \right\} \lambda \geq 0
\]

Then for $\lambda > \lambda_0$, $L(\mu, \lambda)$ is maximized at $\mu = \hat{\mu}_2$, $\hat{\mu}_2 = \hat{\mu}_i^2$, $i = 1, \ldots, m$, where

\[
(A.2) \quad \hat{\mu}_i = \frac{\lambda \mu_i - \sum d_i s_i^2 A_2 x_i}{\lambda - \sum d_i s_i^2}
\]

and

\[
(A.3) \quad L(\hat{\mu}, \lambda) = \lambda K + \sum_{i=1}^{m} \frac{\lambda d_i (A_2 x - \mu_i^2)^2}{\lambda - \sum d_i s_i^2}
\]

From $dL(\hat{\mu}, \lambda)/d\lambda = 0$,

\[
(A.4) \quad K = \sum_{i=1}^{m} \frac{d_i s_i^2 (A_2 x - \mu_i^2)^2}{(\lambda - \sum d_i s_i^2)^2}
\]

and

\[
(A.5) \quad L(\hat{\mu}, \lambda) = \sum_{i=1}^{m} d_i (A_2 x - \mu_i^2)^2 \left( \frac{\lambda}{\lambda - \sum d_i s_i^2} \right)^2
\]

Therefore the problem (3.24) $P'$ is obtained, where the constraint $\lambda > \lambda_0$ is found from property 4.
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References


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