OPTIMAL SELECTION PROBLEM WITH THREE STOPS

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Abstract Optimal selection problem with multiple choices, like secretary, dowry or marriage problems have attracted attention of many applied mathematicians and are also of great significance for those who are looking for “the best partner”. In this paper we consider a variation of the optimal selection problem with three stops allowed which is often referred to as the secretary problem with multiple choices where the objective is to find an optimal stopping rule so as to maximize the probability of selecting three absolute bests.

We also present a set of dynamic programming equations for the problems both of selecting the best object with three stops and of selecting two absolute bests with three stops. For those problems the optimal stopping rules are numerically calculated in aid of computer programming.

1. Introduction Optimal selection problems with multiple choices, like secretary, dowry or marriage problems, have attracted attention not only of many applied mathematicians but also of economists for more than 30 years and are also of great significance for those who are looking for “the best partner”. Starting from the paper by Gilbert and Mosteller [2], various models of the optimal selection problem have been proposed. For example see [3], [4], [5], [8] and [9].

In this paper we consider a variation of the optimal selection problem with three stops, often referred to as the secretary problem which possesses the following framework: The $N$ objects which can be totally ordered according to some criterion and appear in a sequential order at random, which implies that all permutations are equally likely. At each time an object appears the decision maker is able to rank the objects appeared so far according to the above criterion. Whenever an object appears, after observing its relative rank the decision maker must either accept it and then stop or reject it and continue to observe further incoming objects. He is asked to find a stopping rule which maximizes the probability of selecting the best object within three stops.

Even though many versions of the original secretary problem have been studied by several authors like Gilbert and Mosteller [2], Tamaki [4], Stadje [5] and Sakaguchi [8], the “No-Information” secretary problem with three stops has yet been unsolved. The “No-Information” problem as treated later in §2, §3 and §4 is described as follows: The decision maker is allowed to have three stops (choices) and his objective is to choose the three absolute best objects among $N$ objects which randomly appear in a sequential order.

We derive an optimal stopping rule and the maximum probability of selecting the three absolute best objects by using OLA policy which is proposed by Ross [6]. This policy can be defined for a class of Markov decision processes whose optimality equation is given by

$$V(i) = \max\{R(i), -C(i) + \sum_{j=i+1}^{N} p_{ij} V(j)\}, \quad i = 1, 2, \ldots, N - 1, \quad V(N) = R(N).$$
where the state space is finite, \(\{1, 2, \cdots, N\}\) with the particular transition probabilities \(\{p_{ij} \mid p_{ij} \geq 0 \text{ for } i < j, \quad p_{ij} = 0 \text{ for } i \geq j\}\). The decision maker facing state \(i\) must decide whether to accept terminal reward \(R(i)\) and stop, or by paying the cost \(C(i)\), to move to the next state \(j\) according to transition probability \(p_{ij}\). Let \(V(i)\) be the maximum of the expected reward when starting from an initial state \(i\). Define

\[
B \equiv \{N\} \cup \{i \mid R(i) \geq -C(i) + \sum_{j=i+1}^{N} p_{ij} R(j), \quad 1 \leq i \leq N-1\},
\]

which represents the set of states in which immediate stopping is at least as good as continuing for exactly one more period and then stopping. The policy that indicates stopping at \(i\) if and only if \(i \in B\) is called the OLA policy. \(B\) is called closed if there exists an integer \(i^*\) such that \(B = \{i \mid i^* \leq i \leq N\}\). Ross [6] proves that if \(B\) is closed, then such OLA policy is optimal.

In §5 we present a set of dynamic programming (abbreviated by DP) equations for the problem of selecting the best object with three stops and derive the optimal stopping policy and the probability of achieving this object through numerical calculations. Furthermore, we derive a set of DP equations for the two absolute best objects with three stops by analogous numerical approach. Such a numerical approach in aid of computer programming is quite useful in deriving an optimal stopping rule for any \(N\) whenever a set of DP equations is available.

Gilbert & Mosteller [2] provides some numerical results for the problem of selecting the best object with multiple choice, and our numerical results are inspired by their work. It is important to note that for such a type of multi-choice problems, if we derive a set of DP equations, then the computer programming approach gives the optimal stopping policy for any \(N\).

2. Selecting three absolute bests—Model

The \(i\)-th best object among the objects observed so far is called “\(i\)-candidate” and denoted by \(C_i\). We need only \(C_1, C_2\) and \(C_3\) throughout this paper, since other candidates cannot be chosen. Let \(N\) be the number of objects. We are allowed to make three stops and “win” if the accepted are the three absolute best ones. We are asked to find a stopping policy that maximizes the probability of win.

Three kinds of states are defined as follows: Suppose the decision-maker is facing the \(i\)-th object. The state \((i; C_k; 1; \ell)\), \(k = 1, 2, 3; \ell = 1, 2\), means that (1) the \(i\)-th object is \(C_k\), (2) the first two stops were made at \(C_1\) and \(C_\ell\) in this order, and (3) still one stop left to be made. The state \((i; C_k; 2)\), \(k = 1, 2\), means that (1) the \(i\)-th object is \(C_k\) and (2) still two stops left to be made. The state \((i; 3)\) means that (1) the \(i\)-th object is \(C_1\), and (2) three stops left to be made. An optimal policy satisfies the following condition: if an object is better (worse) than some previously accepted (rejected) one, then accept (reject) it. Hence the optimal first stop should be made at \(C_1\) and the optimal second stop should be made either at \(C_1\) or at \(C_2\).

We denote by \(V_1(i; C_k; 1\ell), V_2(i; C_k),\) and \(V_3(i)\), the probability of win under an optimal policy when starting from the state \((i; C_k; 1\ell), (i; C_k; 2)\) and \((i; 3)\), respectively. Given the \(i\)-th object is \(C_1\), the conditional probability that the \(j\)-th object \((i < j \leq N)\) is the earliest \(C_1\) is \(i/j(j-1)\). Given the \(i\)-th object is \(C_1\), the conditional probability that the \(j\)-th object \((i < j \leq N)\) is the earliest \(C_2\) is \(i(i-1)/j(j-1)(j-2)\). And given the \(i\)-th is \(C_1\), the conditional probability that the \(j\)-th object \((i < j \leq N)\) is the earliest \(C_2\) or the earliest \(C_3\) is \(i(i-1)(i-2)/j(j-1)(j-2)(j-3)\). Thus we obtain the
following set of DP equations.

\[ V_1(i, C_k, 1\ell) = \frac{i(i-1)(i-2)}{N(N-1)(N-2)}, \quad (3 \leq i \leq N-1, \ k = 1, 2, \ \ell = 1, 2), \quad (2.1) \]

\[ V_1(N, C_k, 1\ell) = 1. \quad \{ \text{stop is optimal} \} \]

\[ V_1(i, C_3, 1\ell) = \max\left\{ \frac{i(i-1)(i-2)}{N(N-1)(N-2)}, \sum_{j=i+1}^{N} \frac{i(i-1)(i-2)}{(j-1)(j-2)(j-3)} \left[ \sum_{k=1}^{3} V_1(j, C_k, 1\ell) \right] \right\}, \quad (2.2) \]

\[ V_1(N, C_3, 1\ell) = 1. \]

\[ V_2(i, C_1) = \sum_{j=i+1}^{N} \frac{i(i-1)(i-2)}{(j-1)(j-2)(j-3)} \left[ \sum_{k=1}^{3} V_1(j, C_k, 1\ell) \right], \quad 3 \leq i \leq N-1, \quad (2.3) \]

\[ V_2(2, C_1) = \frac{1}{3} \left\{ \frac{12}{N(N-1)(N-2)} + V_1(3, C_3, 1\ell) \right\}. \quad \{ \text{stop is optimal} \} \]

\[ V_2(i, C_2) = \max\left\{ \sum_{j=i+1}^{N} \frac{i(i-1)(i-2)}{(j-1)(j-2)(j-3)} \left[ \sum_{k=1}^{3} V_1(j, C_k, 1\ell) \right], \sum_{j=i+1}^{N} \frac{i(i-1)}{(j-1)(j-2)} \left[ \sum_{k=1}^{2} V_2(j, C_k) \right] \right\}, \quad 3 \leq i \leq N-1, \quad (2.4) \]

\[ V_2(2, C_2) = \max\left\{ \frac{1}{3} \left[ \frac{12}{N(N-1)(N-2)} + V_1(3, C_3, 12) \right], \sum_{j=3}^{N} \frac{2}{(j-1)(j-2)} \left[ \sum_{k=1}^{2} V_2(j, C_k) \right] \right\} \]

\[ V_3(i) = \max\left\{ \sum_{j=i+1}^{N} \frac{i(i-1)}{(j-1)(j-2)} \left[ \sum_{k=1}^{2} V_2(j, C_k) \right], \sum_{j=i+1}^{N} \frac{i}{(j-1)} V_3(j) \right\}, \quad 2 \leq i \leq N-1, \quad V_3(N) = 0, \quad (2.5) \]

\[ V_3(1) = \max\left\{ \frac{1}{6} \left[ \frac{12}{N(N-1)(N-2)} + V_1(3, C_3, 11) \right] + \frac{1}{2} V_2(2, C_2), \sum_{j=2}^{N} \frac{1}{j(j-1)} V_3(i) \right\} \]

The first expression inside the max \{ \cdots \} is the probability of win due to immediate stopping in current state and the second one inside the max \{ \cdots \} is the probability of win due to continuing and behaving optimally hereafter. For computational convenience, we introduce the following notation:

\[ F_1(i) \equiv V_1(i, C_k, 1\ell)/i(i-1)(i-2), \ \ k = 1, 2, \ \ell = 1, 2, \ 3 \leq i \leq N-1, \]

\[ F_2(i) \equiv V_1(i, C_3, 1\ell)/i(i-1)(i-2), \ \ell = 1, 2, \ 3 \leq i \leq N-1, \]

\[ F_3(i) \equiv V_2(i, C_1)/i(i-1)(i-2), \ 3 \leq i \leq N-1, \]

\[ F_4(i) \equiv V_2(i, C_2)/i(i-1)(i-2), \ 3 \leq i \leq N-1, \]

\[ F_5(i) \equiv V_3(i)/i(i-1), \ 2 \leq i \leq N-1, \]

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Then we have the following set of "reduced" DP equations.

\[ F_1(i) = \frac{1}{N(N - 1)(N - 2)}, \quad 3 \leq i \leq N. \quad (R1) \]

\[ F_2(i) = \max \left\{ \frac{1}{N(N - 1)(N - 2)}, \sum_{j=i+1}^{N} \frac{1}{j-3} \frac{2}{N(N - 1)(N - 2)} + F_2(j) \right\}, \quad 3 \leq i \leq N - 1, \quad (R2) \]

\[ F_2(N) = \frac{1}{N(N - 1)(N - 2)}. \]

\[ F_3(i) = \sum_{j=i+1}^{N} \frac{1}{j-3} \frac{2}{N(N - 1)(N - 2)} + F_2(j), \quad 3 \leq i \leq N - 1, \quad (R3) \]

\[ F_3(N) = 0, \quad V_2(2, C_1) = \frac{4}{N(N - 1)(N - 2)} + 2F_2(3). \]

\[ F_4(i) = \max \{ F_3(i), \frac{1}{i-2} \sum_{j=i+1}^{N} [F_3(j) + F_4(j)] \}, \quad 3 \leq i \leq N - 1, \quad (R4) \]

\[ F_4(i) = 0, \quad V_2(2, C_2) = \max \left\{ \frac{4}{N(N - 1)(N - 2)} + 2F_2(3), 2 \sum_{j=3}^{N} [F_3(j) + F_4(j)] \right\}. \]

\[ F_5(i) = \max \left\{ \sum_{j=i+1}^{N} [F_3(j) + F_4(j)], \frac{1}{i-1} \sum_{j=i+1}^{N} F_5(j) \right\}, \quad 2 \leq i \leq N - 1, \quad (R5) \]

\[ F_5(N) = 0, \quad V_3(I) = \max \left\{ \frac{2}{N(N - 1)(N - 2)} + F_2(3) + \frac{1}{2} V_2(2, C_2), \sum_{j=2}^{N} F_5(j) \right\}. \]

3. Selecting three absolute bests—Optimal stopping times

The optimal first stopping time is determined by \((R5)\), the optimal second by \((R3) \& (R4)\), and the optimal third by \((R1) \& (R2)\).

Theorem 1.

The optimal third stop is made at the earliest \(C_1\) after the second stop, at the earliest \(C_2\) after the second stop, or at the earliest \(C_3\) after \(r_1\), where \(r_1\) is given by

\[ r_1 = \min \{ i \mid \sum_{j=1}^{N-1} \frac{1}{j-2} \leq \frac{1}{3}, \quad 3 \leq i \leq N - 1 \}. \quad (3.1) \]

Proof. For \((R2)\), we have the OLA stopping region

\[ B_1 \equiv (N, C_3, 1\ell; 1) \cup \{(i, C_3, 1\ell; 1) \mid \frac{1}{N(N - 1)(N - 2)} \geq \sum_{j=i+1}^{N} \frac{1}{j-3} \frac{3}{N(N - 1)(N - 2)} \}, \quad (3.2) \]

for \(\ell = 1, 2\). Then \(B_1 = \{(i, C_3, 1\ell; 1) \mid r_1 \leq i \leq N\}\) and it turned out to be closed. Thus the theorem is proved. \(\square\)

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For this optimal third stopping policy, we obtain by \((R2)\),

\[
F_2(i) = \begin{cases} 
  \frac{1}{N(N - 1)(N - 2)(i - 2)} \{(r_1 - 3)F_2(r_1 - 1) + 2(r_1 - i - 1)\}, & 3 \leq i \leq r_1 - 1 \\
  \frac{1}{N(N - 1)(N - 2)}r_1 \leq i \leq N, 
\end{cases}
\]

where

\[
F_2(r_1 - 1) = 3 \cdot \sum_{j=r_1-1}^{N-1} \frac{1}{j - 2}.
\]

Let

\[
K(i) = \frac{1}{i - 3} \{F_2(i) + \frac{2}{N(N - 1)(N - 2)}\}
\]

for \(i \geq 4\). Then we have by \((3.3)\),

\[
K(i) = \begin{cases} 
  \frac{r_1 - 3}{N(N - 1)(N - 2)(i - 2)(i - 3)} \{F_2(r_1 - 1) + 2\}, & 4 \leq i \leq r_1 - 1 \\
  \frac{1}{N(N - 1)(N - 2)(i - 3)}, & r_1 \leq i \leq N, 
\end{cases}
\]

and

\[
F_3(i) = \begin{cases} 
  \frac{1}{N(N - 1)(N - 2)(i - 2)} \{(r_1 - 3)F_2(r_1 - 1) + 2(r_1 - i - 1)\}, & 3 \leq i \leq r_1 - 1 \\
  \frac{3}{N(N - 1)(N - 2)} \sum_{j=r_1+1}^{N} \frac{1}{j - 3}, & r_1 \leq i \leq N - 1. 
\end{cases}
\]

For \((R3)\), we have the OLA stopping region,

\[
B_2 \equiv (N, C_2; 2) \cup (N - 1, C_2; 2)
\]

\[
\cup \left\{ (i, C_2; 2) \mid \frac{4}{N(N - 1)(N - 2)} + 2F_2(3) \geq 2 \sum_{j=3}^{N-1} 2 \sum_{k=j+1}^{N} K(k), \quad i = 2 \right\}.
\]

\[
\sum_{j=i+1}^{N} K(j) \geq \frac{1}{i - 2} \sum_{j=i+1}^{N-1} 2 \sum_{k=j+1}^{N} K(k), \quad 3 \leq i \leq N - 2
\]

\[
B_2 \text{ can be written as}
\]

\[
B_2 = (N, C_2; 2) \cup \{(i, C_2; 2) \mid \Phi_1(i) \geq 0, \ 1 \leq i \leq N - 1\}
\]

where

\[
\Phi_1(i) = \begin{cases} 
  2 \sum_{j=4}^{N} (3 - j)K(j) + F_2(3) + \frac{2}{N(N - 1)(N - 2)}, & i = 2, \\
  \frac{1}{N(N - 1)(N - 2)} \{3(r_1 - 3)F_2(r_1 - 1) + 6(2r_1 - N - i - 2)
  \\
  - 2(r_1 - 3)(F_2(r_1 - 1) + 2) \sum_{j=i}^{r_1-2} \frac{1}{j - 1}\}, & 3 \leq i \leq r_1 - 2, \\
  \frac{1}{N(N - 1)(N - 2)} \{9(i - 2) \sum_{j=i}^{N-1} \frac{1}{j - 2} - 6(N - i)\}, & r_1 - 1 \leq i \leq N - 1.
\end{cases}
\]
Theorem 2.

The optimal second stop is made either at the earliest \( C_1 \) after the first stop, or at the earliest \( C_2 \) after \( r_2 \), where \( r_2 \) is given by,

\[
r_2 = \min \{ i : 3(r_1 - 3)F_2(r_1 - 1) + 6(2r_1 - N - i - 2)
- 2(r_1 - 3)(F_2(r_1 - 1) + 2) \sum_{j=i}^{r_2} \frac{1}{j - i} \geq 0, \quad 2 \leq i \leq N - 1 \}.
\] (3.11)

Proof. It is sufficient to show that \( B_2 \) is closed. This statement is equivalent to showing that, if there exists an integer \( r_2 \) such that \( \Phi_1(r_2) \geq 0 \), then \( \Phi_1(i) \) is also non-negative for any \( i \geq r_2 \). Let \( \Phi_1(i) \equiv N(N - 1)(N - 2)\Phi_1(i) \). We have for \( i \geq r_1 - 1 \),

\[
\Phi_1(i) - \Phi_1(i + 1) = 9\left( \frac{5}{3} - \sum_{j=i+1}^{N-1} \frac{1}{j - 2} \right) \geq 0,
\] (3.12)

by the definition of \( r_1 \). Therefore \( \Phi_1(i) \) is a non-increasing function for \( r_1 - 1 \leq i \leq N - 1 \) and \( \Phi_1(N - 1) = 3 > 0 \). On the other hand, we have for \( 3 \leq i \leq r_1 - 3 \),

\[
\Phi_1(i) - \Phi_1(i + 1) = -4\left\{ \frac{r_1 - 3}{i - 1} - \frac{3}{2} + \frac{r_1 - 3}{2(i - 1)}F_2(r_1 - 1) \right\}
\leq -4\left\{ \frac{3}{2} \left( \frac{r_1 - 3}{i - 1} - 1 \right) \right\} < 0,
\] (3.13)

by using the fact \( F_2(r_1 - 1) > 1 \). Therefore \( \Phi_1(i) \) is a non-decreasing function for \( 3 \leq i \leq r_1 - 2 \). Since \( \Phi_1(2) \) can be rewritten as

\[
\Phi_1(2) = \Phi_1(3) - \sum_{j=4}^{N} K(j) - 2 \sum_{j=5}^{N} K(j)
\] (3.14)

and \( K(i) > 0 \) for all \( i \), \( \Phi_1(2) \leq 0 \) if \( \Phi_1(3) \leq 0 \). Thus \( \Phi_1(i) \) is proved to be unimodal. Therefore \( B_2 \) is closed and the states \((r_1, C_3, 1; \ell, 1)\), \( \ell = 1, 2 \), belong to \( B_2 \). Thus the proof is completed.

For this optimal second stopping policy, we obtain

\[
F_4(i) = \begin{cases} 
1 & \frac{1}{N(N - 1)(N - 2)(i - 1)(i - 2)} \{(r_1 - 3)(r_2 - 1 - i)(F_2(r_1 - 1) + 2) \\
- (r_2 - 1 - i)(r_2 + i - 4) + (r_2 - 2)F_4(r_1, r_2)\}, \\
3 \leq i \leq r_2 - 1, \\
1 & \frac{1}{N(N - 1)(N - 2)(i - 2)} \{(r_1 - 3)F_2(r_1 - 1) + 2(r_1 - 1 - i)\}, \\
r_2 \leq i \leq r_1 - 1, \\
3 & \frac{1}{N(N - 1)(N - 2)} \sum_{j=i+1}^{N} \frac{1}{j - 3}, \\
r_1 \leq i \leq N - 1, 
\end{cases}
\] (3.15)

where

\[
F_4(r_1, r_2) \equiv 2(r_1 - 3)(F_2(r_1 - 1) + 2) \sum_{j=r_2}^{N} \frac{1}{j - 2} + 2(3N - 5r_1 + 2r_2 + 3)
- 2(r_1 - 3)F_2(r_1 - 1).
\] (3.16)
Let us consider the optimal first stopping time. We have the OLA stopping region for (R5),

\[ B_3 \equiv (N; 3) \]

\[
\begin{cases}
\begin{aligned}
\frac{2}{N(N-1)(N-2)} + F_2(3) + \frac{1}{2}V_2(2, C_2) & \geq \sum_{i=2}^{N-1} \sum_{k=i+1}^{N} [F_3(k) + F_4(k)], \\
\sum_{j=i+1}^{N} [F_3(j) + F_4(j)] & \geq \frac{1}{i-1} \sum_{j=i+1}^{N-1} \sum_{k=j+1}^{N} [F_3(k) + F_4(k)], \\
\end{aligned}
\end{cases}
\]

and \( B_3 \) can be rewritten as

\[
B_3 = (N; 3) \cup \{(i; 3) \mid \Phi_2(i) \geq 0, 1 \leq i \leq N - 1\},
\]

where

\[
\Phi_2(i) = \begin{cases}
\frac{2}{N(N-1)(N-2)} + F_2(3) + V_2(2, C_2) + \sum_{j=2}^{N} (2-j)[F_3(j) + F_4(j)], \\
\sum_{j=i+1}^{N} (2i-j)[F_3(j) + F_4(j)], \\
\end{cases}
\]

\[
2 \leq i \leq N - 1.
\]

**Theorem 3.**
The optimal first stop is made at the earliest \( C_1 \) after \( r_3 \), where

\[
r_3 = \min \{ i \mid \sum_{j=1+1}^{r_2-1} (2i-j)[F_3(j) + F_4(j)] + 2 \sum_{j=r_2}^{N} (2i-j)F_3(j) \geq 0, 1 \leq i \leq N - 1 \}.
\]

**Proof.** Let \( \Phi'_2(i) \equiv N(N-1)(N-2)\Phi_2(i) \). By (3.7) and (3.17), we have

\[
\Phi'_2(i) = \begin{cases}
6 \sum_{j=i+1}^{r_1-1} \sum_{k=j+1}^{N} \frac{1}{k-3}, \\
2 \sum_{j=i+1}^{r_1-1} \frac{(2i-j)}{j-2} \{(r_1-3)F_2(r_1-1) + 2(r_1-1-j)\}
+ 6 \sum_{j=r_2}^{r_2-1} \sum_{k=j+1}^{N} \frac{1}{k-3}, \\
\sum_{j=r_2}^{r_2-1} \frac{(2i-j)}{j-2} \{(r_1-3)(r_2-2)(F_2(r_1-1) + 2) - (r_2-1-j)(r_2 + i - 4) \}
+ (r_2-2)F_4(r_1, r_2) - 2(j-1)(j-2) \}
+ 2 \sum_{j=r_2}^{r_2-1} \frac{(2i-j)}{j-2} \{(r_1-3)F_2(r_1-1) + 2(r_1-1-j)\}
+ 6 \sum_{j=r_1}^{r_2-1} \sum_{k=j+1}^{N} \frac{1}{k-3}, \\
2 \leq i \leq r_2 - 2.
\end{cases}
\]
It is sufficient to prove that $B_3$ is closed. For $r_1 - 1 \leq i \leq N - 2$, we have

$$
\Phi_2'(i) - \Phi_2'(i + 1) = 3\{9(i - 1) \sum_{j=i+1}^{N-1} \frac{1}{j - 2} - 6(N - i - 1)\} \geq 0,
$$

(3.22)

by using the fact $\Phi_1(i) \geq 0$ for $r_1 - 1 \leq i \leq N - 1$. Therefore $\Phi_2(i)$ is a non-increasing function for $r_1 - 1 \leq i \leq N - 1$ and $\Phi_2(N - 1) = 0$.

For $r_2 - 1 \leq i \leq r_1 - 3$, we have

$$
\Phi_2'(i) - \Phi_2'(i + 1) = -4(r_1 - 3)(F_2(r_1 - 1) + 2) \sum_{j=i+1}^{r_1-2} \frac{1}{j - 1} + 6(r_1 - 3)F_2(r_1 - 1) + 12(2r_1 - N - 3 - i) \geq 0,
$$

(3.23)

by the definition of $r_2$. And we have

$$
\Phi_2'(r_1 - 2) - \Phi_2'(r_1 - 1) = 2\{9(r_1 - 3) \sum_{j=r_1-1}^{N-1} \frac{1}{j - 2} - 6(N - r_1 + 1)\} \geq 0,
$$

(3.24)

by using the fact $\Phi_1(r_1 - 1) \geq 0$. Therefore $\Phi_2(i)$ is a non-increasing function for $r_2 - 1 \leq i \leq N - 1$.

We show that $\Phi_2(i)$ is a non-decreasing function for $2 \leq i \leq r_2 - 2$. We have for $3 \leq i \leq r_2 - 2$

$$
\Phi_2(i - 1) - \Phi_2(i) = (i - 2)[F_3(i) + F_4(i)] - 2 \sum_{j=i+1}^{N} [F_3(j) + F_4(j)].
$$

(3.25)

Since in state $(i, C_2; 2)$ the decision of continuing is optimal for $3 \leq i \leq r_2 - 2$ by Theorem 2, we have

$$
(i - 2)F_4(i) = \sum_{j=i+1}^{N} [F_3(j) + F_4(j)],
$$

(3.26)

and

$$
(i - 2)F_3(i) \leq \sum_{j=i+1}^{N} [F_3(j) + F_4(j)].
$$

(3.27)

Then by (3.25), (3.26) and (3.27) we have

$$
\Phi_2(i - 1) - \Phi_2(i) \leq 0,
$$

Therefore, $\Phi_2(i)$ is a non-decreasing function for $2 \leq i \leq r_2 - 2$. We need to show that if $\Phi_2(2) < 0$ then $\Phi_2(1) < 0$. We show this in the case $r_2 \geq 3$. By using the fact $V_2(2, C_2) = 2 \sum_{j=3}^{N} [F_3(j) + F_4(j)] \geq \frac{1}{N(N-1)(N-2)} + 2F_2(3)$, we obtain

$$
\Phi_2(1) = \Phi_2(2) - \sum_{j=3}^{N} [F_3(j) + F_4(j)] + \frac{2}{N(N-1)(N-2)} + F_2(3) < 0.
$$

Thus the proof is completed.

The proof for the case $r_2 = 2$ is similar and will be omitted.
We are able to calculate the value, \( V_3(1) \), which is the probability of win under the optimal stopping policy, by using (3.3), (3.7) and (3.15).

4. Selecting three absolute bests—Asymptotic behaviors

In this section several asymptotic behaviors are considered.

**Theorem 4.**

Let \( d_i = \lim_{N \to \infty} r_i/N, \ i = 1, 2, 3 \), then we obtain

\[
d_1 = e^{-\frac{1}{3}},
\]

and \( d_2 \approx 0.437 \), which is the unique root \( x \) of the equation

\[
d_1^{-1}(1 + x) - \log x = \frac{17}{6},
\]

and \( d_3 \approx 0.1668 \), which is the unique root \( x \) of the equation

\[
- \log x = \left( \frac{3}{2} x^2 - 6d_1x + 6d_1d_2 - 3d_2^2 - c \right)/(6d_1d_2 - 3d_2^2),
\]

where \( c = \frac{3}{2}d_1^2 - 29d_1d_2 + \frac{27}{2}d_2^2 + 6d_2 + \frac{3}{2} - 3d_2 \log d_2. \)

The asymptotic value of the maximum probability of win is given by

\[
\lim_{N \to \infty} V_3(1) = d_3^3 - 3d_1d_3^2 + (6d_1d_2 - 3d_2^2)d_3 \approx 0.163. \quad (4.4)
\]

**Proof.** The asymptotic values \( d_1, d_2 \) and \( d_3 \) are straightforward by (3.1), (3.11) and (3.26), respectively. \( F_5(i) \) is obtained by Theorem 3, but the complete form of \( F_5(i) \) need not to be obtained.

\[
F_5(i) = \begin{cases} 
\frac{1}{i - 1} \sum_{j=i+1}^{N} F_5(j), & 2 \leq i \leq r_3 - 1 \\
\sum_{j=i+1}^{N} [F_3(j) + F_4(j)], & r_3 \leq i \leq N - 1. 
\end{cases} 
\]

Therefore we obtain for \( 2 \leq i \leq r_3 - 1 \),

\[
F_5(i) = \frac{(r_3 - 2)(r_3 - 1)}{(i - 1)i} F_5(r_3 - 1), \quad (4.7)
\]

and for large \( N \), by (R5)

\[
V_3(1) = 2F_5(2) = (r_3 - 1)(r_3 - 2)F_5(r_3 - 1). \quad (4.8)
\]

Noting that

\[
(r_3 - 2)F_5(r_3 - 1) = \sum_{j=r_3}^{N-1} \sum_{k=j+1}^{N} [F_3(k) + F_4(k)], \quad (4.9)
\]
we can obtain by (3.7), (3.15), (4.8) and (4.9)

\[
V_3(1) = \frac{3}{N(N-1)(N-2)} \left\{ \sum_{j=r_3}^{r_1-1} \frac{j-r_3}{j-2} \left[ (r_1-3)F_2(r_1-1) + 2(r_1-1-j) \right] \\
+ \sum_{j=r_3}^{r_1-1} \frac{j-r_3}{(j-1)(j-2)} \left[ (r_2-1-j)(r_2-4+j) + (r_2-2)F_4(r_1,r_2) \right] \\
+ \sum_{j=r_2}^{r_1-1} \frac{j-r_2}{j-2} \left[ (r_1-3)F_2(r_1-1) + 2(r_1-1-j) \right] \\
+ 6 \sum_{j=r_3}^{N-1} \frac{1}{k-3} \right\}.
\]

(4.10)

Therefore (4.4) is obtained by (4.10).

The following Table 1 gives $r_1, r_2, r_3$ and $V_3(1)$ for some values of $N$.

**Table 1**

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5. The other two problems with three stops

5.1 Selecting the best with three stops

We define that the process is in state $(i;s)$, $s = 1, 2, 3$, if the decision maker still has $s$ stops left to be made and the $i$-th object is $C_1$ and he is facing this $C_1$. Let $V_s(i)$, $s = 1, 2, 3$, be the maximum probability of “win” under an optimal policy when starting from the state $(i;s)$. When the objective is fulfilled, we call this event “win”.

We have the following set of DP equations.

\[
V_s(i) = \max\{ \sum_{j=i+1}^{N} \frac{i}{j(j-1)} V_s - 1(j) + \frac{i}{N} \sum_{j=i+1}^{N} \frac{i}{j(j-1)} V_s - 1(j) \},
\]

\[s \leq i \leq N-1, \quad V_s(N) = 1, \quad s = 1, 2, 3, \quad V_0(j) = 0. \quad (5.1)\]

These equations can be solved numerically and give the optimal stopping policy and the probability of “win” for any $N$.

We have the following optimal policy for any $N$:
Optimal Selection Problem

1st stop should be made at the earliest $C_1$ after $r_3$.
2nd stop should be made either at the earliest $C_1$ after the 1st stop or at the earliest $C_2$ after $r_2$.
3rd stop should be made either at the earliest $C_1$ after the 2nd stop or at the earliest $C_3$ after $r_1$.
Table 2 gives the values $r_1, r_2, r_3$ and $Pr(\text{win}) = V_3(1)$ for some $N$.

Table 2

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These results show the following order relations

$$r_3^N \leq r_2^N \leq r_1^N,$$

and $Pr(\text{win}) = V_3^N(1)$ is a decreasing function of $N$, which hold for any $N$ whenever a set of DP equations is available with the aid of a computer.

5.2 Selecting the two absolute bests with three stops

We define the state $(i, C_k, j; s)$, $k = 1, 2, j = 0, 1, \ldots, 3 - s$, $s = 1, 2, 3$, to mean that:
(1) the decision maker is facing $C_k$ at the i-th period, (2) he has already had $j$ relatively best objects among the accepted ones up to this period $i$, and (3) he has still $s$ left to be made.
We denote by $V_s(i, C_k, j)$ the maximum probability of “win” under an optimal policy when starting from the state $(i, C_k, j; s)$. Then we have the following set of DP equations.

$$V_1(i, C_1, 0) = 0,$$
$$V_1(i, C_2, 0) = 0,$$
$$V_1(i, C_1, 1) = \max\left\{ \frac{i(i-1)}{N(N-1)}, \sum_{j=i+1}^{N} \frac{i(i-1)}{j(j-1)(j-2)} \left[ \sum_{k=1}^{2} V_j(j, C_k, 0) \right] \right\},$$
$$3 \leq i \leq N .$$

$$V_1(i, C_2, 1) = \max\left\{ \frac{i(i-1)}{N(N-1)}, \sum_{j=i+1}^{N} \frac{i(i-1)}{j(j-1)(j-2)} \left[ \sum_{k=1}^{2} V_j(j, C_k, 1) \right] \right\},$$
$$3 \leq i \leq N - 1, V_1(N, C_2, 1) = 1 .$$

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\[ V_1(i, C_1, 2) = \max \left\{ \frac{i(i-1)}{N(N-1)} \sum_{j=1}^{N} \frac{i(i-1)}{j(j-1)(j-2)} \left[ \sum_{k=1}^{2} V_1(j, C_k, 0) \right] \right\} \]

\[ = \frac{i(i-1)}{N(N-1)} \quad 3 \leq i \leq N. \quad (5.6) \]

\[ V_1(i, C_2, 2) = \max \left\{ \frac{i(i-1)}{N(N-1)} \sum_{j=1}^{N} \frac{i(i-1)}{j(j-1)(j-2)} \left[ \sum_{k=1}^{2} V_1(j, C_k, 1) \right] \right\}, \]

\[ 3 \leq i \leq N-1, \quad V_1(N, C_2, 2) = 1. \quad (5.7) \]

\[ V_2(i, C_1, 0) = \max \left\{ \sum_{j=1}^{N} \frac{i(i-1)}{j(j-1)(j-2)} \left[ \sum_{k=1}^{2} V_1(j, C_k, 1) \right] \right\}, \]

\[ \sum_{j=1}^{N} \frac{i(i-1)}{j(j-1)(j-2)} \left[ \sum_{k=1}^{2} V_2(j, C_k, 0) \right], \]

\[ 2 \leq i \leq N-1, \quad V_2(N, C_1, 0) = 0. \quad (5.8) \]

\[ V_2(i, C_2, 0) = \max \left\{ \sum_{j=1}^{N} \frac{i(i-1)}{j(j-1)(j-2)} \left[ \sum_{k=1}^{2} V_1(j, C_k, 0) \right] \right\}, \]

\[ \sum_{j=1}^{N} \frac{i(i-1)}{j(j-1)(j-2)} \left[ \sum_{k=1}^{2} V_2(j, C_k, 0) \right], \]

\[ 2 \leq i \leq N-1, \quad V_2(N, C_2, 0) = 0. \quad (5.9) \]

\[ V_2(i, C_1, 1) = \max \left\{ \sum_{j=1}^{N} \frac{i(i-1)}{j(j-1)(j-2)} \left[ \sum_{k=1}^{2} V_1(j, C_k, 2) \right] + \frac{i(i-1)}{N(N-1)} \right\}, \]

\[ \sum_{j=1}^{N} \frac{i(i-1)}{j(j-1)(j-2)} \left[ \sum_{k=1}^{2} V_2(j, C_k, 0) \right], \]

\[ 2 \leq i \leq N-1, \quad V_2(N, C_1, 1) = 1. \quad (5.10) \]

\[ V_2(i, C_2, 1) = \max \left\{ \sum_{j=1}^{N} \frac{i(i-1)}{j(j-1)(j-2)} \left[ \sum_{k=1}^{2} V_1(j, C_k, 2) \right] + \frac{i(i-1)}{N(N-1)} \right\}, \]

\[ \sum_{j=1}^{N} \frac{i(i-1)}{j(j-1)(j-2)} \left[ \sum_{k=1}^{2} V_2(j, C_k, 1) \right], \]

\[ 2 \leq i \leq N-1, \quad V_2(N, C_2, 1) = 1. \quad (5.11) \]

\[ V_3(i, C_1, 0) = \max \left\{ \sum_{j=1}^{N} \frac{i(i-1)}{j(j-1)(j-2)} \left[ \sum_{k=1}^{2} V_2(j, C_k, 1) \right] \right\}, \]

\[ \sum_{j=1}^{N} \frac{i(i-1)}{j(j-1)(j-2)} \left[ \sum_{k=1}^{2} V_3(j, C_k, 0) \right], \]

\[ 2 \leq i \leq N-1, \quad V_3(N, C_1, 0) = 0. \quad (5.12) \]

\[ V_3(1, C_1, 0) = \max \left\{ \frac{1}{2} \left[ V_2(2, C_1, 1) + V_2(2, C_2, 1) \right] \right\}, \sum_{j=2}^{N} \frac{1}{j(j-1)} V_3(j, C_1, 0). \]

\[ V_3(i, C_2, 0) = \max \left\{ \sum_{j=1}^{N} \frac{i(i-1)}{j(j-1)(j-2)} \left[ \sum_{k=1}^{2} V_2(j, C_k, 0) \right] \right\}, \]
These equations give both the optimal stopping policy and the probability of win for any \( N \) with the aid of a computer.

We have the following optimal stopping policy for any \( N \):

1st stop should be made at the earliest \( C_1 \) after \( r_5 \).

2nd stop should be made either at the earliest \( C_1 \) after \( r_3 \),
or at the earliest \( C_2 \) after \( r_4 \) if \( j = 1 \),
and should be made at the earliest \( C_2 \) after \( r_2 \) if \( j = 0 \).

3rd stop should be made either at the earliest \( C_1 \) after the 2nd stop,
or at the earliest \( C_2 \) after \( r_1 \).

Table 3 gives the values, \( r_1, r_2, r_3, r_4, r_5 \) and \( \Pr(\text{win}) = V_3(1, C_1, 0) \) for some \( N \).

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After computational working, we have the following order relations

\[
\begin{align*}
r_5^N &\leq r_3^N \leq r_2^N \leq r_4^N \leq r_1^N,
\end{align*}
\]

and \( \Pr(\text{win}) = V_3^N(1, C_1, 0) \) is a decreasing function of \( N \), which hold for any \( N \) whenever a set of DP equations is available with the aid of a computer.

7. Remarks

This paper indicates the possibility that the problem of recognizing absolutely \( \ell \) best objects with \( m \) stops \((1 \leq \ell \leq m \leq N)\) is solved by either analytical or computational approach. If we derive a set of DP equations for such problem (each \( \ell \) and \( m \)), then by using OLA policy we will obtain the solution, or numerical solutions of the DP equations will give both the optimal stopping policy and the probability of win for any \( N \).
Acknowledgment

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