A GENERALIZED OPTIMUM REQUIREMENT SPANNING TREE PROBLEM WITH A MONGE-LIKE PROPERTY

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Abstract We consider a generalized optimum requirement spanning tree problem (GORST problem) which is an extension of the problem studied by Hu. In the GORST problem, the degrees of vertices are restricted and the objective function is generalized. We will show that a particular tree $T^*$, which is obtained by a sort of greedy algorithm but is explicitly definable, is a solution to the GORST problem when a condition similar to the Monge property is satisfied. Also, we will define a problem of finding a tree network which minimizes the probability that a request of communication is not realized when the network has $k$ failures (called a "$k$-failure problem"), and show that $T^*$ is an explicit solution to the $k$-failure problem for any $k$ when the maximum degree constraint is imposed and the Monge-like property is satisfied.

1. Introduction

We begin by introducing the optimum requirement spanning tree problem (ORST problem) studied by Hu [7], which has motivated our studies. Let $V = \{0, 1, \ldots, n-1\}$ be a set of $n$ vertices, $\binom{V}{2}$ the set of all pairs of distinct vertices in $V$, and $\mathcal{T}$ the whole set of undirected spanning trees on $V$. A tree $T \in \mathcal{T}$ with an edge set $E$ is denoted by $T = (V, E)$, and the edge $e \in E$ connecting two vertices $v, u \in V$ is denoted by $e = (v, u)$. Assume that a nonnegative value $r_{vu}$ is given to each pair $\{v, u\} \in \binom{V}{2}$, where $r_{vu} = r_{uv}$ holds. Hu [7] defined an ORST as a tree $T \in \mathcal{T}$ which minimizes

$$f(T) = \sum_{\{v, u\} \in \binom{V}{2}} d(v, u; T)r_{vu},$$

where $d(v, u; T)$ is the length of the path between $v$ and $u$ on $T$. ORSTs can be regarded as communication networks of tree type with the minimum average cost when the cost of communication between $v$ and $u$ is proportional to $d(v, u; T)$ and $r_{vu}$ denotes the frequency of communication between $v$ and $u$. Hu [7] showed that a tree minimizing $f$ is obtained by the Gomory-Hu algorithm [5] when the degrees of vertices are not restricted.

The author and his colleagues have extended the ORST problem in various manners. Anazawa, Kodera and Jimbo [2, 3] considered the problem of finding a tree $T \in \mathcal{T}$ which minimizes $f$ under the constraint that, for each vertex $v$, the degree of $v$ in $T$ denoted by $\deg(v; T)$ cannot exceed a given integer $l_v$, that is,

$$\deg(v; T) \leq l_v \text{ holds for all } v \in V,$$

where

$$l_0 \geq l_1 \geq \cdots \geq l_{n-1} \geq 1 \text{ and } \sum_{v=0}^{n-1} l_v \geq 2(n - 1)$$

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Figure 1: \( T^* \) for \( n = 9 \) and \( l_0 = 4, l_1 = 3, l_2 = \cdots = l_8 = 2 \).

are assumed. (The problem for \( l_0 = l_1 = \cdots = l_{n-1} = n-1 \) is equivalent to Hu's one.) And they showed that if

- a positive value \( p_v \) is assigned to each vertex \( v \in V \),
- \( p_0 \geq p_1 \geq \cdots \geq p_{n-1} > 0 \) is satisfied (only strictly-decreasingness is assumed in [2]), and
- \( r_{vu} = cpvp_u \) holds for all \( \{v, u\} \in \binom{V}{2} \) (where \( c \) is a positive constant),

then a particular tree \( T^* = (V, E^*) \) is an explicit solution to the problem. The definition of \( T^* \) is as follows: Assuming that \( l_v \geq 1 \) holds for all \( v \in V \) and \( \sum_{v=0}^{n-1} l_v > 2(n-1) \) holds for all \( v \in \{1, 2, \ldots, n-1\} \) \( (3) \)

(condition (2) implies condition (3), which is proved in Appendix 1), we set \( s_{-1} = 0, s_u = \sum_{v=0}^{n-1} l_v - u \) \((u = 0, 1, \ldots, n-1)\) and let \( N \) be the minimum integer satisfying \( n-1 < s_{N-1} \); also we define a function \( \pi \) on a set \( \{1, 2, \ldots, n-1\} \) by

\[
\pi(v) = \begin{cases} 
  u & \text{if } s_{u-1} + 1 \leq v \leq s_u \text{ for } u = 0, 1, 2, \ldots, N-2 \\
  N-1 & \text{if } s_{N-2} + 1 \leq v \leq n-1
\end{cases}
\]

and let \( E^* = \{e_1, e_2, \ldots, e_{n-1}\} \) where \( e_v = (\pi(v), v) \) \((v = 1, 2, \ldots, n-1)\). Then we obtain \( T^* = (V, E^*) \). Appendix 1 shows that if condition (3) is satisfied then \( \pi \) is definable and \( T^* \) surely is a tree. Roughly speaking, the tree \( T^* \) is constructed by the following procedure: First, to vertex 0, connect the remaining vertices by ascending order of vertex number as many as possible; secondly, to vertex 1, connect the remaining vertices by the same order as many as possible; and continue to connect the remaining vertices in the same manner until all \( n \) vertices are connected. This procedure can be regarded as a sort of “greedy algorithm”. An example of \( T^* \) (for \( n = 9 \) and \( l_0 = 4, l_1 = 3, l_2 = \cdots = l_8 = 2 \)) is shown by Figure 1. Anazawa et al. [2, 3] also gave another interpretation of ORSTs as follows: They minimize the probability that a request of communication is not realized when there is one failure on a vertex or an edge (the probability for \( k \) failures will be shown in Section 5).

Anazawa [1] considered the problem of minimizing \( f \) under constraint (1) satisfying \( l_0 = l_1 = \cdots = l_{n-1} = L \geq 2 \) (where \( L \) is a commonly-given integer), and showed that \( T^* \) is a unique explicit solution under the conditions that

- \( r_{vu} > r_{vu'} \) holds for all \( v, u, u' \in V \) \((u \neq v, u' \neq v, u < u')\), and
- \( r_{vu} + r_{vu'} > r_{vu'} + r_{v'u} \) holds for all \( v, v', u, u' \in V \) \((v < v', u < u')\) such that \( r_{vu}, r_{v'u}, r_{vu'} \) and \( r_{v'u} \) are all defined.
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Since the above inequalities hold if \( r_{vu} = cpvp_u \) (\( c > 0 \)) and \( p_0 > p_1 > \cdots > p_{n-1} > 0 \) are satisfied, the conditions of \( \{r_{vu}\} \) assumed by Anazawa [1] are more general than those considered by Anazawa et al. [2].

The aim of this paper is to generalize the problems and results discussed in the literatures [1, 2, 3]. Let \( g(x) \) be an arbitrary real-valued function of real variable \( x \) such that it is monotone nondecreasing on \([0, n - 1]\), and consider a problem of finding a tree \( T \in \mathcal{T} \) which minimizes a function

\[
f_g(T) = \sum_{\{v,u\} \in \mathcal{E}} g(d(v, u; T)) r_{vu}
\]

subject to constraint (1) satisfying (2). We call this problem a generalized optimum requirement spanning tree problem (GORST problem), and a solution to this problem an \( f_g \)-optimum tree. Our main assertion on the GORST problem in this paper is that \( T^* \) is an \( f_g \)-optimum tree if \( r_{vu} \geq r_{vu'} \) holds for all \( v, u, u' \in V \) \((u \neq v, u' \neq v, u < u')\) and \( r_{vu} + r_{v'u} \geq r_{vu'} + r_{v'u} \) holds for all \( v, v', u, u' \in V \) \((v < v', u < u')\) such that \( r_{vu}, r_{v'u}, r_{vu'} \) and \( r_{v'u} \) are all defined. Further, by introducing dummy vertices \( \{v|v \geq n\} \) and setting \( r_{vu} = 0 \) if \( v \) or \( u \) is dummy, the main assertion can be described more simply as follows:

**Main Theorem** If \( \{r_{vu}\} \) satisfies

\[
r_{vu} + r_{v'u} \geq r_{vu'} + r_{v'u}
\]

for all 4-tuple \( \{v, v', u, u'\} \) \((v < v', u < u')\) such that \( r_{vu}, r_{v'u}, r_{vu'} \) and \( r_{v'u} \) are all defined, then \( T^* \) defined above is \( f_g \)-optimum.

**Remark** (a) By setting \( v' \geq n \) in inequality (4), we have \( r_{vu} \geq r_{vu'} \). Also, we easily find that inequality (4) holds if \( r_{vu} = cpvp_u \) (\( c > 0 \)) and \( p_0 \geq p_1 \geq \cdots \geq p_{n-1} \geq p_n = \cdots = 0 \) are satisfied. Therefore, the condition of \( \{r_{vu}\} \) in Main Theorem is more general than those in the literatures [1, 2, 3]. (b) It is easy to see that if \( r_{vu} + r_{v'u} > r_{vu'} + r_{v'u} \) holds for some 4-tuple \( \{v, v', u, u'\} \) \((v < v', u < u')\) then both \( v \) and \( u \) are non-dummy. In fact, it follows from \( r_{vu} + r_{v'u} > 0 \) and \( r_{vu} \geq r_{vu'} \geq r_{v'u} \geq 0 \) that \( r_{vu} > 0 \) holds.

It is of interest that condition (4) is closely related to the Monge property. A \( m \times n \) matrix \( C = [c_{vu}] \) is called a Monge matrix if it satisfies the Monge property

\[
c_{vu} + c_{v'u} \leq c_{vu'} + c_{v'u} \text{ for all } 0 \leq v < v' \leq m - 1, 0 \leq u < u' \leq n - 1.
\]

The property is named after the French mathematician Gaspard Monge, and is rediscovered by Hoffman [6] (compactly reviewed by Pferschy et al. [8] and Deineko et al. [4]). It is well-known that, in the classical Hitchcock transportation problem, if the cost matrix is Monge then a feasible solution obtained by the north-west-corner rule is optimum for any feasible demand and supply vectors. Also, Monge matrices make some NP-hard problems (ex. travelling salesman problem) efficiently solvable (see [8]). If \( C \) has unspecified elements and satisfies the first inequality in (5) for all 4-tuple \( \{v, v', u, u'\} \) \((v < v', u < u')\) such that \( c_{vu}, c_{v'u}, c_{vu'}, c_{v'u} \) are all specified, then \( C \) is called an incomplete Monge matrix. For \( \{r_{vu}\} \) satisfying the condition in Main Theorem, if \( n \times n \) matrix \( C \) is defined so as to satisfy \( c_{vu} = ar_{n-u-1,v-n-1} + b \) \( \text{ (where } a (0) \text{ and } b \text{ are arbitrary constants) with diagonal elements unspecified, then } C \) is a symmetric incomplete Monge matrix.

In this paper, after giving mathematical preliminaries in Section 2, we will show some properties of the tree \( T^* \) in Section 3. The proof of Main Theorem will be given in Section 4.
As an example of the GORST problem, we will formulate in Section 5 a "k-failure problem" which is an extension of the "one-failure problem" discussed by Anazawa et al. [2, 3], and show that $T^*$ is an explicit solution to the $k$-failure problem for any $k$ ($0 < k \leq 2n - 1$) when constraint (1) with (2) is imposed and the condition in Main Theorem is satisfied.

2. Preliminaries

Throughout this paper, we use the following notation. For a graph $G = (V, E)$ and a subset $U \subseteq V$, a subgraph $G \cap U$ is defined by $G' = (U, E')$, where $E' = \{ (v, u) \in E \mid v, u \in U \}$; while a subgraph $G \setminus U$ is defined by $G'' = (V \setminus U, E'')$, where $E'' = \{ (v, u) \in E \mid v, u \in V \} \setminus U$.

For a rooted tree $T \in \mathcal{T}$, if vertex $v$ lies on a path (root, $\ldots$, $u$) of $T$, then $v$ is called the $d(v, u; T)$-th ancestor of $u$ (note that any vertex is the 0-th ancestor of itself); especially the first ancestor of $u$ is called the parent of $u$. As to $T^*$ defined in Section 1, $\pi(v)$ is the parent of $v$ for $v = 1, \ldots, n - 1$ if vertex 0 is the root. Also, let $\chi(v) = \{ u \mid v \}$ is the parent of $u$. Further, the level of $v$ is defined by $d(v, \text{root}; T)$.

For a path $P = (u_1, u_2, \ldots, u_k)$ of a tree $T = (V, E) \in \mathcal{T}$, let $F$ be a forest defined by

$$F = (V, E \setminus \{(u_1, u_2), (u_2, u_3), \ldots, (u_{k-1}, u_k)\}),$$

and $T(u_i) = (V(u_i), E(u_i))$ ($i = 1, \ldots, k$) the connected components of $F$ each of which contains $u_i$.

An edge $(v, u)$ such that $v$ or $u$ is a dummy vertex is called a dummy edge. For a tree $T = (V, E)$, we will sometimes construct another tree $\tilde{T} = (\tilde{V}, \tilde{E})$ satisfying $\tilde{V} = V \cup \{ \text{dummy vertices} \}$ and $\tilde{E} = E \cup \{ \text{dummy edges} \}$, i.e. $\tilde{T} \setminus \{ \text{dummy vertices} \} = T$. Then it is obvious that $f_\phi(T) = f_\phi(T)$ holds. When constructing a dummies-added tree $\tilde{T} = (\tilde{V}, \tilde{E})$, we do not restrict the degrees of vertices in $\tilde{V}$.

For a tree $T = (V, E) \in \mathcal{T}$ satisfying (1) with (2) and a path $P = (u_1, \ldots, u_k)$ ($k = 2$ or 3) of $T$, we define an isomorphism $\sigma_F$. Let $T(u_i) = (V(u_i), E(u_i))$ be defined for $P$, and $T(u_i) = (\tilde{V}(u_i), \tilde{E}(u_i))$ ($i = 1, k$) be obtained by adding dummies to $T(u_i) = (V(u_i), E(u_i))$ ($i = 1, k$) so that $\tilde{T}(u_1)$ and $\tilde{T}(u_k)$ can be isomorphic and the underlying isomorphism $\sigma_F : \tilde{V}(u_1) \rightarrow \tilde{V}(u_k)$ can satisfy the following two:

(i) $\sigma_F(u_1) = u_k$.

(ii) For any $v \in \tilde{V}(u_1)$, if $\chi(v)$ has at least one dummy vertex, then $\chi(\sigma_F(v))$ has no dummy vertices, where we regard $u_1$ as the root of $\tilde{T}(u_1)$ and $u_k$ as that of $\tilde{T}(u_k)$.

We call such an isomorphism $\sigma_F$ a forced isomorphism for $P$. Appendix 2 shows that $\sigma_F$ can be defined for any tree $T \in \mathcal{T}$ and any path $P = (u_1, \ldots, u_k)$ ($k = 2$ or 3) of $T$. Also, letting $\tilde{T} = (\tilde{V}, \tilde{E})$ where

$$\tilde{V} = \tilde{V}(u_1) \cup \left( \bigcup_{i=2}^{k-1} V(u_i) \right) \cup \tilde{V}(u_k)$$
and
$$\tilde{E} = \tilde{E}(u_1) \cup \left( \bigcup_{i=2}^{k-1} E(u_i) \right) \cup \tilde{E}(u_k) \cup \left( \bigcup_{i=1}^{k-1} \{ (u_i, u_{i+1}) \} \right),$$

we consider the following transformation of $\tilde{T}$ which may reduce the $f_\phi$ value: Let $V_C = \{ v \in \tilde{V}(u_1) \mid v > \sigma_F(v) \}$, and exchange $v$ and $\sigma_F(v)$ for all $v \in V_C$. We call such a transformation biasing with respect to $\sigma_F$. Further, let $T'$ be a tree obtained from $\tilde{T}$ by biasing and $T' = T' \setminus \{ \text{dummy vertices} \}$. (An example of constructing $T'$ from $T$ is illustrated by Figure 2.) The following lemma assures us that $T'$ is also a tree belonging to $\mathcal{T}$ and satisfies constraint (1).

Lemma 1 For $\tilde{T}$ and $\sigma_F$ defined above and for an arbitrary vertex $v \in \tilde{V}(u_1)$, let

$$N = \{ w \in \chi(v) \mid w \leq n - 1 \}, \quad N' = \{ w' \in \chi(\sigma_F(v)) \mid w' \leq n - 1 \},$$
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Biasing

Figure 2: An example of constructing $T'$ from $T$, where broken lines denote dummy vertices and edges. In $T$, we choose a path $P = (u_1, u_2)$ with $u_1 = 0$ and $u_2 = 1$, and define $T(u_1) = (V(u_1), E(u_1))$ and $T(u_2) = (V(u_2), E(u_2))$ for $P$ where $V(u_1) = \{0, 2, 4\}$ and $V(u_2) = \{1, 3, 5, 6\}$. $T$ has two isomorphic dummies-added trees $\tilde{T}(u_1) = (\tilde{V}(u_1), \tilde{E}(u_1))$ and $\tilde{T}(u_2) = (\tilde{V}(u_2), \tilde{E}(u_2))$ where $\tilde{V}(u_1) = \{0, 2, 4, 8, 9\}$ and $\tilde{V}(u_2) = \{1, 3, 5, 6, 7\}$. The underlying forced isomorphism $\sigma_P$ is defined by setting $\sigma_P(0) = 1$, $\sigma_P(2) = 7$, $\sigma_P(4) = 3$, $\sigma_P(8) = 5$ and $\sigma_P(9) = 6$. Hence, we have $V_C = \{4, 8, 9\}$. 

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The inclusion relation of $N$, $D$, $N'$, $D'$ and their subsets. In this figure, the left-right arrow indicates that the elements of $N_c \cup D_c$ and those of $N'_c \cup D'_c$ are exchanged by biasing with respect to $\sigma_P$.

$$N_C = N \cap V_C, \tilde{N}_C = N \setminus N_C, N'_C = N' \cap \sigma_P(V_C) \text{ and } \tilde{N}'_C = N' \setminus N'_C,$$

where $\sigma_P(X) = \{\sigma_P(x) \in \tilde{V}(u_k) | x \in X\}$ for $X \subset \tilde{V}(u_1)$. Then we find that

(i) if $v < n \leq \sigma_P(v)$, then $|\tilde{N}_C| + |N'_C| \leq l_v - 1$ and $|N_c| + |N_C| = 0$ hold,

(ii) if $\sigma_P(v) < n \leq v$, then $|\tilde{N}_C| + |N'_C| \leq l_{\sigma_P(v)} - 1$ and $|\tilde{N}'_C| + |N_C| = 0$ hold,

(iii) if $v < \sigma_P(v) < n$, then $|\tilde{N}_C| + |N'_C| \leq l_v - 1$ and $|\tilde{N}'_C| + |N_C| \leq l_{\sigma_P(v)} - 1$ hold,

(iv) if $\sigma_P(v) < v < n$, then $|\tilde{N}_C| + |N'_C| \leq l_{\sigma_P(v)} - 1$ and $|\tilde{N}'_C| + |N_C| \leq l_v - 1$ hold.

**Proof** Let $D = \chi(v) \setminus N$, $D' = \chi(\sigma_P(v)) \setminus N'$.

$$D_C = D \cap V_C, \tilde{D}_C = D \setminus D_C, D'_C = D' \cap \sigma_P(V_C) \text{ and } \tilde{D}'_C = D' \setminus D'_C.$$ The inclusion relation of $N$, $D$, $N'$, $D'$ and their subsets is illustrated by Figure 3. Note that the following five relations are always satisfied: $|N| + |D| = |N'| + |D'|$, $|N_C| + |D_C| = |N'_C| + |D'_C|$, $|\tilde{N}_C| + |\tilde{D}_C| = |\tilde{N}'_C| + |\tilde{D}'_C|$, $|N| \leq l_v - 1$ and $|N'| \leq l_{\sigma_P(v)} - 1$.

(i) Since $N = \tilde{N}_C$ (i.e. $N_C = \emptyset$) and $N' = \tilde{N}'_C \cup N'_C = \emptyset$, we have $|\tilde{N}_C| + |N'_C| = |N| \leq l_v - 1$ and $|\tilde{N}'_C| + |N_C| = 0$.

(ii) The proof is similar to that of (i).

(iii) In this case, $l_v \geq l_{\sigma_P(v)}$ holds and $D = \emptyset$ or $D' = \emptyset$ is satisfied. When $D = \emptyset$ holds, $D'_C = \emptyset$ is obvious. Then we have $|N_C| = |N'_C|$. Hence, we find that

$$|\tilde{N}_C| + |N'_C| = |\tilde{N}'_C| + |N_C| = |N| \leq l_v - 1$$

are satisfied. When $D' = \emptyset$ holds, we have $|N| \leq |N'|$. Since it is obvious that $\tilde{D}_C = \emptyset$ holds, we obtain $|\tilde{N}_C| = |\tilde{N}'_C|$. Hence, we find that

$$|\tilde{N}_C| + |N'_C| = |\tilde{N}'_C| + |N_C| = |N'| \leq l_{\sigma_P(v)} - 1 \leq l'_v - 1$$

are satisfied.
(iv) The proof is similar to that of (iii).

**Lemma 2** For a tree $T \in \mathcal{T}$ and a path $P = (u_1, \ldots, u_k)$ ($k = 2$ or $3$) of $T$, let $\bar{T}$ be a dummies-added tree on which a forced isomorphism $\sigma_P$ is defined. Also, let $\bar{T}'$ be a tree obtained from $\bar{T}$ by biasing with respect to $\sigma_P$ and $T'' = \bar{T}' \setminus \{\text{dummy vertices}\}$. If $\{r_{vu}\}$ satisfies the condition in Main Theorem, then $f_g(T') \leq f_g(T)$ holds.

**Proof** We have only to show that $f_g(T') \leq f_g(\bar{T})$ holds. Let $\bar{V}_C = \bar{V}(u_1) \setminus V_C$ and $D_{vu} = \{g(d(v, u; T')) - g(d(v, u; T))\}r_{vu}$. First, we consider the case of $k = 2$. Then $f_g(T'') - f_g(\bar{T})$ is expressed by

\[
\sum_{v, u \in V_C} D_{vu} + \sum_{v \in \bar{V}_C, u \in \sigma_P(V_C)} D_{vu} + \sum_{v, u \in \sigma_P(V_C)} D_{vu} + \sum_{v \in V_C, u \in \bar{V}_C} D_{vu} + \sum_{v \in \sigma_P(V_C), u \in \sigma_P(V_C)} D_{vu} + \sum_{v \in \bar{V}_C, u \in \sigma_P(V_C)} D_{vu}. \quad (6)
\]

However, noting that the first six summations are all equal to zero, we have

\[
f_g(T') - f_g(\bar{T}) = \sum_{v \in \bar{V}_C, u \in \sigma_P(V_C)} (D_{vu} + D_{\sigma_P(u)} + D_{\sigma_P(u)} + D_{\sigma_P(u)}). \quad (7)
\]

For two vertices $v \in V_C$ and $u \in \bar{V}_C$, let $\delta_{vu} = d(v, u; \bar{T})$ and $\Delta_{vu} = d(v, u; T')$. Then $\delta_{vu} < \Delta_{vu}$ is obvious. Also, noting that

\[
\delta_{vu} = d(v, u; \bar{T}) = d(\sigma_P(v), \sigma_P(u); \bar{T}) = d(v, \sigma_P(u); \bar{T}) = d(\sigma_P(v), u; T')
\]

and

\[
\Delta_{vu} = d(v, u; T') = d(\sigma_P(v), \sigma_P(u); T') = d(v, \sigma_P(u); T') = d(\sigma_P(v), u; T'),
\]

we obtain

\[
f_g(T') - f_g(\bar{T}) = - \sum_{v \in \bar{V}_C, u \in \sigma_P(V_C)} \{g(\Delta_{vu}) - g(\delta_{vu})\} (r_{\sigma_P(v)} + r_{\sigma_P(u)} - r_{\sigma_P(v)} \sigma_P(u) - r_{vu}). \quad (7)
\]

Due to the assumption of $\{r_{vu}\}$, the second factor of the summand in (7) is always non-negative, and if it is positive then we find from the remark of Main Theorem that both $\sigma_P(v)$ and $u$ are non-dummy, which implies $\Delta_{vu} \leq n - 1$. Hence, we find from the assumption of $g$ that $f_g(T'') - f_g(\bar{T}) \leq 0$ holds for $k = 2$.

In the case of $k = 3$, $f_g(T') - f_g(\bar{T})$ is expressed by

\[
\sum_{v, u \in V(u_2)} D_{vu} + \sum_{v \in \bar{V}(u_1), u \in V(u_2)} D_{vu} + \sum_{v \in V(u_3), u \in V(u_2)} D_{vu} + (6).
\]

However, the first three summations are all equal to zero. Hence, $f_g(T') - f_g(\bar{T}) \leq 0$ is similarly obtained. The proof is just completed.

### 3. Properties of $T^*$

Here, we show some properties of the tree $T^* = (V, E^*)$ defined in Section 1. Suppose that $T^*$ satisfies (2). Let $V_\nu = \{0, 1, \ldots, \nu - 1\}$ ($1 \leq \nu \leq n$) and $T^*_\nu = T^* \cap V_\nu$. Note that $T^*_\nu$ ($1 \leq \nu \leq n$) are subtrees of $T^*$.
Lemma 3 For each $T^*_v$ ($v \geq 2$), let $P = (u_1, u_2, \ldots, u_k)$ be an arbitrary path of $T^*_v$ satisfying $u_i < u_k$, and let $m = \lceil \frac{k}{2} \rceil$ where $\lceil x \rceil$ is the maximum integer not exceeding $x$. Then $u_i < u_{k-i+1}$ and $\deg(u_i; T^*_v) \geq \deg(u_{k-i+1}; T^*_v)$ hold for $i = 1, 2, \ldots, m$.

Proof Let vertex 0 be the root of $T^*_v$. For two vertices $v$ and $u$ on $T^*_v$, we find from the procedure of constructing $T^*$ that

(i) if $v$ is the $d$-th ($d > 0$) ancestor of $u$, then $v < u$ holds,

(ii) if $0 < v < u$, then $\pi(v) \leq \pi(u)$ holds, and

(iii) if $v < u$, then the level of $v$ does not exceed that of $u$.

For the path $P$, if $u_1$ is an ancestor of $u_k$, then we find from (i) that $u_i < u_{k-i+1}$ holds for $i = 1, 2, \ldots, m$. Otherwise, continue to compare $u_i$ with $u_{k-i+1}$ by ascending order of $i$ until we become an ancestor of $u$. Then we find from (ii) that $u_i < u_{k-i+1}$ holds for $i = 1, 2, \ldots, m$.

Lemma 4 Suppose that a tree $T \in T$ satisfies (1) with (2) and contains a subtree $T^*_v$ (i.e. $T \cap V_v = T^*_v$). Let $P = (u_1, \ldots, u_k)$ ($k = 2$ or $3$) be an arbitrary path of $T$. For the tree $T$ and the path $P$, let $T'$ be a dummies-added tree on which a forced isomorphism $\alpha$ is defined, $\tilde{T}'$ a tree obtained from $\tilde{T}$ by biasing with respect to $\sigma_p$, and $T'' = T' \setminus \{\text{dummy vertices}\}$. Then $T''$ also contains $T^*_v$.

Proof We have only to show that $T''$ contains $T^*_v$. Since the proof varies according as where $T^*_v$ lies on $\tilde{T}$, we should consider the following three cases:

(i) $k = 3$ and $V_v \subset V(u_2)$,

(ii) $k = 2$ or $3$, $V_v \cap \tilde{V}(u_1) \neq \emptyset$ and $V_v \cap \tilde{V}(u_k) = \emptyset$,

(iii) $k = 2$ or $3$, $V_v \cap \tilde{V}(u_1) \neq \emptyset$ and $V_v \cap \tilde{V}(u_k) \neq \emptyset$.

In case (i), it is obvious for $\tilde{T}''$ obtained by biasing to contain $T^*_v$. In case (ii), since $v \geq \nu$ holds for all $v \in \tilde{V}(u_k)$, it follows that $V_v \cap \tilde{V}(u_1) \subset V_C$. Hence, $\tilde{T}''$ also contains $T^*_v$. In case (iii), suppose that $u_1 < u_k$ holds without loss of generality. Then we find from Lemma 3 that $V_v \cap \tilde{V}(u_k) \subset \sigma_p(V_v \cap \tilde{V}(u_1))$ holds. We also find that $v < \sigma_p(v)$ holds for any $v \in V_v \cap \tilde{V}(u_1)$. In fact, if $\sigma_p(v) \in V_v \cap \tilde{V}(u_k)$, then $v < \sigma_p(v)$ comes from Lemma 3; otherwise $v < v \leq \sigma_p(v)$ holds. Hence, we have $V_v \cap \tilde{V}(u_1) \subset V_C$ and $V_v \cap \tilde{V}(u_k) \subset \sigma_p(V_C)$, which imply that $T''$ contains $T^*_v$. 

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4. Proof of Main Theorem

Let \( T^* = (V, E^*) \in \mathcal{T} \) be the tree defined in Section 1, and suppose that \( T^* \) satisfies (2). For a tree \( T = (V, E) \in \mathcal{T} \), let

\[
v_T = \begin{cases} 
  \min\{v > 0 | e_v = (\pi(v), v) \in E^*, e_v \notin E \} & \text{if } E \neq E^* \\
  n & \text{if } E = E^*.
\end{cases}
\]

We will show that any \( f_g \)-optimum tree can be transformed into \( T^* \) with the \( f_g \) value unchanged.

Let \( T \) be an \( f_g \)-optimum tree with \( v_T < n \). Note that \( T \cap \{0, 1, \ldots, v_T - 1\} = T^* \) (a subtree of \( T^* \)). Also, let \( v^* = \pi(v_T) \in T^* \). Since \( v_T < n \), we can consider a path \( P' = (v^*, \ldots, v_T) \) of \( T \). Among the vertices on \( P' \), a vertex adjacent to \( v_T \) is denoted by \( v_1 \), and a vertex adjacent to \( v^* \) is denoted by \( v_2 \) (\( v_2 \) may coincide with \( v_1 \)). Then we find that \( v^* < v_1 \) holds. In fact, if \( v^* = v_1 \), then we have \( (v_1, v_T) = (\pi(v_T), v_T) \in E \), which contradicts the definition of \( v_T \); else, if \( v^* > v_1 \), then a certain vertex \( v' \) in \( \{v < v_T | \pi(v) = v_1 \in T^* \} \) is pushed out by \( v_T \), that is, \( (v_1, v') = (\pi(v'), v') \in E^* \) and \( (\pi(v'), v') \notin E \) hold, which contradicts the minimality of \( v_T \). Here, we consider the following two cases: (i) \( v_2 < v_T \) and (ii) \( v_2 > v_T \).

In case (i), let \( P = (u_1, \ldots, u_k) \) be a subpath of \( P' \) satisfying

\[
d(u_1, v_2; T) = d(u_1, v_1; T) = \left[ \frac{d(v', v_1; T) - 1}{2} \right] \quad (\text{then } k = 2 \text{ or } 3).
\]

Let \( \hat{T} \) be a dummies-added tree on which we can define a forced isomorphism \( \sigma_P \) such that \( \sigma_P(v^*) = v_1 \) holds and a vertex \( v** \) adjacent to \( v^* \) satisfies \( \sigma_P(v**) = v_T \) and \( v** > v_T \) (it is obvious that such \( v** \) exists). Also, let \( \hat{T}' \) be the tree obtained from \( \hat{T} \) by biasing with respect to \( \sigma_P \), and \( T'' = (V, E') = \hat{T}' \setminus \{\text{dummy vertices}\} \). Then we find from Lemmas 2 and 4 that \( T'' \) is also \( f_g \)-optimum and satisfies \( T'' \cap \{0, 1, \ldots, v_T - 1\} = T^* \) and \( (v^*, v_T) = (\pi(v_T), v_T) \in E', \) that is, \( v_T > v_T \) holds.

In case (ii), let \( P = (u_1, \ldots, u_k) \) be a subpath of \( P' \) satisfying

\[
d(u_1, v_2; T) = d(u_1, v_T; T) = \left[ \frac{d(v_2, v_T; T) - 1}{2} \right] \quad (\text{then } k = 2 \text{ or } 3).
\]

Similarly to (i), let \( \hat{T} \) be a dummies-added tree on which we can define a forced isomorphism \( \sigma_P \) satisfying \( \sigma_P(v_2) = v_T \). Also, let \( \hat{T}' \) be the tree obtained from \( \hat{T} \) by biasing with respect to \( \sigma_P \), and \( T' = \hat{T}' \setminus \{\text{dummy vertices}\} \). Then we obtain \( v_T > v_T \) in the same way with that of (i).

By continuing this process, we find that \( T^* \) is \( f_g \)-optimum.

5. An Example of the GORST Problem

Finally, we show an example of the GORST problem, which is to find a tree network minimizing the probability that a request of communication is not realized when the network has \( k \) failures (called a "\( k \)-failure problem"). Let vertices be regarded as network hosts, edges as network cables, and \( \{f_m\} \) as relative frequencies of communication. A failure can occur on a vertex or an edge; if even one failure has occurred on a path \( (v, \ldots, u) \), then \( v \) and \( u \) cannot communicate with each other. Let us define the probability that a request of communication is not realized on a tree network \( T = (V, E) \) with \( k \) failures, denoted by \( p(T; k) \). Consider a time interval \( I \) in which the number of failures on \( T \) does not change and at most one
request of communication occurs on $T$. Let $F_T$ denote the number of failures on $T$ in $I$, $R_{vu}$ the event that a request of communication between $v$ and $u$ occurs in $I$, and $R_T$ the event that a request of communication between a certain pair on $T$ occurs in $I$. Then the relative frequency of communication between $v$ and $u$ is expressed by $r_{vu} = \Pr\{R_{vu}\mid R_T\}$. We assume that each vertex (host) has enough ability of processing and, hence, the frequency of failure on each vertex does not depend on the amount of traffic. Also, we observe that an edge failure results mostly from incomplete connection at the connector of a host (rarely from the snapping of a cable) and it occurs independently of a vertex failure. Hence, we assume that

(i) a failure occurs equally often on $n$ vertices, and so does it on $n - 1$ edges, where the probability of vertex failure is not necessarily equal to that of edge failure,

(ii) any two failures occur mutually independently.

From these assumptions, letting

$$\alpha_{ij} = \Pr\{i \text{ vertices and } j \text{ edges are broken down}\mid F_T = i + j\},$$

we can assume that $\alpha_{ij}$'s do not depend on the structure of $T$. Let $v \leftrightarrow u$ be the state that the path $(v, \ldots, u)$ has no failure in $I$, and $\overline{v} \leftrightarrow \overline{u}$ the state that $(v, \ldots, u)$ has at least one failure in $I$. It is easy to see that

$$\Pr\{\overline{v} \leftrightarrow \overline{u}\mid F_T = k\} = 1 - \Pr\{v \leftrightarrow u\mid F_T = k\}$$

$$= 1 - \sum_{i+j=k} \binom{n - (d(v, u; T) + 1)}{i} \binom{(n - 1) - d(v, u; T)}{j} \alpha_{ij}. $$

Then the desired probability is expressed by

$$p(T; k) = \sum_{\{v, u\} \in \binom{\overline{V}}{k}} \Pr\{\overline{v} \leftrightarrow \overline{u} \text{ and } R_{vu}\mid F_T = k \text{ and } R_T\}$$

$$= \sum_{\{v, u\} \in \binom{\overline{V}}{k}} \Pr\{\overline{v} \leftrightarrow \overline{u}\mid F_T = k\} r_{vu}. $$

The $k$-failure problem is to find a tree $T$ minimizing $p(T; k)$ for each $k$.

For a fixed $k$ ($0 < k \leq 2n - 1$), let

$$g(x) = 1 - \sum_{i+j=k} \frac{(n - 1 - x)}{i} \frac{(n - 1 - x)}{j} \alpha_{ij}$$

which is obtained by replacing $d(v, u; T)$ in $\Pr\{\overline{v} \leftrightarrow \overline{u}\mid F_T = k\}$ by $x$. Then it is easy to find that $g(x)$ is monotone increasing on $[0, n - 1]$. Hence, the $k$-failure problem under constraint (1) satisfying (2) is a special case of the GORST problem. Further, if $\{r_{vu}\}$ satisfies the condition in Main Theorem, then $T^*$ defined in Section 1 minimizes $p(T; k)$ for any $k$ ($0 < k \leq 2n - 1$).
Appendix 1

First, we show that condition (2) implies condition (3). Note that condition (3) also satisfies
\[ \sum_{v=0}^{n-1} l_v \geq 2(n-1) \]. In fact, since \[ \sum_{v=0}^{n-2} l_v \geq 2(n-1-1) + 1 = 2n - 3 \] is obtained from (3), it follows from \( l_{n-1} \geq 1 \) that
\[ \sum_{v=0}^{n-1} l_v = \sum_{v=0}^{n-2} l_v + l_{n-1} \geq 2(n-1) \]
holds. Assume that \( \{l_v\} \) with (2) also satisfies \( l_{\nu} \geq \sum_{v=0}^{\nu-1} l_v \leq 2(\nu-1) \) for some \( \nu \). Since \( l_{\nu} = \sum_{v=0}^{\nu-1} l_v + \sum_{v=\nu}^{n-1} l_v \geq 2(n-1-1) \), we have \( 2(\nu-1) + \sum_{v=\nu}^{n-1} l_v \geq 2(n-1-1) \), that is,
\[ \sum_{v=0}^{\nu-1} l_v \geq 2(n-\nu) \]. Then it follows from the monotonicity of \( \{l_v\} \) that \( l_v \geq 2 \) holds. However, we also find that \( \sum_{v=0}^{\nu-1} l_v \geq 2\nu \) holds, which is contradiction.

Secondly, we show that if condition (3) is satisfied then \( \pi \) is definable and \( T^* \) surely is a tree. Since condition (3) implies that \( \sum_{v=0}^{n-1} l_v \geq 2(n-1) \) holds, we have \( s_{u-1} \geq n-1 \) under (3). This means that \( N \) in the definition of \( \pi \) is surely determined. Also, it is easy to see that \( T^* \) is a tree if \( \pi(v) < v \) holds for all \( v \in \{1,2,\ldots,n-1\} \). For any vertex \( v \) satisfying \( \pi(v) = u \) \((0 \leq u \leq N-1 < n)\), we have \( s_{u-1} + 1 \leq v \). On the other hand, we find from condition (3) that
\[ u < \sum_{v=0}^{u-1} l_v - (u-1) + 1 = s_{u-1} + 1 \]
holds. Hence, we obtain \( \pi(v) = u < v \).

Appendix 2

For any tree \( T \in \mathcal{T} \) and any path \( P = (u_1, \ldots, u_k) \) \((k = 2 \text{ or } 3)\) of \( T \), we can simultaneously establish two dummies-added trees \( \tilde{T}(u_i) = (\tilde{V}(u_i), \tilde{E}(u_i)) \) \((i = 1, k)\) and a forced isomorphism \( \sigma_P \) stated in Section 2 by using the following algorithm, where \( T(u_i) = (V(u_i), E(u_i)) \) \((i = 1, \ldots, k)\) are defined for the path \( P \), and \( \tilde{V}(u_i,l) \) is defined by \( \{w \in \tilde{V}(u_i)| \text{the level of } w \text{ is } l\} \).

Procedure Make-\(\tilde{T}(u_1)\) - \( \tilde{T}(u_k)\) - and - \( \sigma_P \) \((T: \text{tree}; P: \text{path})\);
begin
\[ \tilde{V}(u_1) := V(u_1); \tilde{V}(u_k) := V(u_k); \tilde{E}(u_1) := E(u_1); \tilde{E}(u_k) := E(u_k); \]
set \( \sigma_P(u_1) := u_1; \)
dv := n; \{ dv denotes the current number of dummy vertex \}
l := 0;
while \( \tilde{V}(u_1, l) \cup \tilde{V}(u_k, l) \) has a vertex \( v \) with \( \chi(v) \neq \emptyset \) do begin
for all \( v \in \tilde{V}(u_1, l) \) do begin
\( d^* := \max\{\deg(v; \tilde{T}(u_1)), \deg(\sigma_P(v); \tilde{T}(u_1))\}; \)
\{ adding dummy vertices and edges so that \}
\( \deg(v; \tilde{T}(u_1)) = \deg(\sigma_P(v); \tilde{T}(u_1)) = d^* \}
if \( \deg(v; \tilde{T}(u_1)) < d^* \) then
for \( i := 1 \) to \( d^* - \deg(v; \tilde{T}(u_1)) \) do begin
\( \tilde{V}(u_1) := \tilde{V}(u_1) \cup \{dv\}; \)
\( \tilde{E}(u_1) := \tilde{E}(u_1) \cup \{(v, dv)\}; \)
dv := dv + 1;
end
else if \( \deg(\sigma_P(v); \tilde{T}(u_k)) < d^* \) then
end

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for \( i := 1 \) to \( d^* - \deg(\sigma_P(v); T(u_k)) \) do begin
\[
\begin{align*}
\tilde{V}(u_k) &:= V(u_k) \cup \{dv\}; \\
\tilde{E}(u_k) &:= E(u_k) \cup \{(\sigma_P(v), dv)\}; \\
dv &:= dv + 1;
\end{align*}
\]
end;
\{ making one-to-one mapping from \( \chi(v) \) to \( \chi(\sigma_P(v)) \) \}

for all \( w \in \chi(v) \) do begin
\[
\begin{align*}
\text{choose } w' \in \chi(\sigma_P(v)) \text{ to which any vertex in } \chi(v) \\
\text{is not mapped by } \sigma_P;
\end{align*}
\]
set \( \sigma_P(w) = w' \);
end;
l := l + 1;
end;

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