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AN INSPECTION GAME WITH SMUGGLER'S DECISION ON THE AMOUNT OF CONTRABAND

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Abstract This paper deals with an inspection game of Customs and a smuggler. Customs has two options: patrol or no-patrol. The smuggler makes a decision on the amount of contraband to smuggle. In a given period of days, Customs has a limited number of opportunities to patrol while the smuggler can ship any amount of contraband as long as he has not exhausted this supply. When both players take action, there are some possibilities that Customs captures the smuggler and there are also possibilities that the smuggler is successful. If the smuggler is captured or there remains no day for playing the game, the game ends. In this paper, we formulate the problem as a multi-stage two-person zero-sum stochastic game, derive a closed form of equilibrium in a specific case and investigate the properties of the optimal strategies for the players. Nearly all past research has studied the smuggler's strategy with the two choices of smuggling or no-smuggling. This paper focuses on the smuggler's decision as to the amount of contraband.

Keywords: Game theory, inspection game, multi-stage stochastic game, two-person zero-sum

1. Introduction

This paper deals with an inspection game, which can be applied to a variety of inspection problems such as the smuggling problem of contraband, arms-control treaty violation or inspection by the International Atomic Energy Agency (IAEA) for nuclear facilities. The inspection game research originates from Dresher [6], who formulated the compliance problem with the treaty of arms reduction as a multi-stage game. Maschler [12] generalized Dresher's problem. Both Dresher and Maschler considered the game where a player, called violator, wished to violate the treaty in secret and the other player, called inspector, wanted to commit himself to effective inspection. The violator must pay penalty 1 if the violation is exposed by an inspection but he can escape the exposure by side payment of penalty q. Dresher discussed special cases of $q = 1/2$ and $q = 1$, and Maschler did a general case of $0 \leq q \leq 1$.

Their research branched forth to two types of applications. One is the application to the arms-reduction treaty. This application includes the international inspection by the IAEA, which draws international interest related to atomic weapons. The research by Canty et al. [5], Avenhaus et al. [1–3] and Hohzaki [11] contribute to this type of application. Avenhaus et al. [2] surveyed past studies on compliance with regulations and treaties and categorized them into three phases. Canty et al. [5] analyzed a sampling inspection problem for nuclear materials by a sequential game model and proposed an efficient inspection strategy to induce an inspectee to comply with the Treaty on the Non-Proliferation of Nuclear Weapons or related treaties. Avenhaus and Canty [1] embedded two types of errors in the inspection into a sequential game model and analyzed effective inspection under the criterion of timeliness of detecting illegal behavior. Avenhaus and Kilgour [3] discussed a nonzero-sum one-shot game with an inspector and two inspectee countries, where the inspector distributes his inspection resource for the inspection to be executed in two countries in an effective way and the inspectees make their decision about legal or illegal action to pursue their own interest in an egoistic manner. The distribution strategy of inspection resource by the inspector is studied further by Hohzaki [11], who studies an inspection game with many inspectees and derives an optimal plan for dispatching inspection staffs to facilities in inspectee countries.

The other branch of the application of the Dresher's research is the smuggling game which is usually modeled such that it is played by a smuggler and Customs. Thomas and Nisgav [14] dealt with a smuggling game, in which Customs kept a watch on illegal actions of the smuggler by using one or two patrol boat. They formulated the problem as a multi-stage recursive game and numerically derived optimal strategies of players by repeatedly solving a one-stage matrix game step by step. Baston and Bostock [4] gave a closed form of equilibrium for a similar game. Many researchers adopt the so-called perfect-capture assumption that the inspectors or Customs capture the violator or the smuggler when both players meet. Baston and Bostock model the problem into the imperfect capture model and succeed in solving the game by introducing the capture probability depending on the number of patrol boats. But the smuggler is assumed to have at most one opportunity to ship contraband, as in the preceding papers. Garnaev [8] extended their work to a model of three patrol boats.

Sakaguchi [13] first introduced the assumption that the smuggler might take an action several times in the perfect-capture model. He considered two versions of the model. In the first version, the smuggler is compelled to take an illegal action as many times as preplanned. In the second one, he may skip some of preplanned smuggling opportunities. His model is a repeated game with several opportunities of smuggling and then the value of the game is given by multiplying the value of an element game of one stage by the number of the opportunities. Furthermore his discussion is based on the assumption that an optimal solution is always given by not a saddle point but an equilibrium point of mixed strategies. The validity of the assumption was not proved in his paper. Ferguson and Melolidakis [7] is an extension of Sakaguchi model. They introduced the assumption, as seen in some original research, that the smuggler can get rid of the capture by means of side payment $q(\leq 1)$. He must pay the penalty of unit cost 1 only if he is captured when smuggling. The problem in the model is whether the smuggler prefers to pay the penalty on his capture or the side payment in advance.

Hohzaki et al. [9] takes on a position similar to Sakaguchi's in terms of the number of opportunities of smuggling. Unlike Sakaguchi's, however, it is an imperfect capture model, where the encounter of the smuggler and Customs stochastically results in one of three cases: capture, success of smuggling or nothing. In this sense, the paper takes account of probabilities of accomplishing players' aims and their model is not a repeated game but a stochastic game. Furthermore, they introduce another assumption that the capture of the smuggler terminates the game. In the previous models of imperfect capture, the game proceeds to the next stage even though the smuggler is captured. Hohzaki [10] developed another version of the game, where the smuggler is forced to try smuggling a preplanned number of times.

As reviewed above, almost all past research has treated a two-choice strategy of smuggling or non-smuggling as a smuggler strategy. Here we include another strategic issue for the smuggler: how much of the contraband should be shipped at any one time. In this paper, we deal with the smuggling strategy on the amount of contraband, which has not been discussed in the past, and we analyze a stochastic multi-stage game with a smuggler and Customs.

Through this study, we can clarify the characteristics of the smuggler's decision making. The analysis gives us some lessons about when the smuggler is likely to smuggle a lot, how often Customs has to go patrol and how the improvement on the effectiveness of an individual patrol could deter the smuggling. Furthermore, we could consider a good or bad policy against the smuggler's rational behavior, e.g. how the budget cut for patrol affects the income of the smuggler. These are the direction this paper points toward.

In the next section, we describe a smuggling problem and formulate it as a two-person zero-sum stochastic game with multiple stages, considering the transition of states of the game. In Section 3 and 4, we develop a theory for a specific-formatted game. In Section 3, we solve some special cases of the game and derive optimal solutions. These are extended to obtain a closed form of equilibrium for the general problem in Section 4. In Section 5, we take some numerical examples and clarify the properties of optimal strategies of players by our general model.

2. Modeling and Formulation

Here we describe some assumptions about a multi-stage two-person zero-sum inspection game, where Player A patrols to prevent smuggling by Player B, a smuggler.

A1. We consider a multi-stage game, where each of two players takes action once a day during N days. We count the number of stages by residual days.

A2. Player A can patrol at most K times. If $K > N$, the excess opportunities to patrol are of no value. At an initial stage with N residual days, Player B has $X > 0$ contraband on hand and desires to smuggle as much as possible.

A3. At each day or each stage, Player A decides to patrol or not, while Player B determines the integer amount of contraband to smuggle with an upper limit of current contraband on hand.

A4. When Player A patrols and Player B tries to smuggle y contraband on the same day, A can capture B with probability of $q_1(y) > 0$ but B succeeds to smuggle with probability $q_2(y) > 0$, where $q_1(y)+q_2(y) \leq 1$. With the residual probability of $1-q_1(y)-q_2(y)$, there is no capture and no success of smuggling (no-event). Even for no event, the contraband Player B tries to smuggle is lost. If Player A does not patrol, B will certainly succeed in his smuggling.
Probabilities $q_1(y)$ and $q_2(y)$ are assumed to be monotonic nondecreasing and nonincreasing for y, respectively, with initial value $q_1(0) = 0$.

A5. The payoff of the game is zero-sum. Successful smuggling yields Player B a reward of 1 per unit of smuggled contraband while Player A loses the same. If Player A captures B, A gets reward $\alpha > 0$, which is a number relative to the value of contraband. We define the payoff of the game by the reward of Player A.

A6. Unless Player A captures B, the game transfers to the next stage. At the beginning of each stage, Player B can know the action of patrol or no-patrol Player A took at the previous stage because Player A might be a public organization such as the coast guard or Customs and his actions could be comparatively open. Player A also gets the information about the B's past action of smuggling or no-smuggling from his continuous analysis or intelligence activity even though Player A did not patrol. Upon arrest of Player B or the expiration of the preplanned N days, the inspection game ends.

In the multi-stage zero-sum game, Player A behaves as a maximizer and B as a minimizer. Let us consider a state, where n stages remain, Player A has at most k chances to patrol, and Player B has x contraband on hand. Each stage consists of a smuggler's action and a Customs' action. Because both players can recognize a resultant state each time they finish playing a stage game from Assumption A6, an information set is made of one node or one state (n, k, x) at the beginning of the stage n on the game tree. In the state, players are interested in their future reward after the current state whatever happened in the past and we limit our discussion to Markov Nash equilibrium, where the player takes the same strategy in the same state, in this paper. If we denote the stage game starting with state (n, k, x) by $\Gamma(n, k, x)$, we have the following recursive formulation.

$$\Gamma(n,k,x) \equiv$$
$$\begin{array}{c} \\ P \\ NP \end{array} \begin{pmatrix} S\ (y) & NS\ (y=0) \\ \alpha q_1(y) - yq_2(y) + (1-q_1(y))\Gamma(n-1,k-1,x-y) & \Gamma(n-1,k-1,x) \\ -y + \Gamma(n-1,k,x-y) & \Gamma(n-1,k,x) \end{pmatrix} \quad (1)$$

The two rows correspond to two strategies of Player A: $\{Patrol(P),\ No-Patrol(NP)\}$. Columns indicate the entire strategies of Player B about the amount of contraband: $y = x, x-1, \cdots, 1, 0$. There are $x+1$ strategies in total but we adopt notation $S(y)$ for smuggling of y contraband, and $S(0)$ or NS for no-smuggling. Each element in the matrix above is validated as follows.

For the element of P of Player A and $S(y)$ of Player B, reward $\alpha q_1(y) - yq_2(y)$ is expected for Player A at the current stage and the game transfers to the next stage $n-1$ only if Player B is not arrested with probability $1 - q_1(y)$. At the next stage, Player A has already used up a chance of patrolling and the amount of untouched contraband of Player B decreases by y. In the second row where Player A does not patrol, Player A certainly loses reward y by an easy smuggling of Player B and the stage transfers to the next while Player A keeps the same opportunities of patrolling as before.

By replacing $\Gamma(n, k, x)$ in Equation (1) with its value of the game $v(n, k, x)$, we derive a recurrence formula for our stochastic multi-stage game.

$$\begin{aligned} &v(n,k,x) \\ &= val \begin{pmatrix} \alpha q_1(y) - yq_2(y) + (1-q_1(y))v(n-1,k-1,x-y) & v(n-1,k-1,x) \\ -y + v(n-1,k,x-y) & v(n-1,k,x) \end{pmatrix}, \end{aligned} \quad (2)$$

where the first column is for a general strategy of smuggling of y contraband, $S(y)$, and the second for NS strategy. The symbol 'val' indicates the value of the matrix game. Furthermore we have some initial conditions for stage $n = 0$ and boundary ones in special cases.

$$v(0,k,x) = 0, \quad v(n,k,0) = 0, \quad v(n,0,x) = -x \ (n > 0), \quad (3)$$
$$v(n,k,x) = v(n,n,x) \quad (if\ \ k > n) \quad (4)$$

In our modeling, patrols cost nothing and Player A is going to patrol as many times as preplanned. Player A deliberates over when he patrols: patrol now or later. On the other hand, Player B can divide all contraband into portions and try to smuggle as many times as the number of portions. He has to think of the termination of the game by his capture with probability $q_1(y)$, which means the abandonment of the rest of untouched contraband, and the temporal waste of contraband with probability $1 - q_1(y) - q_2(y)$.

We already have a numerical solution method to calculate an equilibrium solution through all stages. Beginning with initial condition (3), we repeat solving the matrix game (2) from Stage $n = 1$ to N. In Section 3 and 4, we are going to develop a theory for a specific-formatted game, where the capture probability and the success probability are constants, i.e. $q_1(y) = p_1$ and $q_2(y) = p_2$ for $y > 0$, and derive closed-forms of optimal strategies and the value of the game using the special setting of parameters. On the other hand, we might give up theoretical development for our general modeling with the dependency of $q_1(y)$ and $q_2(y)$ on the amount of contraband y. We take the general modeling again to analyze optimal strategies of players in Section 5.

Let us confirm the specific-formatted matrix game by applying fixed parameters p_1 and p_2 to Equation (2), as follows.

$$\begin{aligned} &v(n,k,x) \\ &= val \begin{pmatrix} \alpha p_1 - p_2 x & \alpha p_1 - p_2 y + (1-p_1)v(n-1,k-1,x-y) & v(n-1,k-1,x) \\ -x & -y + v(n-1,k,x-y) & v(n-1,k,x) \end{pmatrix}. \end{aligned} \tag{5}$$

The matrix has $x+1$ columns in practice but we abbreviate the matrix by three columns which corresponds to $S(x)$, $S(y)$ and NS, as Equation (5).

3. A Procedure for Solutions and Equilibria for Special Cases

Here we focus on the game with constant probabilities, $q_1(y) = p_1$ and $q_2(y) = p_2$. In Section 3.1, we derive an analytical form of Nash equilibrium in a special case. In Section 3.2, we point out that Nash equilibrium is obtained by recursively solving a difference equation under a basic assumption.

Starting with initial conditions (3), we can recursively solve the matrix game (5) while changing indices like $n = 1, \cdots, N,\ k = 1, \cdots, K,\ x = 1, \cdots, X$ to reach the value of the game in an arbitrary state of N days, K opportunities of patrolling and X amount of contraband. By this procedure, we are given optimal strategies of players at each stage. Let us try to solve some games in special cases, preliminary to general solutions as shown in Section 4.

3.1. Case of $k = n$

Here we are going to derive $v(1,1,x)$ with $x > 0$. From conditions (3) and (4), we have a matrix game (5)

$$v(1,1,x) = val \begin{pmatrix} \alpha p_1 - p_2 x & \leq \alpha p_1 - p_2 y & 0 \\ \vee & \vee & \| \\ -x & < \ -y \ < & 0 \end{pmatrix} \tag{6}$$

for $v(1,1,x)$. In the matrix, inequality symbols $<$ or others are written in between two elements. They show us that the entirely smuggling strategy (ESS for short), $S(x)$, dominates other smuggling strategies or partially smuggling strategies (PSSs for short): $S(y),\ 0 < y < x$. Then P weakly dominates NP for Player A's strategy. These results lead us to a value of the game $v(1,1,x) = \min\{\alpha p_1 - p_2 x,\ 0\}$.

Next let us enumerate all forms of optimal strategies. We denote probabilities of taking strategy P and NP by π and $1-\pi$, respectively. As a mixed strategy of Player B, we take probability ρ for strategy $S(x)$ and $1-\rho$ for NS. By the mixed strategies of both players, the expected payoff $R(\pi, \rho)$ is given by

$$R(\pi, \rho) = \rho\{\pi(\alpha p_1 - p_2 x + x) - x\}. \tag{7}$$

From this, we can derive an optimal response for a player to the other as follows. An optimal strategy $\pi^*(\rho)$ of Player A corresponding to Player B's strategy ρ is (a) arbitrary if $\rho = 0$ and (b) $\pi^*(\rho) = 1$ if $\rho > 0$.

Conversely, an optimal B's strategy $\rho^*(\pi)$ responding to an A's strategy π is as follows. (i) In the case of $\alpha p_1 - p_2 x > 0$: Using

$$\pi_1 = \frac{x}{\alpha p_1 - p_2 x + x}, \tag{8}$$

(a) $\rho^*(\pi) = 0$ if $\pi > \pi_1$, (b) $\rho^*(\pi)$ is arbitrary if $\pi = \pi_1$ and (c) $\rho^*(\pi) = 1$ if $\pi < \pi_1$. (ii) In the case of $\alpha p_1 - p_2 x = 0$: (a) $\rho^*(\pi)$ is arbitrary if $\pi = 1$ and (b) $\rho^*(\pi) = 1$ if $\pi < 1$. (iii) In the case of $\alpha p_1 - p_2 x < 0$: $\rho^*(\pi) = 1$.

By analysis of optimal responses, all combinations of the responses, namely equilibrium points (π^*, ρ^*), are given by

$$(i) \quad If \ \ \alpha p_1 - p_2 x > 0, \quad (\pi^*, \rho^*) = (arbitrary\ satisfying\ \pi \geq \pi_1,\ 0). \tag{9}$$

$$(ii) \quad If \ \ \alpha p_1 - p_2 x = 0, \quad (\pi^*, \rho^*) = (1,\ arbitrary). \tag{10}$$

$$(iii) \quad If \ \ \alpha p_1 - p_2 x < 0, \quad (\pi^*, \rho^*) = (1,\ 1). \tag{11}$$

For $\Gamma(n, n, x)$ in the case of $k = n$, we can take the similar analysis to derive its equilibrium. That is why we detail the derivation of an equilibrium for $(n, k) = (1, 1)$.

Lemma 3.1 (Case of $k = n$)**.** *In the case of $k = n$, in which Player A can afford to patrol every day, the value of the game is*

$$v(n, n, x) = \min\{\alpha p_1 - p_2 x,\ 0\}. \tag{12}$$

Optimal strategies of players are given by (9)∼(11) in the case of $n = 1$. In the case of $n > 1$, we have two types of equilibria.

$$(i) \quad If \ \ \alpha p_1 - p_2 x > 0, \ (\pi^*, \rho^*) = (arbitrary\ satisfying\ \pi \geq \pi_1,\ 0). \tag{13}$$

$$(ii) \quad If \ \ \alpha p_1 - p_2 x \leq 0, \ (\pi^*, \rho^*) = (1,\ arbitrary). \tag{14}$$

Proof. Let us prove the lemma by mathematical induction. As shown above, the lemma is valid for $n = k = 1$. Now we assume the validity of the lemma for $v(n-1, n-1, y)$. Considering condition (4), the matrix game (5) is written in

$$v(n, n, x) = \\ val \begin{pmatrix} \alpha p_1 - x p_2 & \alpha p_1 - p_2 y + (1-p_1) v(n-1, n-1, x-y) & v(n-1, n-1, x) \\ \vee & & \| \\ -x \quad < & -y + v(n-1, n-1, x-y) \quad < & v(n-1, n-1, x) \end{pmatrix}.$$

We can assure a relation $-x < -y + v(n-1, n-1, x-y) < v(n-1, n-1, x)$ between elements in the 2nd row by reasoning as follows. The middle value is realized by the perfect success of smuggling of contraband y against all-day patrolling of Player A. The first value is given by the perfect smuggling of the whole contraband x. The relation remains valid as long as $v(n-1, n-1, y) = \min\{\alpha p_1 - y p_2, 0\} > -y$. The relation in the 1st row $v(n-1, n-1, x) \leq \alpha p_1 - x p_2 < \alpha p_1 - y p_2 + (1-p_1) v(n-1, n-1, x-y)$ is evident. Considering the monotone decrease of two terms in the brace $\{\cdot\}$ of the last expression of

$$\begin{aligned} &\alpha p_1 - y p_2 + (1-p_1) v(n-1, n-1, x-y) \\ &\qquad = \alpha p_1 - y p_2 + (1-p_1) \min\{\alpha p_1 - (x-y) p_2, 0\} \\ &\qquad = \min\{\alpha p_1 + (1-p_1)(\alpha p_1 - x p_2) - y p_1 p_2,\ \alpha p_1 - y p_2\}, \end{aligned}$$

we have $\alpha p_1 - yp_2 + (1-p_1)v(n-1, n-1, x-y) > \alpha p_1 - xp_2$ by the substitution $y = x$. Thus the dominance relation between some strategies gives us $v(n,n,x) = v(n-1, n-1, x)$. Now we come to obtain the expected payoff $R(\pi, \rho) = \rho\{\pi(\alpha p_1 - p_2 x + x) - (x + v(n-1, n-1, x))\} + v(n-1, n-1, x)$. By the similar way that we derive three types of equilibria (9)∼(11), we can derive two types of equilibria (13) and (14) in the case of $n > 1$. □

We notice that a slight difference between two cases of $n = 1$ and $n > 1$ exists in (11) and (14). In the situation that Player A can patrol every day, namely, $k = n$, and $\alpha p_1 - p_2 x \leq 0$, the expected reward of Player B is the same whenever he ships the whole contraband. He may decide just to smuggle at the terminal stage $n = 1$ and not at any other stages. We also notice that the value of $\alpha p_1 - p_2 x$ has an influence on optimal strategies. Let us call the value the discriminant value of the inspection game.

3.2. Basic assumption and a procedure for solution

Here, we rely on the recurrence equation (5) to solve the multi-stage inspection game. Let us proceed further for a general solution with the following additional assumption (called No-Partially-Smuggling (NPS) assumption), the correctness of which will be proved later.

NPS Assumption: Optimal strategy of Player B depends on two pure strategies: ESS and NS.

By this assumption, the game (5) becomes the following simple 2×2 matrix game.

$$v(n,k,x) = val \begin{pmatrix} \alpha p_1 - p_2 x & v(n-1, k-1, x) \\ -x & v(n-1, k, x) \end{pmatrix}$$

We state some self-evident inequalities, as mentioned before.

$$-x \leq -y + v(n-1, k, x-y) \leq v(n-1, k, x), \tag{15}$$

$$-x < \alpha p_1 - p_2 x, \quad v(n-1, k-1, x) \leq v(n-1, k, x), \quad v(n,k,x) \leq 0 \tag{16}$$

The nonpositiveness of the value of the game, which is indicated by the last inequality, is understandable because Player B always makes the payoff zero by persisting with NS strategy. For any matrix game, we have a relation of minimax value ≥ value of the game ≥ maximin value. Therefore, from (16), the following inequalities hold:

$$\begin{aligned} \min\{\alpha p_1 - p_2 x, v(n-1, k, x)\} &\geq v(n,k,x) \\ &\geq \max\{v(n-1, k-1, x), -x\} \\ &= v(n-1, k-1, x). \end{aligned}$$

To derive the second inequality, we use $\alpha p_1 - p_2 x \geq v(n-1, k-1, x)$. We can get this by applying the following inequality, which comes from the former inequality of the above, to $v(n-1, k-1, x)$.

$$\alpha p_1 - p_2 x \geq v(n,k,x), \quad v(n-1, k, x) \geq v(n,k,x). \tag{17}$$

We have to note that inequality (17) is valid based on the NPS assumption. Inequalities (15) and (16) are, however, already proved without the NPS. We will illustrate the inequalities discussed so far in the matrix of the game.

$$v(n,k,x) = val \begin{pmatrix} \alpha p_1 - p_2 x & \geq & v(n-1, k-1, x) \\ \vee & & \wedge\!| \\ -x & \leq & v(n-1, k, x) \end{pmatrix} \tag{18}$$

This points out that the game $\Gamma(n,k,x)$ has no saddle point but a mixed strategy as its equilibrium point. Therefore, an optimal probability of patrol π^* is derived from an equation

$$\pi^*(\alpha p_1 - p_2 x) + (1-\pi^*)(-x) = \pi^* v(n-1,k-1,x) + (1-\pi^*)v(n-1,k,x),$$

which represents the value of the game $v(n,k,x)$. By substituting the optimal strategy

$$\pi^* = \frac{x + v(n-1,k,x)}{\alpha p_1 - p_2 x + x + v(n-1,k,x) - v(n-1,k-1,x)}$$

into the above equation, we obtain the following recursive equation for the value of the game:

$$v(n,k,x) = \frac{(\alpha p_1 - p_2 x)\cdot v(n-1,k,x) + x\cdot v(n-1,k-1,x)}{\alpha p_1 - p_2 x + x + v(n-1,k,x) - v(n-1,k-1,x)}. \tag{19}$$

We can use this equation to generally calculate the value of the game in two ways. First, we fix the number of patrol opportunities $k-1$ and solve the equation as a difference equation between $v(n,k,x)$ and $v(n-1,k,x)$ with given $\{v(n,k-1,x), n=1,\cdots,N\}$. For example, fixing $k=1$ and substituting $v(n-1,k-1,x) = -x$ into Equation (19), we can solve the difference equation of $v(n,1,x)$ and $v(n-1,1,x)$. The procedure is repeated for $k=2,3,\cdots$ to obtain the value of the game for all k.

In the second way, we assume that $\{v(n,n-s,x),\ n=1,2,\cdots,N\}$ are given for a fixed s and regard the equation as a difference equation between $\{v(n,n-(s+1),x)\}$ and $\{v(n-1,n-1-(s+1),x)\}$ to solve and obtain $v(n,n-(s+1),x)$ for $n=1,2,\cdots,N$. From Lemma 3.1, we already know $\{v(n,n,x),\ n=1,\cdots,N\}$, which is used to make a concrete difference equation (19) with $k=n-1$, that is, a difference equation between $v(n,n-1,x)$ and $v(n-1,(n-1)-1,x)$. We next solve it to obtain $\{v(n,n-1,x), n=1,\cdots,N\}$. If we substitute the values into $v(n-1,n-2,x)$ of Equation (19) with $k=n-2$, we have another difference equation for $\{v(n,n-2,x), n=1,\cdots,N\}$.

For convenience sake, we use the following expression as a substitute for $v(n,k,x)$.

$$y(n,k,x) \equiv \frac{1}{v(n,k,x)+x}. \tag{20}$$

Using this symbol, the recurrence equation (19) has a simpler form of $y(n,k,x)$, as follows.

$$\begin{aligned} y(n,k,x) &= y(n-1,k,x) + \frac{1}{\gamma(x)}\left(1 - \frac{y(n-1,k,x)}{y(n-1,k-1,x)}\right) \\ &= \left(1 - \frac{1}{\gamma(x)y(n-1,k-1,x)}\right) y(n-1,k,x) + \frac{1}{\gamma(x)}, \end{aligned} \tag{21}$$

where

$$\gamma(x) \equiv \alpha p_1 - x p_2 + x. \tag{22}$$

Noting that Equation (19) with $k=1$ and $v(n-1,0,x) = -x$ equals the equation with $y(n-1,0,x) = 1/(v(n-1,0,x)+x) = \infty$, we can substitute $y(n,0,x) = \infty$ in some expressions for $k=0$.

From now, we are going to find a sufficient condition that the NPS assumption is valid. Let π and $1-\pi$ be the probabilities of Player A's taking strategy P and NP, respectively. If Player B adopts a pure strategy $S(y)$, the expected payoff is

$$\begin{aligned} R(\pi, S(y)) &= \pi\{\alpha p_1 - p_2 y + (1-p_1)v(n-1,k-1,x-y)\} \\ &\quad +(1-\pi)(-y + v(n-1,k,x-y)). \end{aligned}$$

If we draw a line of the payoff on a plane with π as x-coordinate and $R(\pi, S(y))$ as y-coordinate, it runs through two points $(0,\ -y+v(n-1,k,x-y))$ and $(1,\ \alpha p_1 - p_2 y + (1-p_1)v(n-1,k-1,x-y))$. This mapping of the expected payoff is a general way to calculate a maximin value $\max_\pi \min_{\{S(y),y=0,\cdots,x\}} R(\pi, S(y))$ and find an optimal strategy π^* of Player A. $R(\pi, S(x))$ is a line with a positive inclination running through two points $(0,\ -x)$ and $(1,\ \alpha p_1 - p_2 x)$ while line $R(\pi, NS)$ has a negative inclination and goes at points $(0,\ v(n-1,k,x))$ and $(1,\ v(n-1,k-1,x))$. Intercepts of these three lines at $\pi = 0$ increase in the order of $R(\pi, S(x))$, $R(\pi, S(y))$ and $R(\pi, NS)$, as shown by inequality (15). As we can see from the first inequality of (17), the intercept of line $R(\pi, S(x))$ is larger than that of $R(\pi, NS)$ at $\pi = 1$. From these facts, two lines $R(\pi, S(x))$ and $R(\pi, NS)$ cross each other and the cross point is given by

$$\begin{aligned}\pi_3 &= \frac{v(n-1,k,x)+x}{\gamma(x)+v(n-1,k,x)-v(n-1,k-1,x)} \\ &= \frac{1}{\gamma(x)y(n-1,k,x)+1-y(n-1,k,x)/y(n-1,k-1,x)}. \end{aligned} \tag{23}$$

If the intercept of $R(\pi, S(x))$ is smaller than that of $R(\pi, S(y))$ at $\pi = 1$, that is, $\alpha p_1 - p_2 x < \alpha p_1 - p_2 y + (1-p_1)v(n-1,k-1,x-y)$, two lines do not cross in an interval $0 \le \pi \le 1$. In this case, strategy $S(x)$ dominates $S(y)$ and then we do not need $S(y)$ to derive an optimal strategy of Player B. Conversely, in the case of $\alpha p_1 - p_2 x \ge \alpha p_1 - p_2 y + (1-p_1)v(n-1,k-1,x-y)$, the two lines cross at

$$\pi_4 = \frac{v(n-1,k,x-y)+x-y}{(1-p_2)(x-y)-(1-p_1)v(n-1,k-1,x-y)+v(n-1,k,x-y)} =$$
$$\frac{1}{(1-p_1-p_2)(x-y)y(n-1,k,x-y)-(1-p_1)y(n-1,k,x-y)/y(n-1,k-1,x-y)+1}. \tag{24}$$

As understood by Figure 1, if and only if $\pi_3 \le \pi_4$, an equilibrium point or optimal π is determined by the cross point of only two lines $R(\pi, S(x))$ and $R(\pi, NS)$, and $R(\pi, S(y))$, $y \ne x$ does not affect the equilibrium at all.

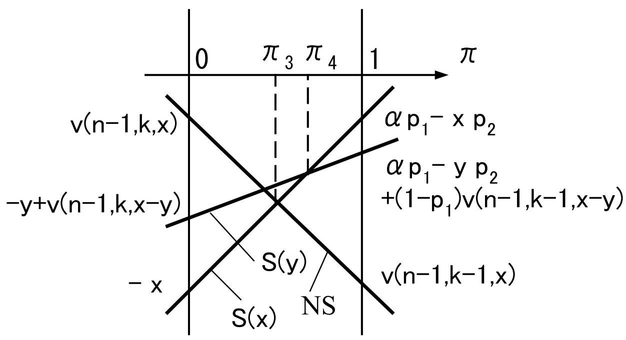

Figure 1: Expected payoff by Player A's mixed strategy π

If we replace $y(n-1,k,x-y)/y(n-1,k-1,x-y)$ in Equations (23) and (24) with $y(n,k,x-y)$ and $y(n-1,k,x-y)$ using Equation (21), π_3 and π_4 is expressed in simpler forms.

$$\pi_3 = \frac{1}{\gamma(x)y(n,k,x)} \tag{25}$$

$$\pi_4 = \frac{1}{(1-p_1)\gamma(z)y(n,k,z)-\{(1-p_1)\gamma(z)-(1-p_1-p_2)z\}y(n-1,k,z)+p_1}, \tag{26}$$

where $z \equiv x - y$.

Summing up what we discussed so far, when there is no dominance among the smuggling strategies, $\pi_3 \le \pi_4$ is necessary and sufficient for an equilibrium to be determined by only ESS and NS. When there is a dominance relation, there could be three cases of $1 < \pi_4$, $\pi_4 < 0$ or $\pi_4 = \pm\infty$ which might happen by the parallel running of lines $R(\pi, S(x))$ and $R(\pi, S(y))$. At least, $\pi_3 \le \pi_4$ is the sufficient condition that only two pure strategies of ESS and NS determine the equilibrium point for the game. That is what the following lemma illustrates.

Lemma 3.2. *A sufficient condition that the equilibrium of matrix game $\Gamma(n,k,x)$ is determined by a mixed strategy of ESS ($S(x)$) and NS is that, for any z ($0 \le z \le x$),*

$$(1-p_1)\gamma(z)y(n,k,z)-\{(1-p_1)\gamma(z)-(1-p_1-p_2)z\}y(n-1,k,z)+p_1 \le \gamma(x)y(n,k,x). \tag{27}$$

If the condition (27) is always correct, not $\pi_4 < 0$ but $1 < \pi_4$ holds whenever there exists the dominance among Player B's strategies. That means that the inclination of line $R(\pi, S(x))$ is larger than that of $R(\pi, S(y))$ and indicates the following inequality

$$\gamma(y) + (1-p_1)v(n-1,k-1,x-y) - v(n-1,k,x-y) \le \gamma(x). \tag{28}$$

We may repeat the results of this subsection for the analysis of game $\Gamma(n,k,x)$. Under the NPS assumption, we have Equations (17)-(21) and the value of the game is given by (19) or (21). If we can verify inequality (27) by calculating $y(n,k,x)$, we can say that the NPS assumption is correct. For the matrix game $\Gamma(n,k,x)$, we are going to derive an optimal strategy of Player B ρ^*, the probability of smuggling the whole contraband. It can be done in the similar way that we obtained Equations (23) or (25) for an optimal strategy of Player A.

$$\pi^* = \frac{v(n-1,k,x)+x}{\gamma(x)+v(n-1,k,x)-v(n-1,k-1,x)} = \frac{1}{\gamma(x)y(n,k,x)} \tag{29}$$

$$\rho^* = \frac{v(n-1,k,x)-v(n-1,k-1,x)}{\gamma(x)+v(n-1,k,x)-v(n-1,k-1,x)} = 1 - \frac{y(n-1,k,x)}{y(n,k,x)}. \tag{30}$$

4. A General Solution for the Game with Constant Probabilities

We solve the difference equation (21) by specifying parameter k to obtain the value of game $\Gamma(n,k,x)$. Here we show you the general form of the value of the game. At the same time, we prove the validity of the NPS assumption to complete finding the equilibrium of our multi-stage inspection game.

Theorem 4.1. *If $\alpha p_1 - x p_2 < 0$, the value of the game $\Gamma(n,k,x)$ is*

$$y(n,k,x) = \frac{n}{k\gamma(x)} \tag{31}$$

$$v(n,k,x) = \frac{k\gamma(x)}{n} - x, \tag{32}$$

where $\gamma(x) \equiv \alpha p_1 - xp_2 + x$. *The optimal patrolling strategy* π^* *and the optimal smuggling strategy* ρ^* *are given by*

$$\pi^* = \frac{k}{n}, \quad \rho^* = \frac{1}{n}. \tag{33}$$

Proof. Let us prove the theorem by induction. In the case of $n = 1$, (32) is valid for $k = 0, 1$ from Equations (3) and (12). Assume that the theorem is correct for $\{v(n-1, k, x),\ k = 1, \cdots, n-1,\ x = 1, 2, \cdots\}$. We can transform the right-hand side of Equation (21) to

$$\left(1 - \frac{k-1}{n-1}\right) \frac{n-1}{k\gamma(x)} + \frac{1}{\gamma(x)} = \frac{n-k}{k\gamma(x)} + \frac{1}{\gamma(x)} = \frac{n}{k\gamma(x)}$$

to see the validity of the theorem for $\{v(n, k, x),\ k = 1, \cdots, n-1,\ x = 1, 2 \cdots\}$. For $k = n$, we can verify that from Equation (12). The optimal strategy (33) is easily calculated from Equations (29) and (30). □

Formula (31) brings us $y(n, 0, x) = \infty$ for $k = 0$, from which we are also given initial value $v(n, 0, x) = -x$ of (3) using (32). We are permitted to use infinity $y(n, k, x)$ for special cases.

Theorem 4.2. *If* $\alpha p_1 - xp_2 \geq 0$, *the value of the game* $\Gamma(n, k, x)$ *is*

$$\begin{aligned} y(n,k,x) &= \frac{\sum_{l=0}^{k} {}_{n-k+l-1}C_l \cdot x^l \gamma(x)^{k-l}}{x \sum_{l=0}^{k-1} {}_{n-k+l-1}C_l \cdot x^l \gamma(x)^{k-l}} \\ &= \frac{1}{x}\left(1 + \frac{{}_{n-1}C_k}{\sum_{l=0}^{k-1} {}_{n-k+l-1}C_l \cdot (\gamma(x)/x)^{k-l}}\right) \end{aligned} \tag{34}$$

$$v(n,k,x) = -\frac{{}_{n-1}C_k \cdot x^{k+1}}{\sum_{l=0}^{k} {}_{n-k+l-1}C_l \cdot x^l \gamma(x)^{k-l}}, \tag{35}$$

where ${}_mC_l$ *is a substitute for combination* $\binom{m}{l}$.

For $k = n$, ${}_{l-1}C_l$ appears in the formulas above. We take the factorial function to be 1 in the case of $l = 0$ and zero in the case of $l > 0$ by ordinary custom. The custom gives us $y(n, n, x) = 1/x$ and $v(n, n, x) = 0$, which are consistent with Equation (12). For $k = 0$, the dominator includes operator $\sum_{l=0}^{-1}$. Mathematical custom tells us that the operator is zero. From the definition, we obtain $y(n, 0, x) = \infty$ or $v(n, 0, x) = -x$, which also consists with initial conditions (3).

Proof. The proof will be done by induction. The remarks in the end of the theorem convince us that theorem holds for $k = 0,\ n$. Let us apply formula (34) to $y(n-1, k, x)$ and $y(n-1, k-1, x)$ on the right-hand side of the recurrence equation (21). First we have a transformation

$$\begin{aligned} 1 - \frac{1}{\gamma(x)y(n-1,k-1,x)} &= 1 - \frac{x\sum_{l=0}^{k-2} {}_{n-k+l-1}C_l x^l \gamma(x)^{k-l-1}}{\gamma(x)\sum_{l=0}^{k-1} {}_{n-k+l-1}C_l x^l \gamma(x)^{k-l-1}} \\ &= \frac{\sum_{l=0}^{k-1} {}_{n-k+l-1}C_l x^l \gamma(x)^{k-l} - \sum_{l=0}^{k-2} {}_{n-k+l-1}C_l x^{l+1} \gamma(x)^{k-l-1}}{\gamma(x)\sum_{l=0}^{k-1} {}_{n-k+l-1}C_l x^l \gamma(x)^{k-l-1}}. \end{aligned}$$

If we now replace index $l+1$ in the second term of the numerator above with l, we have

$$\begin{aligned} numerator &= \sum_{l=0}^{k-1} {}_{n-k+l-1}C_l x^l \gamma(x)^{k-l} - \sum_{l=1}^{k-1} {}_{n-k+l-2}C_{l-1} x^l \gamma(x)^{k-l} \\ &= \gamma(x)^k + \sum_{l=1}^{k-1} {}_{n-k+l-2}C_l x^l \gamma(x)^{k-l} = \sum_{l=0}^{k-1} {}_{n-k+l-2}C_l x^l \gamma(x)^{k-l}. \end{aligned}$$

Consequently, the right-hand side of Equation (21) is transformed to

$$\begin{aligned} &\left(1 - \frac{1}{\gamma(x)y(n-1,k-1,x)}\right) y(n-1,k,x) + \frac{1}{\gamma(x)} \\ &= \frac{\sum_{l=0}^{k} {}_{n-k+l-2}C_l x^l \gamma(x)^{k-l} + \sum_{l=0}^{k-1} {}_{n-k+l-1}C_l x^{l+1} \gamma(x)^{k-l-1}}{x\gamma(x) \sum_{l=0}^{k-1} {}_{n-k+l-1}C_l x^l \gamma(x)^{k-l-1}} \\ &= \frac{\sum_{l=0}^{k} {}_{n-k+l-1}C_l x^l \gamma(x)^{k-l}}{x \sum_{l=0}^{k-1} {}_{n-k+l-1}C_l x^l \gamma(x)^{k-l}} \end{aligned}$$

and it is equal to the expression (34). □

Theorem 4.1 and 4.2 are classified by condition $\alpha p_1 - x p_2 = 0$ on the amount of contraband x. We can imagine that under this condition, both theorems give us the same value of the game. Really, from $\gamma(x) = x$ and a formula $\sum_{l=0}^{k} {}_{n-k+l-1}C_l = {}_nC_{n-k}$, we can verify the equality of Equation (34) to (31) as follows.

$$y(n,k,x) = \frac{\sum_{l=0}^{k} {}_{n-k+l-1}C_l \cdot x^k}{x \sum_{l=0}^{k-1} {}_{n-k+l-1}C_l \cdot x^k} = \frac{\sum_{l=0}^{k} {}_{n-k+l-1}C_l}{x \sum_{l=0}^{k-1} {}_{n-k+l-1}C_l} = \frac{{}_nC_{n-k}}{x {}_{n-1}C_{n-k}} = \frac{n}{kx} = \frac{n}{k\gamma(x)}$$

Both theorems are based on the NPS assumption. Now the validity of the NPS assumption is left to be proved for us.

Theorem 4.3. *The value $y(n,k,x)$ given by Theorem 4.1 and 4.2 satisfies inequality (27). Therefore, optimal smuggling strategy is generated from only two pure strategies ESS and NS.*

Proof. We must prove the theorem in two cases of $\alpha p_1 - x p_2 < 0$ and $\alpha p_1 - x p_2 \ge 0$.
(i) Case of $\alpha p_1 - x p_2 < 0$:
(a) Let us prove condition (27) for any z of $\alpha p_1 - z p_2 < 0$. From Theorem 4.1, the right-hand side of (27) is $\gamma(x) y(n,k,x) = n/k$. For the left-hand side, we have the following transformation.

$$\begin{aligned} &(1-p_1)\gamma(z)\frac{n}{k\gamma(z)} - \{(1-p_1)\gamma(z) - (1-p_1-p_2)z\} \frac{n-1}{k\gamma(z)} + p_1 \\ &= \frac{1-p_1}{k} + (1-p_1-p_2)\frac{(n-1)z}{k\gamma(z)} + p_1 = \frac{1-p_1}{k} + (1-p_1-p_2)\frac{n-1}{k(\alpha p_1/z - p_2 + 1)} + p_1 \\ &\le \frac{1-p_1}{k} + \frac{(1-p_1-p_2)(n-1)}{k(1-p_2)} + p_1 \le \frac{1-p_1}{k} + \frac{n-1}{k}\left(1 - \frac{p_1}{1-p_2}\right) + p_1 \\ &\le \frac{1-p_1}{k} + \frac{n-1}{k}(1-p_1) + p_1 = \frac{n}{k} + p_1\left(1 - \frac{n}{k}\right) \le \frac{n}{k} \end{aligned}$$

and verify the validity of (27).
(b) For any z of $\alpha p_1 - zp_2 \geq 0$, we have another transformation of the left-hand side of (27) from Theorem 4.2.

$$(1-p_1)\left(\frac{{}_{n-1}C_k}{\sum_{l=0}^{k-1}{}_{n-k+l-1}C_{n-k-1}(\gamma(z)/z)^{k-l-1}} - \frac{{}_{n-2}C_k}{\sum_{l=0}^{k-1}{}_{n-k+l-2}C_{n-k-2}(\gamma(z)/z)^{k-l-1}}\right) + (1-p_1-p_2)\left(1+\frac{{}_{n-2}C_k}{\sum_{l=0}^{k-1}{}_{n-k+l-2}C_{n-k-2}(\gamma(z)/z)^{k-l}}\right) + p_1. \quad (36)$$

$\alpha p_1 - zp_2 \geq 0$ leads us to $\gamma(z) \geq z$ and then $\gamma(z)/z \geq 1$. Now we consider the maximization of the function above with respect to newly-defined continuous variable $w \equiv \gamma(z)/z$ for $w \geq 1$. Because z is originally discrete, the maximum value is larger than the limited maximum one under the constraint of the discreteness of z.

In the first parenthesis of (36), there are two fractions. Their denominators, $\sum_{l=0}^{k-1} {}_{n-k+l-1}C_{n-k-1} \cdot w^{k-l-1}$ and $\sum_{l=0}^{k-1} {}_{n-k+l-2}C_{n-k-2} \cdot w^{k-l-1}$, are polynomials of z with the same number of terms. Their coefficients have a relation of ${}_{n-k+l-1}C_{n-k-1} = (n-k+l-1)/(n-k-1) \cdot {}_{n-k+l-2}C_{n-k-2}$ between them, that is, the coefficient of the former denominator is larger than the latter one for any index l except for $l=0$. Thus the first fraction approaches to zero more quickly than the second one. Therefore, at $w=1$, the first term of (36) has its maximum, which is

$$\begin{aligned}(1-p_1)&\left(\frac{{}_{n-1}C_k}{\sum_{l=0}^{k-1}{}_{n-k+l-1}C_{n-k-1}} - \frac{{}_{n-2}C_k}{\sum_{l=0}^{k-1}{}_{n-k+l-2}C_{n-k-2}}\right)\\ &= (1-p_1)\left(\frac{{}_{n-1}C_k}{{}_{n-1}C_{n-k}} - \frac{{}_{n-2}C_k}{{}_{n-2}C_{n-k-1}}\right)\\ &= (1-p_1)\left(\frac{n-k}{k} - \frac{n-k-1}{k}\right)\\ &= \frac{1-p_1}{k}.\end{aligned}$$

It is evident that the second term of (36) becomes maximum at $w=1$. The maximum value is

$$\begin{aligned}(1-p_1-p_2)\left(1+\frac{{}_{n-2}C_k}{\sum_{l=0}^{k-1}{}_{n-k+l-2}C_{n-k-2}}\right) &= (1-p_1-p_2)\left(1+\frac{{}_{n-2}C_k}{{}_{n-2}C_{n-k-1}}\right)\\ &= (1-p_1-p_2)\left(1+\frac{n-k-1}{k}\right)\\ &= (1-p_1-p_2)\frac{n-1}{k}.\end{aligned}$$

As a result, we have the following equation for the maximum of the left-hand side of (27).

$$\frac{1-p_1}{k} + (1-p_1-p_2)\frac{n-1}{k} + p_1 = \frac{n-(n-k)p_1-(n-1)p_2}{k} \leq \frac{n}{k}. \quad (37)$$

Now we have finished the verification of the inequality (27) for any z in the case of $\alpha p_1 - xp_2 < 0$.

(ii) Case of $\alpha p_1 - xp_2 \geq 0$:

Any z of $0 \leq z \leq x$ satisfies $\alpha p_1 - zp_2 \geq 0$ and $\gamma(z)/z = \alpha p_1/z - p_2 + 1$ becomes minimum at $z = x$. We can apply our previous analysis (i) to this case to see that (36) has its maximum value at $z = x$ or $y = 0$, where $\gamma(z)/z$ is minimum. Let us recall that the left-hand side of (27) or (36) comes from crossing point π_4 of Equation (26) of two lines of payoffs given by strategies $S(x)$ and $S(y)$, as discussed just before Lemma 3.2. Even if we bring y to 0, the expected payoff for $S(y)$ does not converge to that of NS. As y goes to 0, the payoff for strategy NP approaches to $v(n-1,k,x)$. The payoff for P goes to not $v(n-1,k-1,x)$, but $\alpha p_1 + (1-p_1)v(n-1,k-1,x)$ while the two values have the relation $\alpha p_1 + (1-p_1)v(n-1,k-1,x) > v(n-1,k-1,x)$.

Figure 2 shows the expected payoff with respect to π as y goes to 0 and other payoffs. From the figure, we can see $\pi_3 \leq \pi_4$ and inequality (27) holds. □

Corollary 4.1. *In the case of $k = 1$, optimal patrolling strategy π^* is equal to optimal smuggling strategy ρ^*.*

Proof. In the case of $\alpha p_1 - xp_2 < 0$, $\pi^* = \rho^* = 1/n$ from Equations (33). In the other case of $\alpha p_1 - xp_2 \geq 0$, we have $y(n,1,x) = (\gamma(x) + (n-1)x)/(x\gamma(x))$ by substituting $k = 1$ in (34), which is substituted into Equations (29) and (30) to obtain the desired result $\pi^* = \rho^* = x/(\gamma(x) + (n-1)x)$. □

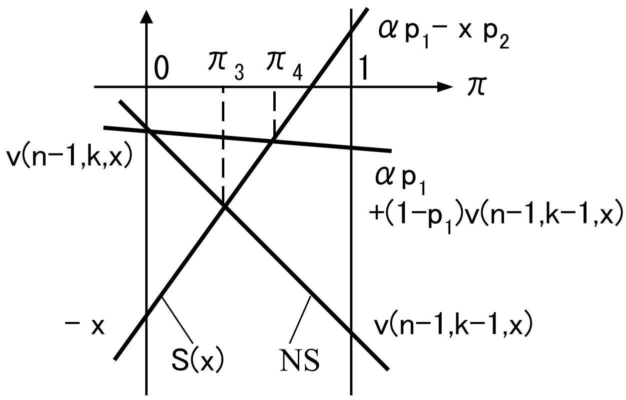

Figure 2: Expected payoff with the limit as $y \to 0$

Corollary 4.2. *In any stage game, optimal patrolling strategy π at the current stage coincides with the probability of patrols estimated for future stages. Optimal smuggling strategy ρ has the same property for x of $\alpha p_1 - p_2 x \leq 0$.*

Proof. Equation (29) gives us the optimal patrolling strategy in state (n,k,x) at stage n. We denote it by $\pi(n,k,x)$. The probability of smuggling tomorrow, T, is calculated conditionally based on today's smuggling strategy as follows.

$$T = \pi(n,k,x)\pi(n-1,k-1,x) + (1-\pi(n,k,x))\pi(n-1,k,x). \tag{38}$$

We prove first that T coincides with $\pi(n,k,x)$.

In the case of $k = 0$, the coincidence is evident from $\pi(n,k,x) = \pi(n-1,k,x) = 0$ and $T = 0$ from the equation above. So we consider the case of $k \neq 0$, for which $y(n,k,x)$ takes a finite value. Applying Equation (29) to (38), we have

$$\begin{aligned} T &= \frac{1}{\gamma(x)y(n,k,x)}\frac{1}{\gamma(x)y(n-1,k-1,x)} + \left(1 - \frac{1}{\gamma(x)y(n,k,x)}\right)\frac{1}{\gamma(x)y(n-1,k,x)} \\ &= \frac{1}{\gamma(x)y(n,k,x)}\left(\frac{1}{\gamma(x)y(n-1,k-1,x)} - \frac{1}{\gamma(x)y(n-1,k,x)}\right) + \frac{1}{\gamma(x)y(n-1,k,x)}. \end{aligned}$$

We divide both sides of recurrence equation (21) by $y(n-1,k,x)$ to get

$$\frac{y(n,k,x)}{y(n-1,k,x)} = 1 - \frac{1}{\gamma(x)y(n-1,k-1,x)} + \frac{1}{\gamma(x)y(n-1,k,x)}.$$

We use this to transform the expression of T and derive

$$T = \frac{1}{\gamma(x)y(n,k,x)}\left(1 - \frac{y(n,k,x)}{y(n-1,k,x)}\right) + \frac{1}{\gamma(x)y(n-1,k,x)} = \frac{1}{\gamma(x)y(n,k,x)} = \pi(n,k,x).$$

Now that we know that the probability of the patrol on the day after tomorrow, namely $n-2$, equals π depending on two states, $(n-1,k,x)$ and $(n-1,k-1,x)$, tomorrow, we calculate the patrolling probability at $n-2$ by Equation (38) to reach the same result that the probability equals to $\pi(n,k,x)$.

We can see that the latter half of the corollary is valid by calculating the probability of smuggling tomorrow, S. Because Player B executes strategy ESS once, as I noted before, smuggling in the future could be done on the condition that it is not tried today. In the following calculation of the probability, we denote an optimal probability of taking strategy ESS for state (n,k,x) by $\rho(n,k,x)$ and use Equation (33) in Theorem 4.1 in the case of $\alpha p_1 - xp_2 \le 0$.

$$\begin{aligned} S &= (1-\rho(n,k,x))\left\{\pi(n,k,x)\rho(n-1,k-1,x) + (1-\pi(n,k,x))\rho(n-1,k,x)\right\} \\ &= \left(1-\frac{1}{n}\right)\left\{\frac{k}{n}\cdot\frac{1}{n-1} + \left(1-\frac{k}{n}\right)\frac{1}{n-1}\right\} = \frac{1}{n}. \end{aligned}$$

This result shows us that the smuggling probability tomorrow coincides with today's probability and the coincidence occurs at any stage in the future. □

Player A surely expends all opportunities of patrol and Player B tries to ship the cargo of all contraband someday if $\alpha p_1 - p_2 x \le 0$. Corollary 4.2 tells us that the uniform possibility of smuggling or patrolling is important for his opponent not to easily anticipate his action. We can show some counter examples about the incorrectness of Corollary 4.2 in the case of $\alpha p_1 - p_2 x > 0$, where Player B does not have the motivation to smuggle any contraband.

5. Numerical Examples

Here we analyze optimal solutions of the game by numerical examples. First we take an example (Example 1) of the specific model, which is discussed in Section 3 and 4. By the exmple, we can confirm general properties of optimal strategy for constant probabilities p_1 and p_2, which are clarified in preceding sections. Secondly, we take the other example (Example 2) in a general case, where the capture probability $q_1(y)$ and the success one $q_2(y)$ depend on the amount of contraband y, to numerically investigate some characteristics of optimal strategy.

For the example (Example 1), we set $\alpha = 2,\ p_1 = 0.6,\ p_2 = 0.3$, the number of stages $n = 1, 2, \cdots, 6$, the number of chances of patrols $k = 0, 1, \cdots, n$ and the amount of contraband $x = 1, 2, \cdots, 5$. Table 1 shows the value of the game $v(n, k, x)$ and Table 2 shows optimal players' strategies at each stage, denoted by a pair of optimal probabilities of patrolling and ESS (π^*, ρ^*). The equilibrium is affected by the sign of $\alpha p_1 - p_2 x$. It is zero for $x = 4$, positive for smaller x and negative for larger amount of contraband than $x = 4$.

We show the matrix form (1) for states $(n, k, x) = (3, 3, 3),\ (6, 2, 3)$ by the following equations, respectively. We attach indices $y = x, x-1, \cdots, 1, 0$ above the matrices.

$$\Gamma(3,3,3) = \begin{array}{c} \begin{matrix} 3 & 2 & 1 & 0 \end{matrix} \\ \begin{pmatrix} 0.3 & 0.6 & 0.9 & 0 \\ -3 & -2 & -1 & 0 \end{pmatrix} \end{array},$$

$$\Gamma(6,2,3) = \begin{array}{c} \begin{matrix} 3 & 2 & 1 & 0 \end{matrix} \\ \begin{pmatrix} 0.30 & 0.33 & 0.29 & -2.35 \\ -3 & -2.39 & -2.03 & -1.71 \end{pmatrix} \end{array}$$

As we prove by Lemma 3.1, strategy ESS is dominant over other smuggling strategies for game $\Gamma(3, 3, 3)$. Patrolling strategy P is also dominant over NP and the game has a saddle point. Concerning $\Gamma(6, 2, 3)$, ESS is dominant over all other strategies of Player B against Player A's strategy NP but the favorite strategy of Player B has the order of $y = 0, 1, 3, 2$ against strategy P. We can see that the strategy PSS of $y = 2$ is dominated by NS but we wonder if there exists another relation of dominance among smuggling strategies. By calculation, we know that the optimal mixed strategy of players is $(\pi^*, \rho^*) = (0.327, 0.162)$ and the PSSs are not used in an optimal mixture of strategies.

Table 1: Value of game (Example 1)

n	k	x				
		1	2	3	4	5
1	1	0	0	0	0	-0.3
2	1	-0.35	-0.87	-1.43	-2.00	-2.65
	2	0	0	0	0	-0.3
3	1	-0.51	-1.21	-1.94	-2.67	-3.43
	2	-0.15	-0.50	-0.91	-1.33	-1.87
	3	0	0	0	0	-0.3
4	1	-0.61	-1.40	-2.20	-3.00	-3.83
	2	-0.29	-0.82	-1.41	-2.00	-2.65
	3	-0.08	-0.32	-0.65	-1.00	-1.48
	4	0	0	0	0	-0.3
5	1	-0.68	-1.51	-2.35	-3.20	-4.06
	2	-0.39	-1.04	-1.71	-2.40	-3.12
	3	-0.17	-0.59	-1.09	-1.60	-2.18
	4	-0.04	-0.22	-0.49	-0.80	-1.24
	5	0	0	0	0	-0.3
6	1	-0.73	-1.59	-2.46	-3.33	-4.22
	2	-0.47	-1.18	-1.92	-2.67	-3.43
	3	-0.26	-0.80	-1.39	-2.00	-2.65
	4	-0.10	-0.44	-0.88	-1.33	-1.87
	5	-0.02	-0.16	-0.39	-0.67	-1.08
	6	0	0	0	0	-0.3

In Table 1, we can find the nonpositiveness of the value of the game, its monotone increase for k and its monotone decrease for x. The value of the game is zero for all x

with nonnegative discriminant value in the case of $n = k$, as shown in Lemma 3.1. Fixing parameters k and x, the value of the game gets smaller for larger n. This means that the increment of the number of stages gives some advantage to Player B, as shown by inequality (17).

Table 2 shows us that the discriminant value has a big influence on optimal strategies of players. If it is positive, the patrolling probability π^* grows larger for more chances of patrol k and Player B decreases the smuggling probability ρ^* to avoid the active patrol. For larger x with nonpositive discriminant value, optimal mixed strategy π^* increases proportional to k but ρ^* stays constant to be $1/n$ for any k, as shown in Theorem 4.1. In the case of smaller x with positive discriminant value, Player B expects some loss of positive expected payoff by the coincidence of his smuggling and his opponent's patrol and keeps the smuggling probability ρ^* low with the reservation of positive possibility to terminate the game with no smuggling. On the other hand, in the case of larger x, he expects negative payoff even though his smuggling encounters the patrol and sets high smuggling probability with the decision to definitely smuggle someday. That is why we are interested in when Player B should execute strategy ESS. We will analyze this problem later. Probabilities π^* and ρ^* are the same in the case of $k = 1$, as Corollary 4.1 showed us. Because optimal strategies change in the way we discussed above for larger k, π^* is always larger than ρ^* for $k > 1$.

Here we are going to analyze the effect of x on equilibria of the game. As long as the discriminant value is positive, the smuggling probability ρ^* and the patrolling probability π^* grow larger as x increases. However the increment stops when the discriminant value becomes zero. For larger x with negative discriminant value, Player B takes the strategy that lets the smuggling probability is estimated to be uniform $1/n$ at any stage n in the future, as shown in Corollary 4.2, and Player A takes a uniform strategy of $\pi^* = k/n$ in reply to the strategy of Player B.

Table 2: Optimal strategy (Example 1)

n	k	x				
		1	2	3	4	5
1	1	(.53, 0)	(.77, 0)	(.91, 0)	(1, 1)	(1, 1)
2	1	(.35, .35)	(.44, .44)	(.48, .48)	(.5, .5)	(.5, .5)
	2	(.53, 0)	(.77, 0)	(.91, 0)	(1, .50)	(1, .50)
3	1	(.26, .26)	(.30, .30)	(.32, .32)	(.33, .33)	(.33, .33)
	2	(.45, .15)	(.58, .25)	(.63, .30)	(.67, .33)	(.67, .33)
	3	(.53, 0)	(.77, 0)	(.91, 0)	(1, .33)	(1, .33)
4	1	(.20, .20)	(.23, .23)	(.24, .24)	(.25, .25)	(.25, .25)
	2	(.38, .16)	(.45, .22)	(.48, .24)	(.5, .25)	(.5, .25)
	3	(.49, .08)	(.65, .16)	(.71, .22)	(.75, .25)	(.75, .25)
	4	(.53, 0)	(.77, 0)	(.91, 0)	(1, .25)	(1, .25)
5	1	(.17, .17)	(.19, .19)	(.20, .20)	(.2, .2)	(.2, .2)
	2	(.32, .15)	(.37, .18)	(.39, .19)	(.4, .2)	(.4, .2)
	3	(.44, .10)	(.54, .16)	(.58, .19)	(.6, .2)	(.6, .2)
	4	(.51, .04)	(.68, .11)	(.76, .16)	(.8, .2)	(.8, .2)
	5	(.53, 0)	(.77, 0)	(.91, 0)	(1, .20)	(1, .20)
6	1	(.15, .15)	(.16, .16)	(.16, .16)	(.17, .17)	(.17, .17)
	2	(.28, .13)	(.31, .15)	(.33, .16)	(.33, .17)	(.33, .17)
	3	(.39, .11)	(.46, .15)	(.49, .16)	(.5, .17)	(.5, .17)
	4	(.47, .06)	(.60, .13)	(.64, .15)	(.67, .17)	(.67, .17)
	5	(.52, .02)	(.71, .08)	(.79, .13)	(.83, .17)	(.83, .17)
	6	(.53, 0)	(.77, 0)	(.91, 0)	(1, .17)	(1, .17)

For larger number of stage n, π^* and ρ^* decrease because players have more options about when to patrol or smuggle. The effect of n on π^* looks clear in the case of nonpositive discriminant value but it becomes vague in the case of positive discriminant value. n has the similar effect on ρ^*.

In the special case of $k = n$, where Player A can patrol every day, Lemma 3.1 exhausts the whole equilibria of the game. Table 2 shows one of the equilibria, though. We should note the following properties of the solution. For smaller x with positive discriminant value, Player B avoids the patrol because of the positive expected payoff. Therefore, even imperfect patrolling strategy $\pi^* < 1$ of Player A perfectly deters Player B from any smuggling. A threshold of π between the perfect deterrence and the not-perfect one is π_1 given by Equation (8). For larger x with negative discriminant value, Player A should take perfect patrol strategy or $\pi^* = 1$ every day.

The next example (Example 2) is taken for the general model with the dependency of probabilities $q_1(y)$ and $q_2(y)$ on the amount of contraband y. Setting $\alpha = 2$ as in the previous example, we change $q_1(y)$ and $q_2(y)$ as shown in Table 3, where $q_1(y) + q_2(y) = 1$. Even if Player B has some danger of capture by a patrol, he can expect negative payoff (positive reward) by 1 or 2 contraband. The preference order of B on the amount is $y = 1, 2$. Especially, $y = 1$ brings Player B comparatively larger reward than others.

Table 3: Setting of $q_1(y)$ and $q_2(y)$ (Example 2)

	y					
	0	1	2	3	4	5
$q_1(y)$	0	0.1	0.45	0.7	0.8	0.85
$q_2(y)$	1	0.9	0.55	0.3	0.2	0.15
$\alpha q_1(y) - yq_2(y)$	0	-0.7	-0.2	0.5	0.8	0.95

By changing n from 1 through 4, k from 1 through n and x from 1 through 5, we obtain the value of the game $v(n, k, x)$ for every combination (n, k, x), which is shown in Table 4. The optimal strategies of players are shown in Table 5, where the probabilities of Player A's choosing P(patrol) or NP(no-patrol) are written as a vector in the upper position and a mixed strategy of combining $S(x), \cdots, S(1), NS$ as a $x+1$-entry vector in the lower position for each combination (n, k, x).

Table 4: Value of the game (Example 2)

n	k	x				
		1	2	3	4	5
1	1	-0.7	-0.7	-0.7	-0.7	-0.7
2	1	-0.85	-1.63	-1.99	-2.31	-2.70
	2	-0.7	-1.33	-1.33	-1.33	-1.33
3	1	-0.9	-1.76	-2.56	-3.09	-3.55
	2	-0.8	-1.54	-2.17	-2.40	-2.52
	3	-0.7	-1.33	-1.90	-1.90	-1.90
4	1	-0.93	-1.82	-2.69	-3.49	-4.12
	2	-0.85	-1.65	-2.39	-3.02	-3.42
	3	-0.78	-1.49	-2.12	-2.65	-2.86
	4	-0.7	-1.33	-1.90	-2.41	-2.41

We itemize some characteristics of the value of the game, as follows.

(1) The value of the game changes in a monotonic nonincreasing manner according to the increase of n or x, but it changes inversely for k. The change is persuadable.

(2) In a special case that Player A can patrol every day, i.e. $k = n$, Player B is going to repeat a smuggling strategy $S(1)$ as times as the amount of contraband x on hand if $x \leq n$ and he would have the total payoff $(\alpha q_1(1) - q_2(1)) \sum_{s=0}^{x-1}(1 - q_1(1))^s$. In the concrete, the value of the game is -0.7, -1.33, -1.90 and -2.41 for $x = 1, 2, 3, 4$. Player B never tries the smuggling of more than a contraband and discard the excess even though x is larger than n from the viewpoint of the expected reward and the capture probability. This fact would be verified by the analysis on optimal strategy of player, which is carried out below.

Table 5: Optimal strategy (Example 2)

n	k	x = 1	2	3	4	5
1	1	(1, 0)	(1, 0)	(1, 0)	(1, 0)	(1, 0)
		(1, 0)	(0, 1, 0)	(0, 0, 1, 0)	(0, 0, 0, 1, 0)	(0, 0, 0, 0, 1, 0)
2	1	(.5, .5)	(.71, .29)	(.36, .64)	(.36, .64)	(.39, .61)
		(.5, .5)	(0, .93, .07)	(0, .29, .71, 0)	(0, .3, 0, .7, 0)	(.3, 0, 0, 0, .7, 0)
	2	(1, 0)	(1, 0)	(1, 0)	(1, 0)	(1, 0)
		(.6, .4)	(0, 1, 0)	(0, 0, 1, 0)	(0, 0, 0, 1, 0)	(0, 0, 0, 0, 1, 0)
3	1	(.33, .67)	(.36, .64)	(.56, .44)	(.23, .77)	(.24, .76)
		(.33, .67)	(0, .6, .4)	(0, 0, .89, .11)	(0, 0, .15, .85, 0)	(.14, 0, 0, 0, .86, 0)
	2	(.67, .33)	(.69, .31)	(1, 0)	(.42, .58)	(.42, .58)
		(.33, .67)	(0, .56, .44)	(0, 0, 1, 0)	(0, 0, .07, .93, 0)	(0, 0, .09, 0, .91, 0)
	3	(1, 0)	(1, 0)	(1, 0)	(1, 0)	(1, 0)
		(.6, .4)	(0, .73, .27)	(0, 0, 1, 0)	(0, 0, 0, 1, 0)	(0, 0, 0, 0, 1, 0)
4	1	(.25, .75)	(.26, .74)	(.29, .71)	(.44, .56)	(.16, .84)
		(.25, .75)	(0, .44, .56)	(0, 0, .63, .37)	(0, 0, 0, .85, .15)	(0, 0, 0, .07, .93, 0)
	2	(.5, .5)	(.51, .49)	(.58, .42)	(.9, .1)	(.29, .71)
		(.25 .75)	(0, .44, .56)	(0, 0, .61, .39)	(0, 0, 0, .81, .19)	(0, 0, 0, .03, .97, 0)
	3	(.75, .25)	(.76, .24)	(.84, .16)	(1, 0)	(1, 0)
		(.25, .75)	(0, .42, .58)	(0, 0, .52, .48)	(0, 0, 0, 1, 0)	(0, 0, 0, 0, 1, 0)
	4	(1, 0)	(1, 0)	(1, 0)	(1, 0)	(1, 0)
		(.6, .4)	(0, .73, .27)	(0, 0, .74, .26)	(0, 0, 0, 1, 0)	(0, 0, 0, 0, 1, 0)

From Table 5, we have some lessons about optimal strategy of player.

(1) In a special case of $k = n$, Player A always patrols and Player B always smuggles unit contraband if he has enough contraband, i.e. $x \geq n$.

(2) In more general case of $k < n$, Player B chooses one of $S(1)$ or NS and never tries smuggling more contraband if $0 \leq x < n$. The probability of choosing $S(1)$ becomes larger as x increases and it reaches a maximum at $x = n$ in almost all cases. For example, the probability is 1 for $(n, k, x) = (3, 2, 3)$ and $(4, 3, 4)$. For larger $x(> n)$, B keeps the probability for $S(1)$ comparatively high and mixes two strategies of $S(1)$ and more-contraband smuggling. The probability for the more-contraband smuggling is larger in the case of smaller k than larger k because he aims for high reward from the smuggling a lot of contraband in the absence of patrol.
Corresponding to the optimal strategy of Player A, explained above, the probability for patrol grows larger as x gets larger in the case of $0 \leq x < n$ and reaches a maximum

at $x = n$. Comparing with $x = n$, A is much less likely to choose patrol in the case of $x > n$ but with less variance. From these observation, A is interested in covering the execution of smuggling by his patrol when the amount of contraband is not enough, i.e. $x < n$, and focuses on covering the more-contraband smuggling by his patrol when B certainly tries smuggling in the case of $x > n$.

6. Conclusion

The most important result of this paper is Theorem 4.3 for the game with constant probabilities. The aim of the paper was that the decision on the amount of contraband could give the smuggler the flexibility for easy smuggling by dividing the contraband into some parts. However the smuggler should ship the whole contraband when he smuggles, as shown in Theorem 4.3. The situation that the smuggler smuggles once was taken as an assumption of their mathematical model so far in the other studies. The paper makes it clear that the one-shot smuggling is valid for optimal strategy even though it is not assumed. Also the paper shows that the sign of the discriminant value $\alpha p_1 - p_2 x$ affects the solution of the game. This fact is located on the extension of our previous paper [9], where we assume that the discriminant value is positive but the smuggler has several opportunities to smuggle.

We also deal with a general model, where the probability of the capture of smuggler or the success in smuggling depends on the amount of contraband, and propose a procedure to numerically calculate the value of the game and optimal strategies. For the general model, we could not develop analytical theory as we could do for the model with constant probabilities p_1 and p_2, but we clarify some characteristics of optimal players' strategies by a numerical example.

We can think of the other situation. First, time or stage might have an influence on the reward by the capture of the smuggler α and on the probability of capture or the success probability of the smuggling. The minor revision could be done by introducing new parameters $\alpha(n)$, $q_1(n, y)$ or $q_2(n, y)$ depending on the stage number n as substitutes for α, $q_1(y)$ or $q_2(y)$ in our formulation. We may assume some discount rate β on the reward of the game in the long term. We could carry out the modification by introducing the discount rate in Equation (2), as follows.

$$
\begin{aligned}
&v(n, k, x) \\
&= val \begin{pmatrix} \alpha q_1(y) - y q_2(y) + (1 - q_1(y))\beta v(n-1, k-1, x-y) & \beta v(n-1, k-1, x) \\ -y + \beta v(n-1, k, x-y) & \beta v(n-1, k, x) \end{pmatrix}.
\end{aligned}
$$

As shown above, additional assumptions do not give our formulation by the recursive equation of matrix games any essential change.

If the contraband is continuously divisible, we must treat its amount y as a continuous variable. For the continuous model, a threshold amount of making the discriminant value zero would become important as we imagine from Theorem 4.2. In our future work, we will extend our modeling to a nonzero-sum game with multi-criteria of players. Especially, inspection games on nuclear power facilities or disarmament [3, 11] have been studied as the nonzero-sum game. Their models could help us to approach the nonzero-sum smuggling game. We also have to discuss the inspection game with incomplete information, where players do not perfectly know the past strategies their opponents took, and evaluate the value of information in the game.

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