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AN IMPLICIT FORMULATION OF MATHEMATICAL PROGRAM WITH COMPLEMENTARITY CONSTRAINTS FOR APPLICATION TO ROBUST STRUCTURAL OPTIMIZATION

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Abstract This paper discusses an implicit reformulation of a class of MPEC (mathematical program with equilibrium constraints) problems. We particularly focus on an MPEC problem arising from the robust optimization of elastic structures subjected to the uncertain external load. We first review the relation between the worst-case detection and robust constraint satisfaction of the structural responses, and then derive an MPEC formulation of the robust structural optimization. Since a standard constraint qualification is not satisfied at any feasible solution of an MPEC problem, we propose a reformulation based on the smoothed Fischer–Burmeister function, in which the smoothing parameter is treated as an independent variable. Numerical examples of robust structural optimization are presented to demonstrate that the presented formulation can be solved by using a standard nonlinear programming approach.

Keywords: Nonlinear programming, mathematical program with equilibrium constraints, complementarity condition, structural optimization, robust optimization

1. Introduction

Recently, both methodologies and numerical techniques for robust structural optimization have received increasing attention in structural and mechanical engineering. Since structures built in the real world inevitably encounter various uncertainties caused by manufacturing variability, aging, limitation of knowledge of input disturbance, etc., the notion of robust structural design is desired naturally and earnestly [4, 9, 12, 15, 23, 29, 33, 39, 46].

The two major frameworks for treating uncertain properties of structural systems are probabilistic and non-probabilistic approaches. Optimization methods of structures based on a probabilistic uncertainty approach have been well developed as the reliability-based optimization (see, e.g., [12, 39, 46], and the references therein). However, particularly for a complex structural system, this approach may often face a difficulty such that the statistical property of the system may not be known in detail, while a complex statistical analysis cannot be justified in the absence of detailed statistical input data. In contrast, a non-probabilistic uncertainty model is often less information-intensive than a probabilistic model, because less details of statistical properties are required [6, 18].

Within the framework of a non-probabilistic approach, uncertain parameters are treated as the so-called *unknown-but-bounded* parameters. One of well-known methods with a non-probabilistic uncertainty model is the *convex model* method [8]. Au *et al.* [4] and Ganzerli and Pantelides [23] proposed numerical algorithms for robust structural optimization by using the convex model method. Note that the convex model method is valid only if the magnitude of uncertainty is small enough, because this approach is essentially based on the first-order approximation of the structural response with respect to the uncertain parameters. Lee and Park [33] also presented a method for robust structural optimization based on the first-order approximation.

There exist two closely related methodologies with which we can address arbitrary large magnitude of uncertainty in a structural system. The one is the notion of *robust counterpart* of optimization problem proposed by Ben-Tal and Nemirovski [10, 11], and another the *info-gap decision theory* proposed by Ben-Haim [7]. A unified methodology of robust counterpart was presented for a broader class of convex optimization problems [11], in which the given data of an optimization problem are supposed to possess non-probabilistic uncertainties. This methodology was applied to the compliance minimization problem of a truss structure subjected to the uncertain external load [9, 45]. A min-max formulation of a robust compliance design was presented for continua by Cherkaev and Cherkaev [15]. Kočvara *et al.* [32] performed a free-material design under multiple loadings by using a cascading technique. The info-gap decision theory introduces the *robustness function* as a quantitative measure of robustness. Specifically, the robustness function represents the greatest level of uncertainty at which any failure cannot occur [7]. By using the robustness function, Kanno and Takewaki [29] proposed a robustness maximization problem of a truss structure subjected to uncertainty in the external load.

In this paper, we consider a non-probabilistic uncertainty in the static external load of a structural system. In accordance with the notion of robust constraint satisfaction which is shared by the robust optimization methodology [11] and the info-gap theory [7], we formulate a robust structural optimization problem, in which any constraint on the mechanical performance cannot be violated at the given magnitude of uncertainty. We clarify the relations between the robust constraint satisfaction, worst-case detection, and robust structural optimization; see Section 2.

The robust structural optimization problem is formulated as a semi-infinite programming problem [40], which can further be reformulated as an MPEC (*mathematical program with equilibrium constraints*) in the complementarity form. Since an MPEC does not satisfy any standard constraint qualification [34], standard nonlinear programming approaches are likely to fail for this problem. For overcoming this difficulty there exist two main approaches. The one is to develop specialized algorithms dealing with MPEC problems (see, e.g., [34, 36]), and the other is to reformulate an MPEC problem into a tractable form so that standard nonlinear programming approaches are applicable. As a method following the latter methodology, this paper presents an implicit reformulation of MPEC by using the smoothed Fischer–Burmeister function [22, 31]; see Section 3.

It is known that various problems in structural and mechanical engineering can be formulated as MPEC problems [2, 20, 28, 42, 43]. Although this paper focuses on the application of MPEC to the robust structural optimization, the presented implicit reformulation can be applied to any MPEC problem in the complementarity form. Numerous smoothing methods, as well as regularization schemes, were proposed for MPEC [13, 17, 19, 22, 25, 26, 28, 38, 42, 44]. An MPEC problem in the complementarity form involves the complementarity conditions

$$g_i(\boldsymbol{x}) \geq 0, \quad h_i(\boldsymbol{x}) \geq 0, \quad g_i(\boldsymbol{x}) h_i(\boldsymbol{x}) = 0, \qquad i = 1, \dots, M \tag{1.1}$$

as its constraints, where g_i, $h_i : \mathbb{R}^N \to \mathbb{R}$ $(i = 1, \dots, M)$. In a typical regularization approach to MPEC, the complementarity constraints in (1.1) are relaxed as [38]

$$g_i(\boldsymbol{x}) \geq 0, \quad h_i(\boldsymbol{x}) \geq 0, \quad g_i(\boldsymbol{x}) h_i(\boldsymbol{x}) \leq \varepsilon, \qquad i = 1, \dots, M, \tag{1.2}$$

where $\varepsilon > 0$ is a constant. Then a sequence of the relaxed optimization problem with (1.2), instead of (1.1), is solved by using a standard nonlinear programming approach, e.g., the SQP method, by gradually decreasing $\varepsilon \searrow 0$. In contrast, in a smoothing method for MPEC, the constraints in (1.1) are replaced with [22, 42]

$$\psi(g_i(\boldsymbol{x}), h_i(\boldsymbol{x}); \varepsilon) = 0, \qquad i = 1, \dots, M, \tag{1.3}$$

where $\varepsilon > 0$ is a constant, $\psi(\,\cdot\,; \varepsilon)$ is continuously differentiable for any $\varepsilon > 0$, and $\psi(\,\cdot\,; 0) : \mathbb{R}^2 \to \mathbb{R}$ is a complementarity function; i.e., $\psi(a, b; 0) = 0$ holds if and only if $a \geq 0$, $b \geq 0$, and $ab = 0$. In a manner similar to the relaxation method, the smooth optimization problem with (1.3) is solved sequentially by decreasing ε.

It seems that there exists neither an appropriate guideline for the choice of an initial value for ε nor a proper decreasing strategy of ε. Note that the convergence of algorithms can be proved theoretically irrelevant to the initial value of ε and the decreasing strategy. However, the computational efficiency certainly depends on those choices. Specifically, too rapid reduction of ε is not adequate to avoid the nonsmooth property of the complementarity function, but unnecessary iterations might be spent if we decrease ε too slowly; see Remark 3.4 for details.

This observation motivates us to propose a new reformulation of MPEC in the complementarity form. The key idea is stated as follows. We add an additional constraint to (1.3) so that ε can work as a measure of the residual of the complementarity constraints. The smoothing parameter ε in (1.3) is regarded as a free variable, and the reformed problem is solved by using a standard nonlinear optimization method. Then ε is automatically adjusted according to the residual of the complementarity constraints; i.e., as the residual of the complementarity constraints becomes smaller, the smoothing parameter ε becomes smaller. At the convergent solution, it is guaranteed that ε vanishes automatically, and hence the complementarity constraints are satisfied exactly. This is a key idea presented in this paper.

This paper is organized as follows. Section 2 introduces the robust structural optimization formulated as the semi-infinite programming problem, and then reduces it to an MPEC problem. An implicit reformulation of the MPEC is presented in Section 3. Numerical results are shown in Section 4; an illustrative truss optimization in Section 4.1 and the optimization of a frame structure in Section 4.2. Finally, conclusions are drawn in Section 5.

A few words regarding our notation: All vectors are assumed to be column vectors. The $(m+n)$-dimensional column vector $(\boldsymbol{u}^{\mathrm{T}}, \boldsymbol{v}^{\mathrm{T}})^{\mathrm{T}}$ consisting of $\boldsymbol{u} \in \mathbb{R}^m$ and $\boldsymbol{v} \in \mathbb{R}^n$ is often written simply as $(\boldsymbol{u}, \boldsymbol{v})$. We denote by $\mathcal{S}^n \subset \mathbb{R}^{n \times n}$ the set of all $n \times n$ real symmetric matrices. For a vector $\boldsymbol{p} = (p_i) \in \mathbb{R}^n$, we use $\|\boldsymbol{p}\|_\infty$ to denote its maximum-norm, i.e., $\|\boldsymbol{p}\|_\infty = \max_{i \in \{1,\dots,n\}} |p_i|$. By $\boldsymbol{p} \geq \boldsymbol{0}$ we mean $p_i \geq 0$ $(i = 1, \dots, n)$. We use I and $\boldsymbol{1}$ to denote the $n \times n$ identity matrix and the vector $(1, \dots, 1)^{\mathrm{T}} \in \mathbb{R}^n$, respectively, without specifying n, unless it is not clear from the context.

2. Motivation: Robust Structural Optimization

Consider a finite-dimensional linear elastic structure subjected to the static nodal forces $\boldsymbol{f} \in \mathbb{R}^d$, where d is the number of degrees of freedom of displacements. Small displacements and small strains are assumed. See, e.g., [5] for fundamentals of static analysis of structures.

Let $\boldsymbol{x} \in \mathbb{R}^m$ denote the vector of design variables, where m is the number of design variables. For example, x_i is the member cross-sectional area for a truss, or the element thickness for a plate discretized into finite elements. Throughout the paper, the geometry of the structure is fixed and only sizes of structural elements are considered as the design variables to be optimized.

Let $K(\boldsymbol{x}) \in \mathcal{S}^d$ denote the stiffness matrix. The displacement vector $\boldsymbol{u} \in \mathbb{R}^d$ is found from the equilibrium equation written as

$$K(\boldsymbol{x})\boldsymbol{u} = \boldsymbol{f}, \tag{2.1}$$

where $\boldsymbol{f}$ is assumed to be independent of $\boldsymbol{x}$.

2.1. Structural optimization without uncertainty

We recall the conventional structural optimization problem before discussing the robust structural optimization. Readers may refer to [3, 16] for fundamentals of structural optimization.

Consider a mechanical performance of a structure which can be expressed as a constraint in terms of the displacements $\boldsymbol{u}$. For example, we consider the upper bound constraint of the displacement in a specified direction, or the upper bound constraint of the maximum effective stress of each structural element. In this paper, we restrict ourselves to the constraints written in the form of

$$g_j(\boldsymbol{u}) \geq 0, \qquad j = 1, \dots, n^{\mathrm{c}}, \tag{2.2}$$

where $g_j : \mathbb{R}^d \to \mathbb{R}$ is a differentiable convex function, and n^{c} is the number of the constraints. Note that $\boldsymbol{u}$ implicitly depends on the design variables $\boldsymbol{x}$ through the equilibrium equation (2.1).

Let $\mathcal{X} \subseteq \mathbb{R}^m$ denote the set of admissible design variables. For example, we often consider the lower and upper bound constraints for x_i's. We denote by $v(\boldsymbol{x})$ the total structural volume. It is often in structural optimization that we attempt to minimize the total weight of a structure, which is proportional to $v(\boldsymbol{x})$ if all the structural elements consist of the same material. Thus the conventional structural optimization problem under the constraints in (2.2) is formulated as

$$\left.\begin{array}{ll} \displaystyle\min_{\boldsymbol{x}\in\mathcal{X},\boldsymbol{u}} & v(\boldsymbol{x}) \\ \mathrm{s.\,t.} & K(\boldsymbol{x})\boldsymbol{u} = \boldsymbol{f}, \\ & g_j(\boldsymbol{u}) \geq 0, \qquad j = 1, \dots, n^{\mathrm{c}}. \end{array}\right\} \tag{2.3}$$

2.2. Non-probabilistic uncertainty model

A non-probabilistic model for uncertainty in the external load is introduced.

For real-world structures, it is usual that the external load $\boldsymbol{f}$ in (2.1) cannot be known precisely. Throughout the paper we assume that the uncertainty exists only in $\boldsymbol{f}$, and that the other parameters in a structural system, e.g., the parameters representing the stiffness or the geometry, are known precisely. Let $\tilde{\boldsymbol{f}} \in \mathbb{R}^d$ denote the nominal value, or the best estimate, of $\boldsymbol{f}$. We describe the uncertainty of $\boldsymbol{f} = (f_j) \in \mathbb{R}^d$ by using unknown parameters $\boldsymbol{\zeta} = (\zeta_p) \in \mathbb{R}^k$, where $k \leq d$. Assume that $\boldsymbol{f}$ depend on $\boldsymbol{\zeta}$ affinely as

$$\boldsymbol{f} \in \mathcal{F}(\alpha) := \{\boldsymbol{f} \mid \boldsymbol{f} = \tilde{\boldsymbol{f}} + F_0\boldsymbol{\zeta},\ \alpha \geq \|\boldsymbol{\zeta}\|_\infty\}, \tag{2.4}$$

where $F_0 \in \mathbb{R}^{d\times k}$ is a constant matrix having full column rank. Roughly speaking, α represents the "level" of uncertainty; i.e., the greater the value of α, the greater the range of possible variations of $\boldsymbol{f}$. Therefore, α is called the *uncertainty parameter* [7]. The matrix F_0 represents the relationship of the uncertainties among $f_1, \dots, f_d$.

Remark 2.1. Some uncertainty models other than that in (2.4) can be chosen for describing the uncertainty in the external load, though we postulate that $\mathcal{F}(\alpha)$ is convex for any $\alpha \geq 0$. For example, we may use the standard Euclidean norm, instead of the maximum-norm, to bound $\boldsymbol{\zeta}$ as is done in the conventional ellipsoidal uncertainty model [7–9, 29]. Although the uncertainty model defined by using the maximum-norm may sometimes be more pessimistic than the ellipsoidal uncertainty model, we restrict ourselves to (2.4) for the simple presentation of our reformulation explored in Section 3. ■

In what follows, we assume that $\mathcal{X}$ and $\mathcal{F}$ satisfy

$$\forall \boldsymbol{x} \in \mathcal{X}: \quad \operatorname{Im} K(\boldsymbol{x}) \supseteq \mathcal{F}(\alpha), \tag{2.5}$$

so that (2.1) has a solution for any $\boldsymbol{f} \in \mathcal{F}(\alpha)$. This assumption is required to guarantee that the extremal case of structural response is correctly detected by solving the worst-case detection problem, which minimizes $g_j(\boldsymbol{u})$ in (2.2) under (2.1) and (2.4); see Remark 2.2 for more account. Note that $K(\boldsymbol{x})$ can possibly become singular if some structural elements are removed as a result of optimization.

2.3. Robust structural optimization

When the external load $\boldsymbol{f}$ takes every value in the uncertainty set $\mathcal{F}(\alpha)$ defined in (2.4), the displacement vector $\boldsymbol{u}$ is running through the set $\{\boldsymbol{u} \mid K(\boldsymbol{x})\boldsymbol{u} = \boldsymbol{f},\ \boldsymbol{f} \in \mathcal{F}(\alpha)\}$. Thence, it is natural to require that the constraint on the mechanical performance in (2.2) should be satisfied by all realizations of $\boldsymbol{u}$. Thus, the robust counterpart of the constraint (2.2) is introduced under the assumption (2.5) as

$$\forall \boldsymbol{u} \in \{\boldsymbol{u} \mid K(\boldsymbol{x})\boldsymbol{u} = \boldsymbol{f},\ \boldsymbol{f} \in \mathcal{F}(\alpha)\}: \quad g_j(\boldsymbol{u}) \geq 0 \ (j = 1, \dots, n^{\mathrm{c}}). \tag{2.6}$$

Recall that the nominal structural optimization problem is formulated as (2.3). A robust structural optimization problem is defined by replacing constraint (2.2) with its robust counterpart (2.6) as

$$\left.\begin{array}{ll} \min\limits_{\boldsymbol{x} \in \mathcal{X}} & v(\boldsymbol{x}) \\ \mathrm{s.\,t.} & g_j(\boldsymbol{u}) \geq 0 \ (\forall \boldsymbol{u} \in \{\boldsymbol{u} \mid K(\boldsymbol{x})\boldsymbol{u} = \boldsymbol{f},\ \boldsymbol{f} \in \mathcal{F}(\alpha)\}), \qquad j = 1, \dots, n^{\mathrm{c}}. \end{array}\right\} \tag{2.7}$$

Problem (2.7) is a so-called *generalized semi-infinite programming* (GSIP) problem [40], which consists of a finite number of variables and an infinite number of inequality constraints. It is known that problem (2.7) can be reduced to an MPEC problem [40]; see Section 2.4 for details.

For a fixed $\boldsymbol{x} \in \mathcal{X}$, consider the optimization problem

$$\left.\begin{array}{ll} \min\limits_{\boldsymbol{u},\boldsymbol{f}} & g_j(\boldsymbol{u}) \\ \mathrm{s.\,t.} & K(\boldsymbol{x})\boldsymbol{u} = \boldsymbol{f}, \\ & \boldsymbol{f} \in \mathcal{F}(\alpha), \end{array}\right\} \tag{2.8}$$

where $\boldsymbol{u}$ and $\boldsymbol{f}$ are the variables. Suppose that problem (2.8) has an optimal solution, which is denoted by $(\boldsymbol{u}^{\mathrm{wc}}, \boldsymbol{f}^{\mathrm{wc}})$. From definition (2.6), we see that the robust satisfaction of the jth constraint in (2.6) becomes most critical at $\boldsymbol{u}^{\mathrm{wc}}$. Hence, $\boldsymbol{f}^{\mathrm{wc}}$ and $\boldsymbol{u}^{\mathrm{wc}}$ are called the *worst-case* external load and displacement, respectively, for the jth constraint. Problem (2.8) is referred to as the *worst-case detection problem* [24, 30]. Certainly the worst-case external load depends on $\boldsymbol{x}$.

Remark 2.2. We say that a structure is *kinematically indeterminate*, or *unstable*, if there exists an infinitesimal displacement vector compatible to a set of rigid-body motions of some structural elements. If the structure described by $\boldsymbol{x}$ is kinematically indeterminate, then the solution of problem (2.8) does not necessarily corresponds to the most critical case. See, e.g., [27, Section 4] for an illustrative example. In contrast, assumption (2.5) ensures that the optimal solution of problem (2.8) corresponds to the most critical case among $\boldsymbol{f} \in \mathcal{F}(\alpha)$. ■

The robust constraint introduced in (2.6) requires that the performance constraint (2.2) should be satisfied in the worst case. Hence, problem (2.7) is equivalently rewritten as

$$\left.\begin{array}{ll} \displaystyle\min_{\boldsymbol{x}\in\mathcal{X}} & v(\boldsymbol{x}) \\ \mathrm{s.\,t.} & \displaystyle\min_{\boldsymbol{u},\boldsymbol{\zeta}} \{ g_j(\boldsymbol{u}) \mid K(\boldsymbol{x})\boldsymbol{u} = \boldsymbol{f},\ \boldsymbol{f} \in \mathcal{F}(\alpha) \} \ge 0, \qquad j = 1,\dots,n^{\mathrm{c}}. \end{array}\right\} \tag{2.9}$$

Thus the robust structural optimization problem can alternatively be formulated as a *bi-level optimization* problem.

2.4. MPEC formulation

An MPEC reformulation of problem (2.9) is presented explicitly.

By using (2.4), the lower-level problem included in the constraint of problem (2.9) is explicitly written as

$$\min_{\boldsymbol{u},\boldsymbol{\zeta}} \left\{ g_j(\boldsymbol{u}) \mid K(\boldsymbol{x})\boldsymbol{u} = \tilde{\boldsymbol{f}} + F_0\boldsymbol{\zeta},\ \alpha \ge \|\boldsymbol{\zeta}\|_\infty \right\}. \tag{2.10}$$

For a fixed $\boldsymbol{x}$, problem (2.10) is a convex optimization problem in the variables $\boldsymbol{u}$ and $\boldsymbol{\zeta}$. Moreover, assumption (2.5) implies that problem (2.10) satisfies the Slater constraint qualification. Hence, the optimality condition for problem (2.10) is derived as the standard Karush–Kuhn–Tucker conditions; i.e., $(\boldsymbol{u}^*, \boldsymbol{\zeta}^*)$ is an optimal solution of problem (2.10) if and only if there exists a Lagrange multipliers vector $(\boldsymbol{\mu}^*, \boldsymbol{\lambda}^*, \boldsymbol{\tau}^*)$ satisfying

$$\begin{array}{l} K(\boldsymbol{x})\boldsymbol{u}^* = \tilde{\boldsymbol{f}} + F_0\boldsymbol{\zeta}^*, \\ \nabla g_j(\boldsymbol{u}^*) - K(\boldsymbol{x})\boldsymbol{\mu}^* = \boldsymbol{0}, \\ F_0^{\mathrm{T}}\boldsymbol{\mu}^* + \boldsymbol{\lambda}^* - \boldsymbol{\tau}^* = \boldsymbol{0}, \\ \alpha - \zeta_p^* \ge 0, \quad \lambda_p^* \ge 0, \quad \lambda_p^*(\alpha - \zeta_p^*) = 0, \qquad p = 1,\dots,k, \\ \alpha + \zeta_p^* \ge 0, \quad \tau_p^* \ge 0, \quad \tau_p^*(\alpha + \zeta_p^*) = 0, \qquad p = 1,\dots,k. \end{array}$$

The constraint of (2.9) is then reduced to $g_j(\boldsymbol{u}^*) \ge 0$ for each $j = 1,\dots,n^{\mathrm{c}}$.

Consequently, problem (2.9) is equivalently rewritten as

$$\left.\begin{array}{ll} \min & v(\boldsymbol{x}) \\ \mathrm{s.\,t.} & \boldsymbol{x} \in \mathcal{X}, \\ & \forall j = 1,\dots,n^{\mathrm{c}} : \\ & \quad \left\{\begin{array}{l} K(\boldsymbol{x})\boldsymbol{u}_j - \tilde{\boldsymbol{f}} - F_0\boldsymbol{\zeta}_j = \boldsymbol{0}, \\ \nabla g_j(\boldsymbol{u}_j) - K(\boldsymbol{x})\boldsymbol{\mu}_j = \boldsymbol{0}, \\ g_j(\boldsymbol{u}_j) \ge 0, \\ F_0^{\mathrm{T}}\boldsymbol{\mu}_j + \boldsymbol{\lambda}_j - \boldsymbol{\tau}_j = \boldsymbol{0}, \\ \alpha - \zeta_{pj} \ge 0,\ \lambda_{pj} \ge 0,\ \lambda_{pj}(\alpha - \zeta_{pj}) = 0, \qquad p = 1,\dots,k, \\ \alpha + \zeta_{pj} \ge 0,\ \tau_{pj} \ge 0,\ \tau_{pj}(\alpha + \zeta_{pj}) = 0, \qquad p = 1,\dots,k, \end{array}\right. \end{array}\right\} \tag{2.11}$$

where $\boldsymbol{x} \in \mathbb{R}^m$, $\boldsymbol{u}_j \in \mathbb{R}^d$, $\boldsymbol{\zeta}_j = (\zeta_{pj}) \in \mathbb{R}^k$, $\boldsymbol{\mu}_j \in \mathbb{R}^d$, $\boldsymbol{\lambda}_j = (\lambda_{pj}) \in \mathbb{R}^k$, and $\boldsymbol{\tau}_j = (\tau_{pj}) \in \mathbb{R}^k$ $(j = 1, \dots, n^{\mathrm{c}})$ are the variables to be optimized.

Since problem (2.11) includes complementarity conditions in its constraints, it is called the *mathematical program with complementarity constraints*, or the *mathematical program with equilibrium constraints* (MPEC) *in the complementarity form* [34]. It is well known that any feasible solution of (2.11) does not satisfy a standard constraint qualification (see, e.g., [34]), and hence conventional nonlinear programming approaches are likely to fail for this problem. In Section 3 we propose a reformulation, almost all feasible solutions of which satisfy the *linear independence constraint qualification* (LICQ).

Remark 2.3. The extra variables $\boldsymbol{u}_1, \dots, \boldsymbol{u}_{n^{\mathrm{c}}}$ and $\boldsymbol{\zeta}_1, \dots, \boldsymbol{\zeta}_{n^{\mathrm{c}}}$ included in problem (2.11) have physical interpretation as follows. Let $\boldsymbol{u}_j^*$ and $\boldsymbol{\zeta}_j^*$ be optimal values of $\boldsymbol{u}_j$ and $\boldsymbol{\zeta}_j$, respectively. Then the worst-case external load for the jth constraint is written as $\boldsymbol{f}^{\mathrm{wc}} = \tilde{\boldsymbol{f}} + F_0 \boldsymbol{\zeta}_j^*$. Moreover, $\boldsymbol{u}_j^*$ is the displacement vector corresponding to $\boldsymbol{f}^{\mathrm{wc}}$. Note that the worst-case load differs between the constraints, i.e., $\boldsymbol{\zeta}_{j_1}^* \neq \boldsymbol{\zeta}_{j_2}^*$ $(j_1 \neq j_2)$ in general. For robust structural optimization, the optimal solution often has several worst cases, in each of which the corresponding constraint becomes active. In Section 4 we present numerical examples of robust optimal solutions which have multiple active worst-case constraints. ■

3. Implicit Reformulation of MPEC

For the reformulation of the MPEC problem (2.11) into a tractable form, we make use of a smoothed Fischer–Burmeister function $\phi : \mathbb{R}^3 \to \mathbb{R}$ defined by

$$\phi(y, z, \rho) = y + z - \sqrt{y^2 + z^2 + 2\rho^2}. \tag{3.1}$$

The smoothed Fischer–Burmeister function was originally proposed by Kanzow [31] for solving linear complementarity problems. Subsequently, smooth nonlinear programming approaches for MPEC were proposed by using this function [22, 25, 26, 42].

It should be clear that a main feature of our new reformulation of MPEC is to regard the smoothing parameter ρ in (3.1) as an independent variable. We here establish the key result for reformulating a complementarity condition.

Proposition 3.1. *Let e denote Euler's constant. Then $\boldsymbol{y} = (y_i) \in \mathbb{R}^n$, $\boldsymbol{z} = (z_i) \in \mathbb{R}^n$, and $\rho \in \mathbb{R}$ satisfy*

$$\boldsymbol{y} \geq \boldsymbol{0}, \quad \boldsymbol{z} \geq \boldsymbol{0}, \quad \boldsymbol{y}^{\mathrm{T}} \boldsymbol{z} = 0, \quad \rho = 0 \tag{3.2}$$

if and only if they satisfy

$$\phi(y_i, z_i, \rho) = 0, \qquad i = 1, \dots, n, \tag{3.3}$$

$$\boldsymbol{y}^{\mathrm{T}} \boldsymbol{z} = n(e^\rho - 1). \tag{3.4}$$

Proof. Observe that, for each $i = 1, \dots, n$, the equation $\phi(y_i, z_i, \rho) = 0$ holds if and only if y_i, z_i, and ρ satisfy $y_i \geq 0$, $z_i \geq 0$, and $\rho^2 = y_i z_i$. Hence, the necessity, i.e., the "only if" part, is obtained immediately.

If $\rho = 0$ in (3.4), then $\boldsymbol{y}^{\mathrm{T}} \boldsymbol{z} = 0$. Hence, it remains to show that (3.3) and (3.4) imply $\rho = 0$. Since $\rho^2 = y_i z_i$ $(i = 1, \dots, n)$ hold if (3.3) is satisfied, we see that (3.3) implies

$$n\rho^2 = \sum_{i=1}^{n} y_i z_i = \boldsymbol{y}^{\mathrm{T}} \boldsymbol{z}. \tag{3.5}$$

Substitution of (3.5) into (3.4) yields $\rho^2 = e^\rho - 1$, which holds if and only if $\rho = 0$. □

Note again that in (3.4) not only $\boldsymbol{y}$ and $\boldsymbol{z}$ but also the smoothing parameter ρ are regarded as the independent variables. Moreover, with (3.4) ρ can play a role of a measure of the residual of the complementarity constraints. This point is further discussed in Remark 3.2 below.

It follows from Proposition 3.1 that the MPEC problem (2.11) is equivalently rewritten as the following implicit formulation:

$$\left.\begin{array}{ll} \min & v(\boldsymbol{x}) \\ \mathrm{s.t.} & \boldsymbol{x} \in \mathcal{X}, \\ & \forall j = 1,\dots,n^{\mathrm{c}} : \left\{ \begin{array}{ll} K(\boldsymbol{x})\boldsymbol{u}_j - \tilde{\boldsymbol{f}} - F_0 \boldsymbol{\zeta}_j = \boldsymbol{0}, & \\ \nabla g_j(\boldsymbol{u}_j) - K(\boldsymbol{x})\boldsymbol{\mu}_j = \boldsymbol{0}, & \\ g_j(\boldsymbol{u}_j) \geq 0, & \\ F_0^{\mathrm{T}} \boldsymbol{\mu}_j + \boldsymbol{\lambda}_j - \boldsymbol{\tau}_j = \boldsymbol{0}, & \\ \phi(\alpha - \zeta_{pj}, \lambda_{pj}, \rho) = 0, & p = 1,\dots,k, \\ \phi(\alpha + \zeta_{pj}, \tau_{pj}, \rho) = 0, & p = 1,\dots,k, \end{array}\right. \\ & \displaystyle\sum_{j=1}^{n^{\mathrm{c}}} \sum_{p=1}^{k} [(\alpha - \zeta_{pj})\lambda_{pj} + (\alpha + \zeta_p)\tau_{pj}] + (2n^{\mathrm{c}}k)(1 - e^{\rho}) = 0, \end{array}\right\} \quad (3.6)$$

where $\boldsymbol{x}$, ρ, $\boldsymbol{u}_j$, $\boldsymbol{\zeta}_j$, $\boldsymbol{\mu}_j$, $\boldsymbol{\lambda}_j$, and $\boldsymbol{\tau}_j$ ($j = 1,\dots,n^{\mathrm{c}}$) are the variables. Problem (3.6) is regarded as a conventional nonlinear programming (NLP) problem in the sense that it does not contain complementarity conditions in its constraints. We call problem (3.6) an *implicit reformulation* of the MPEC problem (2.11), because the smoothing parameter ρ is included as one of the variables. When we solve problem (3.6), ρ is to be updated simultaneously together with the other variables at each iteration of a standard nonlinear programming algorithm.

Remark 3.2. In problem (3.6), the complementarity conditions in problem (2.11) are rewritten by using the smoothed Fischer–Burmeister functions and the additional equality constraint in the form of (3.4). We may understand the role of the extra variable ρ more clearly by rewriting (3.4) as

$$\log\left(\frac{1}{n}\boldsymbol{y}^{\mathrm{T}}\boldsymbol{z} + 1\right) = \rho. \quad (3.7)$$

A key observation in (3.7) is that ρ can be regarded as a measure of the residual of the complementarity condition $\boldsymbol{y}^{\mathrm{T}}\boldsymbol{z} = 0$. An alternative candidate for a measure of the residual, for example, might be

$$\frac{1}{n}\boldsymbol{y}^{\mathrm{T}}\boldsymbol{z} = \rho, \quad (3.8)$$

which seems to be simpler than (3.7). However, the function (3.8) does not provide us with an implicit reformulation. Indeed, (3.3) and (3.8) are solely necessary conditions for (3.2). Thus, to obtain a necessary and sufficient condition shown in Proposition 3.1, we choose the logarithm function in (3.7) as a measure of the residual. ■

Remark 3.2 also suggests an appropriate choice of the initial value $\rho_{(0)}$ for the variable ρ when we solve the implicit reformulation with a nonlinear programming algorithm. Let $\boldsymbol{y}_{(0)}$ and $\boldsymbol{z}_{(0)}$ be the initial values of $\boldsymbol{y}$ and $\boldsymbol{z}$ satisfying $0 \neq \boldsymbol{y}_{(0)}^{\mathrm{T}}\boldsymbol{z}_{(0)} > -1/n$. Then, it might be reasonable to put $\rho_{(0)} := \log[(\boldsymbol{y}_{(0)}^{\mathrm{T}}\boldsymbol{z}_{(0)}/n) + 1]$, where $\boldsymbol{y}$ and $\boldsymbol{z}$ correspond to $\boldsymbol{y} = (\alpha\mathbf{1}-\boldsymbol{\zeta}_1,\dots,\alpha\mathbf{1}-\boldsymbol{\zeta}_k,\alpha\mathbf{1}+\boldsymbol{\zeta}_1,\dots,\alpha\mathbf{1}+\boldsymbol{\zeta}_k)$ and $\boldsymbol{z} = (\boldsymbol{\lambda}_1,\dots,\boldsymbol{\lambda}_k,\boldsymbol{\tau}_1,\dots,\boldsymbol{\tau}_k)$, respectively, in problem (3.6).

Remark 3.3. For understanding the role which the smoothing parameter ρ plays, it is interesting to compare problem (3.6) with the following problem:

$$\left.\begin{array}{ll} \min & v(\boldsymbol{x}) \\ \mathrm{s.t.} & \boldsymbol{x} \in \mathcal{X}, \\ & \forall j = 1, \dots, n^{\mathrm{c}} : \left\{\begin{array}{ll} K(\boldsymbol{x})\boldsymbol{u}_j - \tilde{\boldsymbol{f}} - F_0\boldsymbol{\zeta}_j = \boldsymbol{0}, & \\ \nabla g_j(\boldsymbol{u}_j) - K(\boldsymbol{x})\boldsymbol{\mu}_j = \boldsymbol{0}, & \\ g_j(\boldsymbol{u}_j) \ge 0, & \\ F_0^{\mathrm{T}}\boldsymbol{\mu}_j + \boldsymbol{\lambda}_j - \boldsymbol{\tau}_j = \boldsymbol{0}, & \\ \phi(\alpha - \zeta_{pj}, \lambda_{pj}, 0) = 0, & p = 1, \dots, k, \\ \phi(\alpha + \zeta_{pj}, \tau_{pj}, 0) = 0, & p = 1, \dots, k, \end{array}\right. \end{array}\right\} \tag{3.9}$$

which is also equivalent to the MPEC problem (2.11). In our preliminary numerical experiments, the SQP method (`fmincon` [41]) fails to converge to a solution of (3.9), even for moderately small number of complementarity constraints. This may be due to the nonsmooth property of Fischer–Burmeister functions in problem (3.9), i.e., $\phi(\alpha - \zeta_{pj}, \lambda_{pj}, 0)$ and $\phi(\alpha+\zeta_{pj}, \tau_{pj}, 0)$. In contrast, in (3.6) we use the smoothed Fischer–Burmeister function with the smoothing parameter ρ. As discussed in Remark 3.2, ρ in (3.6) roughly corresponds to the residual of the complementarity conditions, and it is usual that the complementarity constraints are not satisfied exactly during the optimization procedure before it converges. Hence, we may expect that $\rho \neq 0$ holds at intermediate solutions of the optimization procedure. If this is the case, then the constraints of (3.6) are differentiable at any intermediate solution, which explains the advantage of the implicit formulation (3.6) over (3.9). ■

Remark 3.4. The idea of regarding a smoothing parameter ρ as an independent variable can also be found in [25, 26], but the formulations presented there are different from ours as follows. The method in [25, 26] is based on the fact that the complementarity conditions in (3.2) are equivalent to

$$\phi(y_i, z_i, \rho) = 0, \qquad i = 1, \dots, n, \tag{3.10}$$

$$e^{\rho} = 1. \tag{3.11}$$

Then the smooth SQP method was applied to the problem including (3.10) and (3.11) in its constraints instead of the complementarity conditions. Thus, the value of the smoothing parameter ρ is irrelevant to the variables $\boldsymbol{y}$ and $\boldsymbol{z}$ in the method in [25, 26], while in our reformulation we attempt to adjust ρ to the residual of the complementarity conditions as discussed in Remark 3.2. In other words, we expect that $|\rho|$ is relatively large at the earlier stage of the optimization procedure, and $|\rho|$ can be smaller as the residual of the complementarity constraints is reduced considerably. In contrast, in the method using (3.10) and (3.11), there may possibly exist two disadvantageous situations: (i) ρ is almost equal to 0 even if the residual of complementarity conditions is still relatively large, then the smoothing effect of the Fischer–Burmeister function might not work properly, which may cause the divergence of the optimization algorithm (see Remark 3.3); (ii) ρ is still far from 0 when the residual of complementarity conditions is sufficiently small, then unnecessary iterations might be spent to reduce the residual of (3.11). ■

Remark 3.5. The methodology of the implicit reformulation presented in Proposition 3.1 can also be applied to smoothed complementarity functions other than the smoothed Fischer–Burmeister function. For example, consider the CHKS (Chen–Harker–Kanzow–Smale) smoothing function defined by [14]

$$\phi_{\mathrm{CHKS}}(y, z, \rho) = y + z - \sqrt{(y-z)^2 + 4\rho^2}.$$

Indeed, it is easy to see that $\boldsymbol{y} = (y_i) \in \mathbb{R}^n$, $\boldsymbol{z} = (z_i) \in \mathbb{R}^n$, and $\rho \in \mathbb{R}$ satisfy (3.2) if and only if they satisfy

$$\phi_{\mathrm{CHKS}}(y_i, z_i, \rho) = 0, \qquad i = 1, \dots, n \tag{3.12}$$

and (3.4). Therefore, in problem (3.6), we can replace the smoothed Fischer–Burmeister function ϕ with the CHKS smoothing function without changing the optimal solution. See Section 4.2.2 for the numerical experiments of the formulation with the CHKS smoothing function. ■

Recall that the difficulty in dealing with the MPEC formulation (2.11) arises from the fact that MPEC fails to satisfy any standard constraint qualification. We next investigate the constraint qualification of the implicit reformulation (3.6). For simplicity, we denote by $\boldsymbol{\xi}$ the vector of the variables of problem (3.6), i.e.,

$$\boldsymbol{\xi} = \left(\boldsymbol{x}, \rho, \left((\boldsymbol{u}_j, \boldsymbol{\zeta}_j, \boldsymbol{\mu}_j, \boldsymbol{\lambda}_j, \boldsymbol{\tau}_j) \mid j = 1, \dots, n^{\mathrm{c}}\right)\right).$$

Under the assumption that the MPEC problem (2.11) satisfies MPEC-LICQ [34], Proposition 3.6 below shows that the LICQ for problem (3.6) is satisfied. Note that we assume the strict complementarity conditions at a feasible solution $\bar{\boldsymbol{\xi}}$ so that the smoothed Fischer–Burmeister function is differentiable at $\bar{\boldsymbol{\xi}}$.

Proposition 3.6. *Suppose that a feasible solution $\bar{\boldsymbol{\xi}}$ of problem* (2.11) *satisfies the strict complementarity, i.e.,*

$$(\alpha - \bar{\zeta}_{pj}) + \bar{\lambda}_{pj} > 0, \quad (\alpha + \bar{\zeta}_p) + \bar{\tau}_{pj} > 0, \qquad p = 1, \dots, k;\ j = 1, \dots, n^{\mathrm{c}}.$$

If problem (2.11) *satisfies MPEC-LICQ at $\bar{\boldsymbol{\xi}}$, then problem* (3.6) *satisfies LICQ at $\bar{\boldsymbol{\xi}}$.*

Proof. For simplicity, we consider the complementarity conditions

$$\mathbb{R}^2 \ni \boldsymbol{y} \geq \boldsymbol{0}, \quad \mathbb{R}^2 \ni \boldsymbol{z} \geq \boldsymbol{0}, \quad \boldsymbol{y}^{\mathrm{T}}\boldsymbol{z} = 0, \tag{3.13}$$

i.e., we put $n = 2$ in Proposition 3.1. Define $\hat{\phi}_1$, $\hat{\phi}_2$, $\delta : \mathbb{R}^5 \to \mathbb{R}$ by

$$\begin{aligned} \hat{\phi}_i(\boldsymbol{y}, \boldsymbol{z}, \rho) &= \phi(y_i, z_i, \rho), \qquad i = 1, 2, \\ \delta(\boldsymbol{y}, \boldsymbol{z}, \rho) &= \boldsymbol{y}^{\mathrm{T}}\boldsymbol{z} - 2(e^{\rho} - 1). \end{aligned}$$

Then Proposition 3.1 implies that $(\boldsymbol{y}, \boldsymbol{z}, \rho)$ satisfies (3.13) and $\rho = 0$ if and only if

$$\hat{\phi}_1(\boldsymbol{y}, \boldsymbol{z}, \rho) = \hat{\phi}_2(\boldsymbol{y}, \boldsymbol{z}, \rho) = \delta(\boldsymbol{y}, \boldsymbol{z}, \rho) = 0. \tag{3.14}$$

Suppose that $(\bar{\boldsymbol{y}}, \bar{\boldsymbol{z}}, \bar{\rho})$ satisfies (3.14) and the strict complementarity conditions. Particularly, we assume

$$\bar{y}_1 > 0, \quad \bar{z}_2 > 0, \quad \bar{y}_2 = \bar{z}_1 = \bar{\rho} = 0.$$

Without loss of generality, it suffices to show that $\nabla\hat{\phi}_1(\bar{\boldsymbol{y}}, \bar{\boldsymbol{z}}, \bar{\rho})$, $\nabla\hat{\phi}_2(\bar{\boldsymbol{y}}, \bar{\boldsymbol{z}}, \bar{\rho})$, and $\nabla\delta(\bar{\boldsymbol{y}}, \bar{\boldsymbol{z}}, \bar{\rho})$ are linearly independent. Simple calculations yield

$$\nabla\hat{\phi}_1(\bar{\boldsymbol{y}}, \bar{\boldsymbol{z}}, \bar{\rho}) = \begin{bmatrix} 1 - \bar{y}_1/|\bar{y}_1| \\ 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \quad \nabla\hat{\phi}_2(\bar{\boldsymbol{y}}, \bar{\boldsymbol{z}}, \bar{\rho}) = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 1 - \bar{z}_2/|\bar{z}_2| \\ 0 \end{bmatrix}, \quad \nabla\delta(\bar{\boldsymbol{y}}, \bar{\boldsymbol{z}}, \bar{\rho}) = \begin{bmatrix} 0 \\ \bar{z}_2 \\ \bar{y}_1 \\ 0 \\ -2 \end{bmatrix},$$

which are linearly independent. The proof in the general case in the assertion of this proposition can be obtained similarly, but is omitted for simple presentation. □

Proposition 3.6 guarantees that the implicit formulation (3.6) satisfies LICQ, which allows to solve (3.6) by using a standard nonlinear optimization approach. In contrast, the constraints are not differentiable at a feasible solution which does not satisfy the strict complementarity. However, as discussed in Remark 3.3, we may expect that $\rho \neq 0$ at almost all intermediate solutions in the course of optimization. If this is the case, then the constraints are differentiable. Throughout the numerical experiments presented in Section 4, it is confirmed that a standard smooth nonlinear programming approach can solve problem (3.6) within moderately small number of iterations.

4. Numerical Experiments

The robust optimal designs of structures subjected to the uncertain static loads are found by solving problem (3.6). Computation was carried out on Core 2 Duo (2.26 GHz with 4.0 GB memory) with MATLAB R2009b. We solve problem (3.6) by using the MATLAB built-in function `fmincon` [41], which implements the SQP method and the interior-point method for nonlinear programming problems. The gradients of the objective and constraint functions are provided as the user-defined functions.

It is known that `fmincon` is not superior to other available nonlinear programming solvers. In fact, it is known that some robust nonlinear programming solvers can solve a quite large class of MPEC problems [21], while `fmincon` usually fails in our preliminary numerical experiments. This drawback of `fmincon`, however, is adequate for our aim to examine the presented regularization scheme as discussed in [38]; i.e., if regularized MPEC problems can be solved by `fmincon` then it is confirmed that the regularization scheme has an advantage to enhance the robustness of a well-developed nonlinear programming solver when it is applied to an MPEC problem.

4.1. 2-bar truss

Consider a two-bar plane truss illustrated in Figure 1. The initial lengths of members (i) and (ii) are 1.0 m and $\sqrt{2}$ m, respectively. The cross-sectional areas of these members are considered as design variables, i.e., $m = 2$. Nodes (b) and (c) are pin-supported, while node (a) is free. Therefore, the number of degrees of freedom of displacements is $d = 2$. The elastic modulus is 200 GPa.

As the nominal external load, $\tilde{\boldsymbol{f}}$, the horizontal force of 10.0 kN is applied at node (a). In accordance with (2.4), we define the uncertainty model in the external load as

$$\boldsymbol{f} = \tilde{\boldsymbol{f}} + F_0\boldsymbol{\zeta}, \quad \alpha \geq |\zeta_j| \ (j = 1, 2) \tag{4.1}$$

with $F_0 = 1.0\,(\mathrm{kN}) \times I_2$ and $k = 2$. Therefore, ζ_1 and ζ_2 correspond to the uncertain external forces (in kN) in the x- and y-directions, respectively.

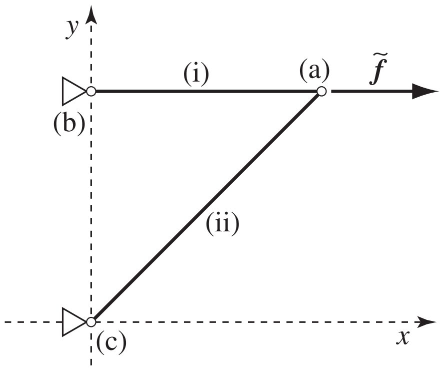


Figure 1: A 2-bar truss

(a) The nominal case

(b) The robust optimal solution

Figure 2: The optimal solutions of the 2-bar truss example

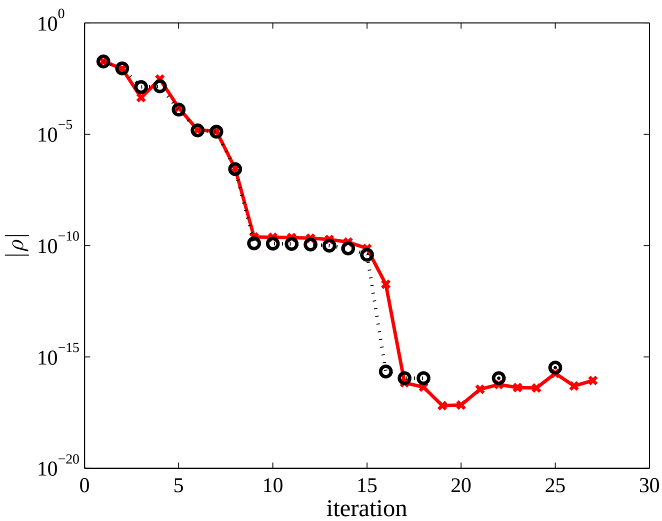


Figure 3: Convergence histories of the smoothing parameter and the residual of the complementarity constraints for the 2-bar truss example ("——": ρ; "······": $\log((\boldsymbol{y}^{\mathrm{T}}\boldsymbol{z}/n)+1)$ in (3.7))

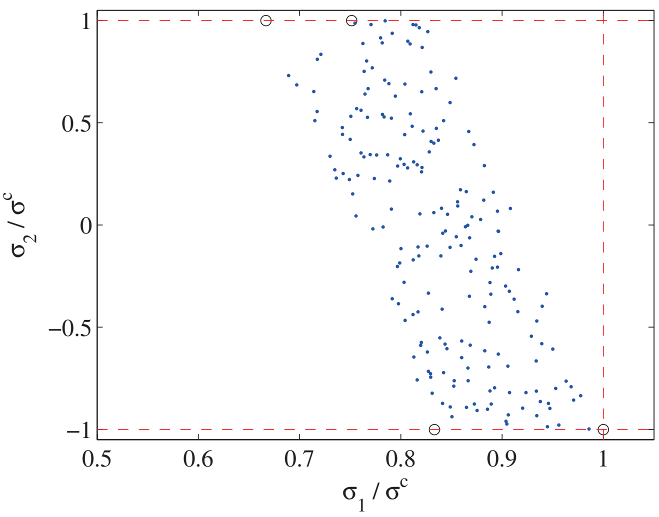


Figure 4: Stress states of the robust optimal design of the 2-bar truss ("∘": stresses in the worst cases; "·": stresses computed from randomly generated loads)

As for the mechanical performance constraints in (2.2), we consider the constraints on the member stresses. Note that the member stress, denoted σ_i, of a truss is a linear function of the displacement $\boldsymbol{u}$. The stress constraint for each member is written as $|\sigma_i| \le \sigma^{\rm c}$ $(i = 1, 2)$, where $\sigma^{\rm c} = 10.0\,{\rm MPa}$ is the upper bound for the member stresses.

Note that the stress constraint requires a special treatment in the structural optimization; see, e.g., [37]. It is often that some members vanish as the result of optimization. In such a case, the stress constraints should be satisfied only by the existing members, and the constraints do not have to be satisfied by the members with vanishing cross-sectional areas. In other words, the stress constraint should be treated as the so-called *vanishing constraint* [1], or the *design-dependent constraint* [37]. In this example, however, the robust optimal solution of this example cannot be kinematically indeterminate, which means that both members should exist at the robust optimal solution. Hence, we do not treat the stress constraints as the vanishing constraints in this example. A precise treatment of the stress constraints in the robust structural optimization can be found in [27].

As the initial solution for a nonlinear programming solver, the initial value for $\boldsymbol{x}$ is given as $(2000.0, 4000.0)\,{\rm mm}^2$. Then we put $\boldsymbol{u}_j = K(\boldsymbol{x})^{-1}\tilde{\boldsymbol{f}}$, $\boldsymbol{\zeta}_j = \boldsymbol{0}$, $\boldsymbol{\mu}_j = -K(\boldsymbol{x})^{-1}\boldsymbol{a}_j$, $\boldsymbol{\tau}_j = \max\{F_0\boldsymbol{\mu}_j, \boldsymbol{0}\}$, and $\boldsymbol{\lambda}_j = \min\{-F_0\boldsymbol{\mu}_j, \boldsymbol{0}\}$ $(j = 1, \dots, n^{\rm c})$, where the max-operator and the min-operator for a vector are understood component-wisely. The initial value for ρ is computed by using (3.7).

The robust optimization problem (3.6) is solved for $\alpha = 1.0$ by using `fmincon` with default settings (i.e., the SQP method). The obtained optimal solution is $\boldsymbol{x}^* = (1200.0, 200.0)$ ${\rm mm}^2$ as illustrated in Figure 2(b), where the width of each member is proportional to its cross-sectional area. Figure 3 shows the convergence history of the smoothing parameter ρ and the residual of the complementarity constraints, both of which decrease almost monotonically. It is observed in Figure 3 that ρ is automatically adjusted to the residual of the complementarity constraints. Note that `fmincon` fails to solve the formulation (3.9); it stops because no feasible solution can be found.

If we do not consider any uncertainty, i.e., in the case of $\alpha = 0$, then the robust structural optimization (3.6) is reduced to the conventional structural optimization (2.3) subjected to

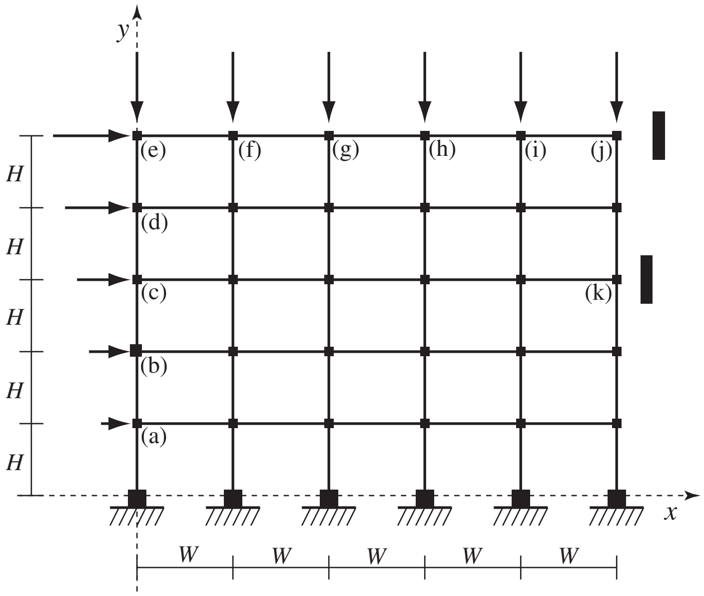


Figure 5: A 5-story frame

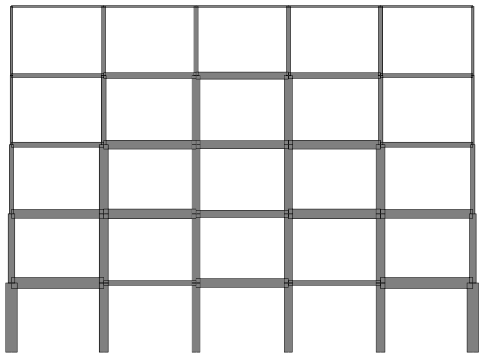
the nominal load. The optimal solution in this case is $\boldsymbol{x}' = (1000.0, 0.0)\,\mathrm{mm}^2$, which is illustrated in Figure 2(a). Since we consider only the horizontal force applied at node (a), member (ii) vanishes at the optimal solution when we impose the stress constraint only on member (i). As a consequence, the nominal optimal solution is kinematically indeterminate (unstable) as seen in Figure 2, whereas the robust optimal solution is kinematically determinate (stable).

Next we randomly generate a number of external loads $\boldsymbol{f}$ satisfying (4.1) with $\alpha = 1.0$, and compute the corresponding member stresses of the robust optimal design. Figure 4 depicts the obtained member stresses $(\sigma_1/\sigma^{\mathrm{c}}, \sigma_2/\sigma^{\mathrm{c}})$ normalized by the specified upper bound σ^{c}. It is observed from Figure 4 that the stress constraints are satisfied for all randomly generated loads. The stress states corresponding to the worst cases are also shown in Figure 4 by the open circles. Note that there exist four extremal cases of stresses, i.e., the maximum and minimum stresses of members (i) and (ii). It is observed from Figure 4 that for each member there exists the worst case in which the stress constraint becomes active.

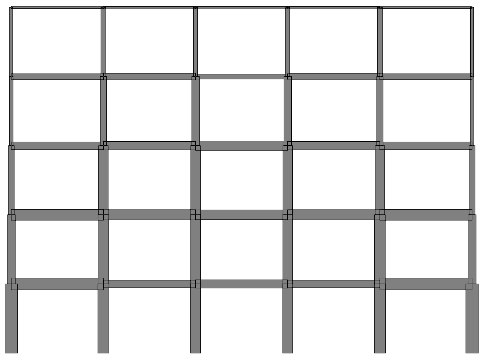
4.2. Plane frame under displacement constraint

As a moderately large example, consider a 5-story 5-bay plane frame structure shown in Figure 5, where $W = 4\,\mathrm{m}$ and $H = 3\,\mathrm{m}$. The frame has 55 members, which are the Euler–Bernoulli beam elements. All the bottom nodes are rigidly supported. Note that for a frame structure the rotation of each free node is also considered as one of the degrees of freedom of (generalized) displacements $\boldsymbol{u} \in \mathbb{R}^d$, and hence $d = 90$. The cross-sectional area, denoted by x_i, of each cross section is chosen as a design variable, i.e., $m = 55$. The elastic modulus is 200 GPa.

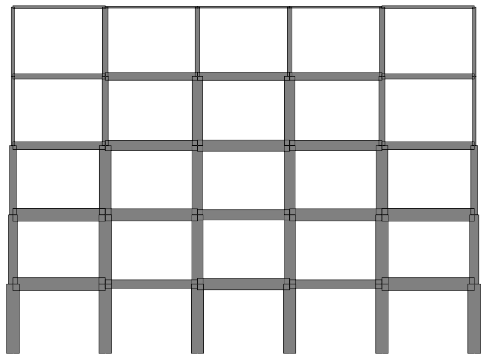
As the nominal load $\tilde{\boldsymbol{f}}$, horizontal forces of 1.0, 2.0, 3.0, 4.0, and $5.0 \times 10^3\,\mathrm{kN}$ are applied at nodes (a)–(e), respectively, in the positive direction of the x-axis. The vertical force of $12.5 \times 10^3\,\mathrm{kN}$ is applied at each of nodes (e)–(j) in the negative direction of the y-axis. Suppose that the uncertain external forces are applied to all free nodes, while no external



(a) The nominal case



(b) The robust optimal solution for $\alpha = 0.1$



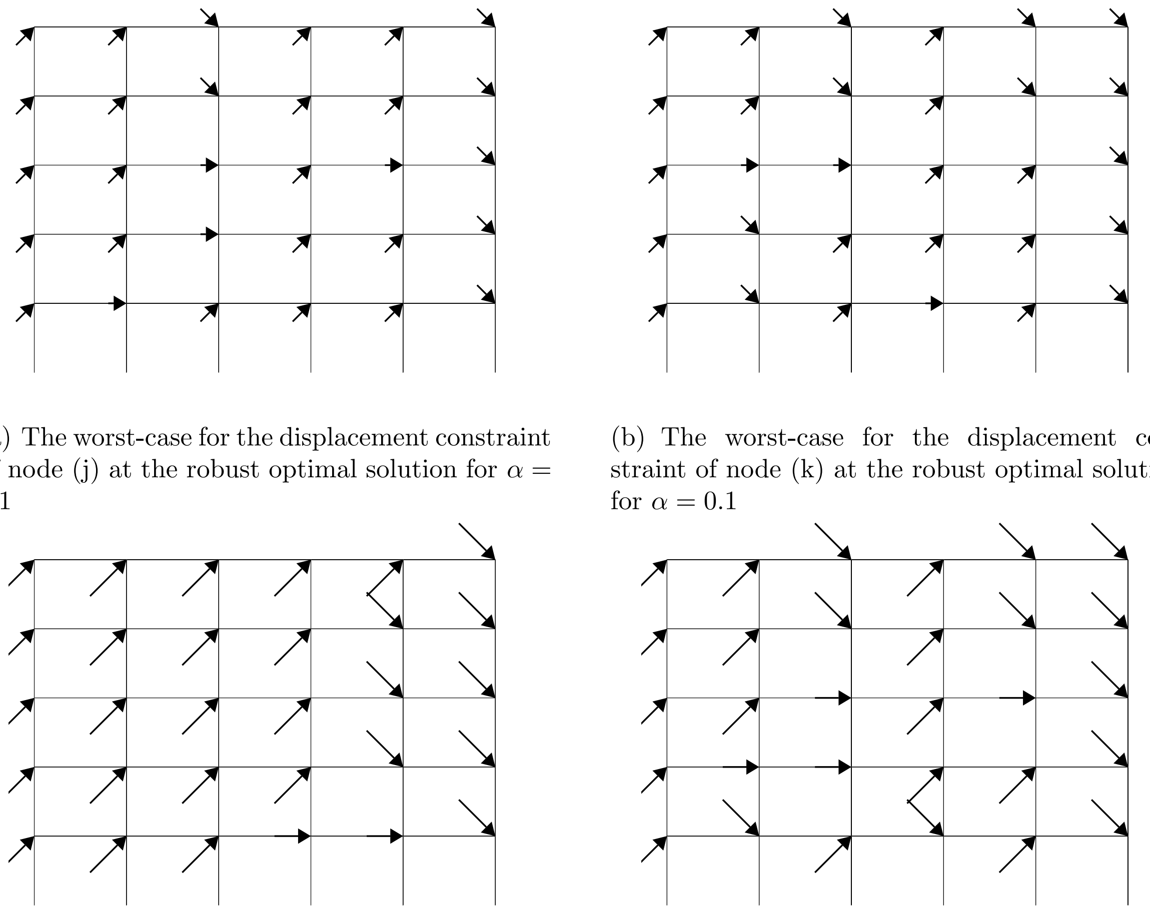
(c) The robust optimal solution for $\alpha = 0.2$

Figure 6: The optimal solutions of the frame example

moments are applied. This uncertainty model is represented as (2.4) by putting $k = 60$ and

$$F_0 = 1.0\,(\text{kN}) \times \left[\begin{array}{c|c|c} \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix} & O & O \\ \hline O & \ddots & O \\ \hline O & O & \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix} \end{array} \right] \in \mathbb{R}^{90 \times 60}.$$

Each of beams and columns is assumed to have a sandwich cross-section; i.e., the second moment of inertia t_i is assumed to be proportional to the cross-sectional area x_i as $t_i = r^2 x_i$, where $2r = 400\,\text{mm}$ is the distance between two flanges. As the admissible set $\mathcal{X}$ of $\boldsymbol{x}$ in (3.6), we consider the lower bound constraints, i.e., $\mathcal{X} = \{\boldsymbol{x} \in \mathbb{R}^m \mid x_i \geq \underline{x}_i \ (i = 1, \dots, m)\}$ with $\underline{x}_i = 2000\,\text{mm}^2$, to guarantee that assumption (2.5) is satisfied. For obtaining a practically acceptable optimal solution, we introduce additional constraints such that the cross-sectional area of a column must not be larger than that at the lower story. Moreover, symmetry located members, with respect to the symmetry axis of the configuration in Figure 5, are supposed to have the same cross-section. Note that the assumptions above, i.e., the sandwich cross-sections, the monotonicity of the cross-sections of columns, and the symmetry of configuration, are quite commonly used in optimization of steel frame



(a) The worst-case for the displacement constraint of node (j) at the robust optimal solution for $\alpha = 0.1$

(b) The worst-case for the displacement constraint of node (k) at the robust optimal solution for $\alpha = 0.1$

(c) The worst-case for the displacement constraint of node (j) at the robust optimal solution for $\alpha = 0.2$

(d) The worst-case for the displacement constraint of node (k) at the robust optimal solution for $\alpha = 0.2$

Figure 7: The external loads in the worst cases for the robust optimal solutions in Figure 6

structures; see, e.g., [35].

As the constraint on the mechanical performance in (2.2), we consider the upper bound constraints for the horizontal displacements of nodes (j) and (k), i.e., $n^{\rm c} = 2$. The upper bounds are $u_1^{\rm c} = 5H/200$ and $u_2^{\rm c} = 3H/200$ for the displacements of nodes (j) and (k), respectively. Consequently, problem (3.6) has 776 variables in total and 240 complementarity constraints.

4.2.1. Results of the proposed method

The robust optimization problem (3.6) is solved by using the interior-point method implemented in `fmincon` [41]. The initial solution is given as $x_i = 10000\,\mathrm{mm}^2$ $(i = 1, \dots, m)$, $\rho = 0.1$, and for the other variables the initial values are given in a manner similar to Section 4.1. Note that in this example the design variables vector $\boldsymbol{x}$ with sufficiently large x_i's is feasible for problem (3.6). The computational results are listed in Table 1. The nominal optimal solution obtained by solving problem (2.3) is shown in Figure 6(a). Figures 6(b) and 6(c) depict the optimal solutions obtained for $\alpha = 0.1$ and $\alpha = 0.2$, respectively. For these robust optimal solutions, Figure 7 shows the worst-case loads for the displacement constraints, which are obtained by solving the worst-case detection problem (2.8). Note that the nominal external load, illustrated in Figure 5, is not depicted in Figure 7. Each

Table 1: Computational results of the plane-frame example

α	Volume (m^3)	CPU (s)	Iter.
nominal	14.1948	13.4	66
0.05	15.3665	238.1	122
0.10	16.5896	118.8	57
0.15	17.7392	283.1	144
0.20	18.9299	245.6	125

Table 2: Computational results for the CHKS smoothing function: the plane-frame example

α	Volume (m^3)	CPU (s)	Iter.
0.05	15.4871	217.3	88
0.10	16.5499	421.0	187
0.15	17.7378	258.1	128
0.20	18.9760	712.9	298

displacement constraint becomes active in the corresponding worst case shown in Figure 7.

4.2.2. Results using the CHKS smoothing function

As discussed in Remark 3.5, the methodology of the implicit reformulation proposed in Section 3 can also be applied to the CHKS smoothing function straightforwardly. We here examine the computational efficiency of the formulation using the CHKS smoothing function.

In a manner similar to Section 4.2.1, the robust optimization problem of the plane frame is solved with the interior-point method implemented in `fmincon` [41]. The same initial solutions are chosen as those in Section 4.2.1. The computational results for the CHKS smoothing function are listed in Table 2. Note that the obtained solution for each value of α is different from the solution obtained in Section 4.2.1, although the objective values are very similar. It is well known that the optimization problem of a frame structure often possesses many local optimal solutions; see, e.g., [35]. It is observed from Tables 1 and 2 that the computational efficiency is not improved with the CHKS smoothing function in general, compared with the smoothed Fischer–Burmeister function.

4.2.3. Results without implicit control of ρ

As mentioned in Remark 3.4, use of the independent variable ρ for constructing an implicit reformulation of MPEC is found in literature [25, 26]. A potential advantage of our formulation is that the smoothing parameter ρ is related to the residual of the complementarity conditions as discussed in Remark 3.2, and hence it is expected that ρ is automatically adjusted to the feasibility of the complementarity conditions. In contrast, the formulation proposed in [25, 26] is based on (3.10) and (3.11), where ρ is irrelevant to the residuals of the complementarity constraints. In this section, we examine the formulation using (3.10) and (3.11) to compare the computational efficiency with the results for our formulation presented in Section 4.2.1.

The same problem of the plane frame is solved. An initial value for ρ, denoted $\rho^{(0)}$, is chosen as $\rho^{(0)} = 0.0.1$, 0.1, or 1.00, while initial values for the other variables are given in a

Table 3: Computational results for the formulation using (3.10) and (3.11): the plane-frame example

α	$\rho^{(0)}$	Volume (m^3)	CPU (s)	Iter.
0.05	0.1	15.4598	707.1	339
0.10	0.01	16.5996	2367.9	1094
0.10	0.1	16.6487	733.9	362
0.10	1.0	16.5637	814.8	383
0.15	0.1	17.8811	1809.6	884
0.20	0.01	19.0099	1432.5	669
0.20	0.1	—	—	($>$ 2100)
0.20	1.0	18.9467	1562.6	752

manner similar to Section 4.2.1. The computational results are listed in Table 3. In the case of $\alpha = 0.2$ and $\rho^{(0)} = 0.1$, `fmincon` does not converge within 2100 iterations. It is observed from Table 3 that the computational efficiency of the formulation using (3.10) and (3.11) is drastically inferior to that of the proposed method listed in Table 1. Furthermore, for all the values α examined, the objective values of the solutions obtained in this section are larger than the solutions obtained by using the proposed method.

5. Concluding Remarks

For a structure subjected to a non-probabilistic uncertainty in the static external load, the robust structural optimization problem is formulated as an MPEC (mathematical program with equilibrium constraints) problem in the complementarity form. In this paper, we have explored an implicit reformulation of the MPEC based on the smoothed Fischer–Burmeister function, which enables us to apply a standard nonlinear programming approach. In the implicit formulation, the parameter ρ for smoothing the Fischer–Burmeister function is considered as an independent variable, and we have introduced an extra equality constraint with which ρ vanishes at the optimal solution. As a consequence, ρ decreases automatically as the optimization algorithm approaches to convergence. The applicability of the proposed methodology to the CHKS smoothing function has also been discussed.

On the other hand, in the proposed reformulation, the complementarity constraints, together with the associated inequality constraints, are treated as equality constraints. This treatment might potentially cause difficulties in numerical solution, because, from the practical point of view, an inequality constraint is more tractable than an equality constraint for an optimization algorithm in general. In the numerical experiments in Section 4, feasible solutions have successfully been found by using a standard nonlinear programming approach.

Acknowledgments

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References

[1] W. Achtziger and C. Kanzow: Mathematical programs with vanishing constraints: Optimality conditions and constraint qualifications. *Mathematical Programming*, **A114** (2008), 69–99.

[2] R. Ardito, G. Cocchetti, and G. Maier: On structural safety assessment by load factor maximization in piecewise linear plasticity. *European Journal of Mechanics, A/Solids*, **27** (2008), 859–881.

[3] J.S. Arora: *Introduction to Optimum Design (2nd ed.)* (Elsevier, San Diego, 2004).

[4] F.T.K. Au, Y.S. Cheng, L.G. Tham, and G.W. Zheng: Robust design of structures using convex models. *Computers & Structures*, **81** (2003), 2611–2619.

[5] O.A. Bauchau and J.I. Craig: *Structural Analysis* (Springer, Dordrecht, 2009).

[6] Y. Ben-Haim: Uncertainty, probability and information-gaps. *Reliability Engineering and System Safety*, **85** (2004), 249–266.

[7] Y. Ben-Haim: *Information-gap Decision Theory: Decisions under Severe Uncertainty (2nd ed.)* (Academic Press, London, 2006).

[8] Y. Ben-Haim and I. Elishakoff: *Convex Models of Uncertainty in Applied Mechanics* (Elsevier, New York, 1990).

[9] A. Ben-Tal and A. Nemirovski: Robust truss topology optimization via semidefinite programming. *SIAM Journal on Optimization*, **7** (1997), 991–1016.

[10] A. Ben-Tal and A. Nemirovski: Robust convex optimization. *Mathematics of Operations Research*, **23** (1998), 769–805.

[11] A. Ben-Tal and A. Nemirovski: Robust optimization—methodology and applications. *Mathematical Programming*, **B92** (2002), 453–480.

[12] H.-G. Beyer and B. Sendhoff: Robust optimization—a comprehensive survey. *Computer Methods in Applied Mechanics and Engineering*, **196** (2007), 3190–3218.

[13] M. Bruggi: On an alternative approach to stress constraints relaxation in topology optimization. *Structural and Multidisciplinary Optimization*, **36** (2008), 125–141.

[14] B. Chen and P.T. Harker: A non-interior-point continuation method for linear complementarity problems. *SIAM Journal on Matrix Analysis and Applications*, **14** (1993), 1168–1190.

[15] E. Cherkaev and A. Cherkaev: Minimax optimization problem of structural design. *Computers & Structures*, **86** (2008), 1426–1435.

[16] P.W. Christensen and A. Klarbring: *An Introduction to Structural Optimization* (Springer, 2009).

[17] J.-P. Dussault, P. Marcotte, S. Roch, and G. Savard: A smoothing heuristic for a bilevel pricing problem. *European Journal of Operational Research*, **174** (2006), 1396–1413.

[18] I. Elishakoff: Essay on uncertainties in elastic and viscoelastic structures: From A.M. Freudenthal's criticisms to modern convex modeling. *Computers & Structures*, **56** (1995), 871–895.

[19] F. Facchinei, H. Jiang, and L. Qi: A smoothing method for mathematical programs with equilibrium constraints. *Mathematical Programming*, **85** (1999), 107–134.

[20] M.C. Ferris and F. Tin-Loi: Limit analysis of frictional block assemblies as a mathematical program with complementarity constraints. *International Journal of Mechanical Sciences*, **43** (2001), 209–224.

[21] R. Fletcher and S. Leyffer: Solving mathematical programs with complementarity constraints as nonlinear programs. *Optimization Methods and Software*, **19** (2004), 15–40.

[22] M. Fukushima, Z.-Q. Luo, and J.-S. Pang: A globally convergent sequential quadratic programming algorithm for mathematical programs with linear complementarity constraints. *Computational Optimization and Applications*, **10** (1998), 5–34.

[23] S. Ganzerli and C.P. Pantelides: Optimum structural design via convex model superposition. *Computers & Structures*, **74** (1998), 639–647.

[24] X. Guo, W. Bai, and W. Zhang: Extreme structural response analysis of truss structures under material uncertainty via linear mixed 0–1 programming. *International Journal for Numerical Methods in Engineering*, **76** (2008), 253–277.

[25] J.-B. Jian: A superlinearly convergent implicit smooth SQP algorithm for mathematical programs with nonlinear complementarity constraints. *Computational Optimization and Applications*, **31** (2005), 335–361.

[26] H. Jiang and D. Ralph: Smooth SQP methods for mathematical programs with nonlinear complementarity constraints. *SIAM Journal on Optimization*, **10** (2000), 779–808.

[27] Y. Kanno and X. Guo: A mixed integer programming for robust truss topology optimization with stress constraints. *International Journal for Numerical Methods in Engineering*, **83** (2010), 1675–1699.

[28] Y. Kanno and J.A.C. Martins: Arc-length method for frictional contact problems using mathematical programming with complementarity constraints. *Journal of Optimization Theory and Applications*, **131** (2006), 89–113.

[29] Y. Kanno and I. Takewaki: Sequential semidefinite program for robust truss optimization based on robustness functions associated with stress constraints. *Journal of Optimization Theory and Applications*, **130** (2006), 265–287.

[30] Y. Kanno and I. Takewaki: Worst-case plastic limit analysis of trusses under uncertain loads via mixed 0–1 programming. *Journal of Mechanics of Materials and Structures*, **2** (2007), 245–273.

[31] C. Kanzow: Some noninterior continuation methods for linear complementarity problems. *SIAM Journal on Matrix Analysis and Applications*, **17** (1996), 851–868.

[32] M. Kočvara, J. Zowe, and A. Nemirovski: Cascading—an approach to robust material optimization. *Computers & Structures*, **76** (2000), 431–442.

[33] K.-H. Lee and G.-J. Park: Robust optimization considering tolerances of design variables. *Computers & Structures*, **79** (2001), 77–86.

[34] Z-.Q. Luo, J.-S. Pang, and D. Ralph: *Mathematical Programs with Equilibrium Constraints* (Cambridge University Press, Cambridge, 1996).

[35] M. Ohsaki: Local and global searches of approximate optimal designs of regular frames. *International Journal for Numerical Methods in Engineering*, **67** (2006), 132–147.

[36] J. Outrata, M. Kočvara, and J. Zowe: *Nonsmooth Approach to Optimization Problems with Equilibrium Constraints* (Kluwer, Dordrecht, 1998).

[37] G.I.N. Rozvany: On design-dependent constraints and singular topologies. *Structural and Multidisciplinary Optimization*, **21** (2001), 164–172.

[38] S. Scholtes: Convergence properties of a regularization scheme for mathematical programs with complementarity constraints. *SIAM Journal on Optimization*, **11** (2001), 918–936.

[39] G.I. Schuëller: Developments in stochastic structural mechanics. *Archive of Applied Mechanics*, **75** (2006), 755–773.

[40] O. Stein and G. Still: Solving semi-infinite optimization problem with interior point techniques. *SIAM Journal on Control and Optimization*, **42** (2003), 769–788.

[41] The MathWorks, Inc.: *Using MATLAB* (The MathWorks, Inc., Natick, 2009).

[42] F. Tin-Loi: A smoothing scheme for a minimum weight problem in structural plasticity. *Structural Optimization*, **17** (1999), 279–285.

[43] F. Tin-Loi and N.S. Que: Parameter identification of quasibrittle materials as a mathematical program with equilibrium constraints. *Computer Methods in Applied Mechanics and Engineering*, **190** (2001), 5819–5836.

[44] T. Yan: A class of smoothing methods for mathematical programs with complementarity constraints. *Applied Mathematics and Computation*, **186** (2007), 1–9.

[45] K. Yonekura and Y. Kanno: Global optimization of robust truss topology via mixed integer semidefinite programming. *Optimization and Engineering*, **11** (2010), 355–379.

[46] C. Zang, M.I. Friswell, and J.E. Mottershead: A review of robust optimal design and its application in dynamics. *Computers & Structures*, **83** (2005), 315–326.

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