

LOWER BOUNDS FOR THE MAXIMUM NUMBER OF SOLUTIONS GENERATED BY THE SIMPLEX METHOD

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Abstract Kitahara and Mizuno [3] get upper bounds for the maximum number of different basic feasible solutions generated by the simplex method with the most negative pivoting rule. In this paper, we obtain lower bounds of the maximum number to show how tight the upper bounds are.

Part of the results in this paper are shown in Kitahara and Mizuno [4] as a quick report without proof. They present a simple variant of Klee-Minty's LP and get a lower bound. In this paper, we explain and prove the properties of the variant more precisely. We also show a new lower bound by using a simple example of LP.

Keywords: Linear programming, simplex method, basic feasible solutions, Klee-Minty's LP

1. Introduction

Kitahara and Mizuno [3] generalize Ye's analysis [6] for the Markov Decision problem to LPs (linear programming problems), and they get upper bounds for the number of different BFSs (basic feasible solutions) generated by Dantzig's simplex method [1] (the simplex method with the most negative pivoting rule) for the standard form of LP

$$\begin{aligned} \min \quad & \mathbf{c}^\top \mathbf{x}, \\ \text{subject to} \quad & \mathbf{A}\mathbf{x} = \mathbf{b}, \mathbf{x} \geq \mathbf{0}, \end{aligned} \tag{1.1}$$

where the number of equality constraints is m , the number of variables is n , $\mathbf{A} \in \Re^{m \times n}$, $\mathbf{b} \in \Re^m$, $\mathbf{c} \in \Re^n$, and $\mathbf{x} \in \Re^n$. One of the upper bounds they get is

$$n \left\lceil m \frac{\gamma}{\delta} \log \left(m \frac{\gamma}{\delta} \right) \right\rceil, \tag{1.2}$$

where δ and γ are the minimum and the maximum values of all the positive elements of BFSs, respectively, and $\lceil a \rceil$ denotes the smallest integer bigger than $a \in \Re$. The size of the bound highly depends on the ratio γ/δ . A question one may have is how tight the bound is. We will partially answer to this question by showing lower bounds.

Let $M(m, n, \delta, \gamma)$ be the maximum number of different BFSs generated by Dantzig's simplex method for any standard form of LP (1.1) and any initial BFS, where the number of equality constraints is at most m , the number of variables is at most n , the minimum value of all the positive elements of BFSs is at least δ , and the maximum value of all the positive elements of BFSs is at most γ . We are interested in the size of $M(m, n, \delta, \gamma)$, but it is usually very difficult to get it. So it is important to get lower and upper bounds of it such that the gap or the ratio between them is small. Kitahara and Mizuno [3] show that (1.2) is an upper bound of $M(m, n, \delta, \gamma)$. In this paper, we obtain lower bounds of $M(m, n, \delta, \gamma)$.

Part of the results in this paper are shown in Kitahara and Mizuno [4] as a quick report without proof. They present a simple variant of Klee-Minty's LP [5] and show that the number of BFSs generated by Dantzig's simplex method for it is just γ/δ . Hence γ/δ is a lower bound of $M(m, n, \delta, \gamma)$, that is,

$$M(m, n, \delta, \gamma) \geq \gamma/\delta.$$

In this paper, we explain and prove the properties of the variant of Klee-Minty's LP more precisely.

We also give a simple example of LP such that $\gamma = \delta$ and Dantzig's simplex method generates m BFSs, and we show that there exists no $s \in [0, 1)$ such that the inequality

$$M(m, n, \delta, \gamma) \leq m^s \frac{\gamma}{\delta}$$

holds for any $m \geq 1$, $n \geq m$, $\delta > 0$, and $\gamma \geq \delta$. Hence the upper bound (1.2) is fairly good.

2. A Simple Variant of Klee-Minty's LP

Klee-Minty's LP presented in Dantzig and Thapa [2] is

$$\begin{aligned} \max \quad & \sum_{i=1}^m 10^{m-i} x_i, \\ \text{subject to} \quad & x_1 \leq 1, \\ & 2 \sum_{i=1}^{k-1} 10^{k-i} x_i + x_k \leq 100^{k-1} \quad \text{for } k = 2, 3, \dots, m, \\ & \mathbf{x} \geq \mathbf{0}, \end{aligned}$$

where $\mathbf{x} = (x_1, x_2, \dots, x_m)^\top$ is a variable vector. Note that not only the right hand side constants but the coefficients of the objective function and the constraints are exponentially big in this LP. Kitahara and Mizuno [4] present a simple variant of Klee-Minty's LP, which is represented as

$$\begin{aligned} \max \quad & \sum_{i=1}^m x_i, \\ \text{subject to} \quad & x_1 \leq 1, \\ & 2 \sum_{i=1}^{k-1} x_i + x_k \leq 2^k - 1 \quad \text{for } k = 2, 3, \dots, m, \\ & \mathbf{x} \geq \mathbf{0}, \end{aligned} \tag{2.1}$$

where the coefficients of the objective function and the constraints are very simple (0, 1, or 2). The standard form using a vector $\mathbf{y} = (y_1, y_2, \dots, y_m)^\top$ of slack variables is written as

$$\begin{aligned} \max \quad & \sum_{i=1}^m x_i, \\ \text{subject to} \quad & x_1 + y_1 = 1, \\ & 2 \sum_{i=1}^{k-1} x_i + x_k + y_k = 2^k - 1 \quad \text{for } k = 2, 3, \dots, m, \\ & \mathbf{x} \geq \mathbf{0}, \mathbf{y} \geq \mathbf{0}. \end{aligned} \tag{2.2}$$

The number of equality constraints of this problem is m and the number of variables is $n = 2m$.

Lemma 2.1. *The variant (2.2) has the following properties*

1. for each $k \in \{1, 2, \dots, m\}$ at any BFS (basic feasible solution), exactly one of x_k and y_k is a basic variable,
2. the problem has 2^m BFSs,
3. each component of any BFS is an integer,
4. the problem is nondegenerate,
5. the optimal BFS is

$$\begin{aligned} \mathbf{x}^* &= (0, 0, \dots, 0, 2^m - 1)^\top, \\ \mathbf{y}^* &= (1, 2^2 - 1, \dots, 2^{m-1} - 1, 0)^\top, \end{aligned}$$

and the optimal value is $(2^m - 1)$,

6. the minimum and the maximum values of all the positive elements of BFSs are 1 and $(2^m - 1)$, that is,

$$\delta = 1, \quad \gamma = 2^m - 1.$$

Proof. We easily get from the equality constraints of (2.2) that

$$x_k \leq 2^k - 1 \text{ and } y_k \leq 2^k - 1 \tag{2.3}$$

for any feasible solution (\mathbf{x}, \mathbf{y}) and any $k \in \{1, 2, \dots, m\}$. Next we will show that

$$x_k + y_k \geq 1. \tag{2.4}$$

From the first equality constraint in (2.2), we have (2.4) for $k = 1$. When $k \geq 2$, we have from the $(k - 1)$ -th and k -th constraints that

$$\begin{aligned} 2 \sum_{i=1}^{k-2} x_i + x_{k-1} + y_{k-1} &= 2^{k-1} - 1, \\ 2 \sum_{i=1}^{k-1} x_i + x_k + y_k &= 2^k - 1. \end{aligned}$$

Subtracting the first equality from the second equality, we get

$$x_k + y_k + x_{k-1} - y_{k-1} = 2^{k-1}. \tag{2.5}$$

Hence we obtain (2.4) from (2.3), (2.5), and $y_{k-1} \geq 0$.

Let (\mathbf{x}, \mathbf{y}) be any BFS of (2.2). From (2.4), either x_k or y_k for any $k \in \{1, 2, \dots, m\}$ must be a basic variable. Since the number of basic variables of any BFS is m , exactly one of x_k and y_k is a basic variable.

If we choose one of x_k and y_k as a basic variable for each $k \in \{1, 2, \dots, m\}$, a basic solution $(\mathbf{x}', \mathbf{y}')$ is determined. The value of x'_1 or y'_1 is 1 from the equality constraint $x'_1 + y'_1 = 1$, and the values of x'_k or y'_k for $k = 2, 3, \dots, m$ are easily computed from (2.5) in this order. Then we can show that the value of each basic variable x'_k or y'_k is a positive integer between 1 and $2^k - 1$ from (2.5) inductively. So the basic solution $(\mathbf{x}', \mathbf{y}')$ is feasible and nondegenerate. The number of such basic feasible solutions is obviously 2^m . We have proved the properties from 1 to 4.

Then we will prove the last two properties. We get from the last equality constraint of (2.2) that

$$\sum_{i=1}^m x_i \leq 2 \sum_{i=1}^{m-1} x_i + x_m + y_m = 2^m - 1.$$

Thus the object function value is at most $(2^m - 1)$. When y_i ($i \in \{1, 2, \dots, m-1\}$) and x_m are basic variables, the BFS is computed as in property 5 of the lemma. Since the object function value is $(2^m - 1)$, it is an optimal solution. The last property 6 in the lemma is obvious from the discussion above. \square

3. Dantzig's Simplex Method for the Variant

Suppose that the initial BFS of the problem (2.2) is $\mathbf{x} = \mathbf{0}$, $\mathbf{y} = (1, 2^2 - 1, \dots, 2^m - 1)^\top$, and we generate a sequence of BFSs by Dantzig's simplex method [1], in which

- a nonbasic variable, whose reduced cost is maximum (in the case of a maximization problem), is chosen as an entering variable,
- if two or more nonbasic variables have the maximal reduced cost, then a variable (either x_i or y_i) of the minimum index in them is chosen as an entering variable.

At first, we explain the case when $m = 1$. Since $x_1 = 0$, $y_1 = 1$ at the initial BFS, y_1 is the basic variable and the dictionary of (2.2) for the initial BFS is

$$\begin{aligned} z &= x_1, \\ y_1 &= 1 - x_1, \end{aligned} \tag{3.1}$$

where z represents the objective function. Then Dantzig's simplex method updates the dictionary as follows: x_1 is chosen as the entering variable from (3.1), and the second dictionary is calculated as

$$\begin{aligned} z &= 1 - y_1, \\ x_1 &= 1 - y_1. \end{aligned}$$

This dictionary is optimal. We get an optimal solution $x_1 = 1$, $y_1 = 0$ and an optimal value 1.

Then we explain the case when $m = 2$. Since $\mathbf{x} = \mathbf{0}$, $\mathbf{y} = (1, 3)^\top$ at the initial BFS, y_1 and y_2 are the basic variables and the dictionary for the initial BFS is

$$\begin{aligned} z &= x_1 + x_2, \\ y_1 &= 1 - x_1, \\ y_2 &= 3 - 2x_1 - x_2, \end{aligned} \tag{3.2}$$

where z represents the objective function. Then Dantzig's simplex method updates the dictionaries as follows: x_1 is chosen as the entering variable from (3.2), and the second dictionary is calculated as

$$\begin{aligned} z &= 1 - y_1 + x_2, \\ x_1 &= 1 - y_1, \\ y_2 &= 1 + 2y_1 - x_2. \end{aligned}$$

The third dictionary is

$$\begin{aligned} z &= 2 + y_1 - y_2, \\ x_1 &= 1 - y_1, \\ x_2 &= 1 + 2y_1 - y_2. \end{aligned}$$

The 4-th dictionary is

$$\begin{aligned} z &= 3 - x_1 - y_2, \\ y_1 &= 1 - x_1, \\ x_2 &= 3 - 2x_1 - y_2. \end{aligned} \tag{3.3}$$

This dictionary is optimal. We get an optimal solution $(x_1, x_2) = (0, 3)$, $(y_1, y_2) = (1, 0)$ and an optimal value $(2^2 - 1) = 3$. The \mathbf{x} -parts of BFSs generated are

$$\mathbf{v}^0 = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \mathbf{v}^1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \mathbf{v}^2 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \mathbf{v}^3 = \begin{pmatrix} 0 \\ 3 \end{pmatrix}.$$

We observe that

- the number of iterations is $(2^2 - 1)$,
- any reduced cost of every dictionary is 1 or -1 ,
- the objective function value increases by 1 at each iteration.

Now we consider the following variant of Klee-Minty's LP

$$\begin{aligned} \max \quad & (2^m - 1) - \sum_{i=1}^m x_i, \\ \text{subject to} \quad & x_1 \leq 1, \\ & 2 \sum_{i=1}^{k-1} x_i + x_k \leq 2^k - 1 \quad \text{for } k = 2, 3, \dots, m, \\ & \mathbf{x} \geq \mathbf{0}, \end{aligned}$$

where the constraints are the same as the problem (2.1) but the objective function has the opposite sign of (2.1). The standard form using a vector $\mathbf{y} = (y_1, y_2, \dots, y_m)^\top$ of slack variables is written as

$$\begin{aligned} \max \quad & (2^m - 1) - \sum_{i=1}^m x_i, \\ \text{subject to} \quad & x_1 + y_1 = 1, \\ & 2 \sum_{i=1}^{k-1} x_i + x_k + y_k = 2^k - 1 \quad \text{for } k = 2, 3, \dots, m, \\ & \mathbf{x} \geq \mathbf{0}, \mathbf{y} \geq \mathbf{0}. \end{aligned} \tag{3.4}$$

It is easy to see that this problem (3.4) also has the properties in Lemma 2.1 except for the optimal solution. The optimal solution of (3.4) is $\mathbf{x} = \mathbf{0}$, $\mathbf{y} = (1, 2^2 - 1, \dots, 2^m - 1)$, and the optimal value is $(2^m - 1)$.

Suppose that the initial BFS is $\mathbf{x} = (0, 0, \dots, 0, 2^m - 1)^\top$, $\mathbf{y} = (1, 2^2 - 1, \dots, 2^{m-1} - 1, 0)^\top$, and we generate a sequence of BFSs by Dantzig's simplex method for the problem (3.4). When $m = 1$, Dantzig's simplex method finds the optimal solution in one iteration just as the case of the problem (2.2). Next we explain the case when $m = 2$. Since $\mathbf{x} = (0, 3)^\top$, $\mathbf{y} = (1, 0)$ at the initial BFS, y_1 and x_2 are the basic variables and the dictionary for the initial BFS is

$$\begin{aligned} w &= 0 & +x_1 & +y_2, \\ y_1 &= 1 & -x_1, \\ x_2 &= 3 & -2x_1 & -y_2. \end{aligned} \tag{3.5}$$

where w represents the objective function. Then Dantzig's simplex method updates the dictionaries as follows: x_1 is chosen as the entering variable from (3.5), and the second dictionary is calculated as

$$\begin{aligned} w &= 1 & -y_1 & +y_2, \\ x_1 &= 1 & -y_1, \\ x_2 &= 1 & +2y_1 & -y_2. \end{aligned}$$

The third dictionary is

$$\begin{aligned} w &= 2 & +y_1 & -x_2, \\ x_1 &= 1 & -y_1, \\ y_2 &= 1 & +2y_1 & -x_2. \end{aligned}$$

The 4-th dictionary is

$$\begin{aligned} w &= 3 & -x_1 & -x_2, \\ y_1 &= 1 & -x_1, \\ y_2 &= 3 & -2x_1 & -x_2. \end{aligned} \tag{3.6}$$

This dictionary is optimal. We get an optimal solution $(x_1, x_2) = (0, 0)$, $(y_1, y_2) = (1, 3)$ and an optimal value 3. The \mathbf{x} -parts of BFSs generated are \mathbf{v}^3 , \mathbf{v}^2 , \mathbf{v}^1 , and \mathbf{v}^0 , which are opposite in order for the case of the problem (2.2). We observe that

- the number of iterations is $(2^2 - 1)$,
- any reduced cost of every dictionary is 1 or -1 ,
- the objective function value increases by 1 at each iteration.

We can get the same properties for any m as shown in the next theorem.

Theorem 3.1. *If we generate a sequence of BFSs by Dantzig's simplex method for the problem (2.2) from an initial BFS $\mathbf{x} = \mathbf{0}$, $\mathbf{y} = (1, 2^2 - 1, \dots, 2^m - 1)^\top$, then*

- the number of iterations is $(2^m - 1)$,
- any reduced cost of every dictionary is 1 or -1 ,
- the objective function value increases by 1 at each iteration.

If we generate a sequence of BFSs by Dantzig's simplex method for the problem (3.4) from an initial BFS $\mathbf{x} = (0, 0, \dots, 0, 2^m - 1)^\top$, $\mathbf{y} = (1, 2^2 - 1, \dots, 2^{m-1} - 1, 0)^\top$, then

- the number of iterations is $(2^m - 1)$,
- any reduced cost of every dictionary is 1 or -1 ,
- the objective function value increases by 1 at each iteration.

We will prove this theorem by using induction in the next section. Here we state two corollaries, which are obtained from the second and third results for each problem in the theorem.

Corollary 3.1. *The sequence of BFSs generated by the simplex method for the problem (2.2) with the minimum index rule is the same as the case of Dantzig's rule.*

Corollary 3.2. *Let $\{(\mathbf{v}^i, \mathbf{u}^i) \in \mathfrak{R}^{2m} : i = 0, 1, 2, \dots, 2^m - 1\}$ be the sequence of BFSs generated by Dantzig's simplex method for the problem (2.2) from an initial BFS $\mathbf{x} = \mathbf{0}$, $\mathbf{y} = (1, 2^2 - 1, \dots, 2^m - 1)^\top$. Then the sequence of BFSs generated by Dantzig's simplex method for the problem (3.4) from an initial BFS $\mathbf{x} = (0, 0, \dots, 0, 2^m - 1)^\top$, $\mathbf{y} = (1, 2^2 - 1, \dots, 2^{m-1} - 1, 0)^\top$ is just the opposite order, that is, $\{(\mathbf{v}^{2^m-1-i}, \mathbf{u}^{2^m-1-i}) \in \mathfrak{R}^{2m} : i = 0, 1, 2, \dots, 2^m - 1\}$.*

4. Proof of Theorem 3.1

In this section, we prove Theorem 3.1 by using induction.

Proof. We have gotten all the results of Theorem 3.1 when $m = 1, 2$. Assume that the results are true when $m = k$. When $m = k + 1$, the first dictionary of the problem (2.2) at

the initial BFS $\mathbf{x} = \mathbf{0}$, $\mathbf{y} = (1, 2^2 - 1, \dots, 2^{k+1} - 1)^\top$ is

$$\begin{aligned} z &= && x_1 & \cdots & +x_k & +x_{k+1}, \\ y_1 &= && 1 & -x_1, \\ & \vdots \\ y_k &= &(2^k - 1) & -2x_1 & \cdots & -x_k, \\ y_{k+1} &= &(2^{k+1} - 1) & -2x_1 & \cdots & -2x_k & -x_{k+1}. \end{aligned} \tag{4.1}$$

Note that the nonbasic variable x_{k+1} does not appear except for the objective function and the last constraint in (4.1), and that the reduced cost of x_{k+1} is 1. So the variable x_{k+1} is not chosen as an entering variable until the reduced costs of the other nonbasic variables become less than 1. While the variable x_{k+1} is nonbasic, the dictionaries except for the last constraint are updated just as in the case when $m = k$. All the reduced costs become less than 1 only when an optimal BFS for $m = k$ is obtained, because of the assumption. Thus the first $(2^k - 1)$ iterations are just like the case when $m = k$. After $(2^k - 1)$ iterations, the basic variables are $y_1, \dots, y_{k-1}, x_k, y_{k+1}$, and we get the 2^k -th dictionary

$$\begin{aligned} z &= &(2^k - 1) & -x_1 & \cdots & -y_k & +x_{k+1}, \\ y_1 &= &1 & -x_1, \\ & \vdots \\ x_k &= &(2^k - 1) & -2x_1 & \cdots & -y_k, \\ y_{k+1} &= &1 & +2x_1 & \cdots & +2y_k & -x_{k+1}. \end{aligned} \tag{4.2}$$

This dictionary except for the last row and the last column is just the same as the dictionary of the optimal BFS for $m = k$. At this dictionary, x_{k+1} is chosen as the entering variable, and the $(2^k + 1)$ -th dictionary is

$$\begin{aligned} z &= &2^k & +x_1 & \cdots & +y_k & -y_{k+1}, \\ y_1 &= &1 & -x_1, \\ & \vdots \\ x_k &= &(2^k - 1) & -2x_1 & \cdots & -y_k, \\ x_{k+1} &= &1 & +2x_1 & \cdots & +2y_k & -y_{k+1}. \end{aligned} \tag{4.3}$$

We see that the objective function value increases by 1, and each reduced cost is 1 or -1 in this case too. The nonbasic variable y_{k+1} does not appear except for the objective function and the last constraint in (4.3), and the reduced cost of y_{k+1} is -1 . So the variable y_{k+1} does not chosen as an entering variable until the algorithm stops. Hence iterations are just like the case when $m = k$. The dictionary (4.3) except for the last column, the last row, and the constant in z is the same as the initial dictionary of the problem (3.4) when $m = k$. Thus the next $(2^k - 1)$ iterations are just like the case of the problem (3.4) when $m = k$. After $(2^k - 1)$ iterations, we get the 2^{k+1} -th dictionary

$$\begin{aligned} z &= &(2^{k+1} - 1) & -x_1 & \cdots & -x_k & -y_{k+1}, \\ y_1 &= &1 & -x_1, \\ & \vdots \\ y_k &= &(2^k - 1) & -2x_1 & \cdots & -x_k, \\ x_{k+1} &= &(2^{k+1} - 1) & -2x_1 & \cdots & -2x_k & -y_{k+1}. \end{aligned}$$

This dictionary is optimal and the optimal value is $(2^{k+1} - 1)$. It is easy to check that all the results for the problem (2.2) in the theorem are true when $m = k + 1$.

In the similar way, we can also prove that all the results for the problem (3.4) in the theorem are true when $m = k + 1$. □

5. Lower Bounds

Recall that $M(m, n, \delta, \gamma)$ denotes the maximum number of different BFSs generated by Dantzig's simplex method for any standard form of LP (1.1) and any initial BFS, where the number of equality constraints is at most m , the number of variables is at most n , the minimum value of all the positive elements of BFSs is at least δ , and the maximum value of all the positive elements of BFSs is at most γ . From Theorem 3.1, we easily get the next result.

Theorem 5.1. *When $n = 2m$ and $\gamma = (2^m - 1)\delta$, we have that*

$$M(m, n, \delta, \gamma) \geq \frac{\gamma}{\delta}.$$

Note that this theorem is valid only when $n = 2m$ and $\gamma = (2^m - 1)\delta$. In a general case, we get a lower bound in the next corollary.

Corollary 5.1. *For any m, n, δ , and γ such that $n \geq m \geq 1$, $n \geq 2$, and $\gamma \geq 2\delta > 0$, we have that*

$$M(m, n, \delta, \gamma) \geq \min \left\{ 2^m - 1, 2^{\lceil \frac{n}{2} - 1 \rceil} - 1, \frac{1}{2} \left(\frac{\gamma}{\delta} - 1 \right) \right\}, \quad (5.1)$$

where $\lceil a \rceil$ denotes the smallest integer bigger than $a \in \mathfrak{R}$.

Proof. We prove the corollary by considering four cases.

Case 1: When $n \geq 2m$ and $\gamma/\delta \geq 2^m - 1$, set $n' = 2m \leq n$ and $\gamma' = (2^m - 1)\delta \leq \gamma$. From the definition of $M(m, n, \delta, \gamma)$ and Theorem 5.1, we have that

$$M(m, n, \delta, \gamma) \geq M(m, n', \delta, \gamma') \geq \frac{\gamma'}{\delta} = 2^m - 1.$$

Case 2: When $n \geq 2m$ and $\gamma/\delta < 2^m - 1$, there exists $m' \leq m - 1$ such that

$$2^{m'} - 1 \leq \frac{\gamma}{\delta} < 2^{m'+1} - 1.$$

From the result of Case 1, we have that

$$M(m, n, \delta, \gamma) \geq M(m', n, \delta, \gamma) \geq 2^{m'} - 1 \geq \frac{1}{2} \left(\frac{\gamma}{\delta} - 1 \right).$$

Case 3: When $2m > n$ and $\gamma/\delta \geq 2^{\lceil \frac{n}{2} - 1 \rceil} - 1$, set $m' = \lceil \frac{n}{2} - 1 \rceil \leq m$, $n' = 2m' \leq n$, and $\gamma' = (2^{m'} - 1)\delta \leq \gamma$. From the definition of $M(m, n, \delta, \gamma)$ and Theorem 5.1, we have that

$$M(m, n, \delta, \gamma) \geq M(m', n', \delta, \gamma') \geq \frac{\gamma'}{\delta} = 2^{\lceil \frac{n}{2} - 1 \rceil} - 1.$$

Case 4: When $2m > n$ and $\gamma/\delta < 2^{\lceil \frac{n}{2} - 1 \rceil} - 1$, there exists $m' \leq \lceil \frac{n}{2} - 1 \rceil - 1$ such that

$$2^{m'} - 1 \leq \frac{\gamma}{\delta} < 2^{m'+1} - 1.$$

Since $2m' \leq 2(\lceil \frac{n}{2} - 1 \rceil - 1) \leq n$, we have from the result of Case 1 that

$$M(m, n, \delta, \gamma) \geq M(m', n, \delta, \gamma) \geq 2^{m'} - 1 \geq \frac{1}{2} \left(\frac{\gamma}{\delta} - 1 \right).$$

Combining the results of these four cases, we get (5.1). □

The lower bound in Corollary 5.1 is less interesting when γ/δ is small. We will get a different lower bound when $\gamma/\delta = 1$ by presenting a very simple example. The example is the following LP on an n -dimensional cube

$$\begin{aligned} \max \quad & \sum_{i=1}^m x_i, \\ \text{subject to} \quad & x_k \leq 1 \quad \text{for } k = 1, 2, \dots, m, \\ & \mathbf{x} \geq \mathbf{0}. \end{aligned} \tag{5.2}$$

The standard form using a vector $\mathbf{y} = (y_1, y_2, \dots, y_m)^\top$ of slack variables is written as

$$\begin{aligned} \max \quad & \sum_{i=1}^m x_i, \\ \text{subject to} \quad & x_k + y_k = 1 \quad \text{for } k = 1, 2, \dots, m, \\ & \mathbf{x} \geq \mathbf{0}, \mathbf{y} \geq \mathbf{0}. \end{aligned}$$

The number of equality constraints of this problem is m , the number of variables is $n = 2m$, and the optimal solution is

$$\mathbf{x}^* = (1, 1, \dots, 1)^\top, \mathbf{y}^* = (0, 0, \dots, 0)^\top.$$

Since all the components of BFSs are 0 or 1, we have that

$$\delta = 1, \gamma = 1.$$

Let

$$\mathbf{x}^0 = (0, 0, \dots, 0)^\top, \mathbf{y}^0 = (1, 1, \dots, 1)^\top$$

be the initial BFS. Then Dantzig's simplex method generates BFSs as follows

$$\begin{aligned} \mathbf{x}^1 &= (1, 0, \dots, 0)^\top, & \mathbf{y}^1 &= (0, 1, \dots, 1)^\top, \\ \mathbf{x}^2 &= (1, 1, \dots, 0)^\top, & \mathbf{y}^2 &= (0, 0, \dots, 1)^\top, \\ & & & \vdots \\ \mathbf{x}^m &= (1, 1, \dots, 1)^\top, & \mathbf{y}^m &= (0, 0, \dots, 0)^\top, \end{aligned}$$

so we get the optimal solution in m iterations and the number of BFSs generated is also m . Hence we obtain the next result.

Theorem 5.2. *When $n = 2m$ and $\gamma = \delta$, we have that*

$$M(m, n, \delta, \gamma) \geq m.$$

In a general case, we can easily get the next corollary, which has advantage over Corollary 5.1 when γ/δ is small.

Corollary 5.2. *For any m, n, δ , and γ such that $n \geq m \geq 1, n \geq 2$, and $\gamma \geq \delta > 0$, we have that*

$$M(m, n, \delta, \gamma) \geq \min \left\{ m, \left\lceil \frac{n}{2} - 1 \right\rceil \right\}.$$

From Theorem 5.2, we also obtain the next corollary.

Corollary 5.3. *There exists no $s \in [0, 1)$ such that the inequality*

$$M(m, n, \delta, \gamma) \leq m^s \frac{\gamma}{\delta} \tag{5.3}$$

holds for any $m \geq 1$, $n \geq m$, $\delta > 0$, and $\gamma \geq \delta$.

Proof. If there exists $s \in [0, 1)$ such that the inequality (5.3) holds, then we have

$$m^s \frac{\gamma}{\delta} \geq M(m, n, \delta, \gamma) \geq m$$

from Theorem 5.2. We have contradiction when $m \geq 2$ and $\gamma = \delta$. □

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