

RELAXED DIVISOR METHODS AND THEIR SEAT BIASES

Tetsuo Ichimori
Osaka Institute of Technology

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Abstract A class of methods of apportionment called the relaxed divisor methods is introduced here. It is theoretically shown that their biases between small and large states decrease and eventually approach zero as the house size increases. Because the methods of Hill and Webster are examples of the class, they both are practically unbiased for the house size large enough. However, computer simulations performed here strongly suggest that Webster is the first method to practically attain a state of unbiasedness when the house size increases from 200 to 43,500.

Keywords: Discrete optimization, apportionment, divisor methods, seat bias

1. Introduction

The apportionment problem is defined by Article I, Section 2 of the U.S. Constitution, which requires that seats of representatives shall be apportioned among the states, based proportionally on their population. Because seats are not dividable, it is impossible to completely satisfy the constitutional requirement and some deviation from strict proportionality is unavoidable. This fact leads to a long debate over 200 years.

In the history of apportionment, the discovery of the Alabama paradox* was a terrible blow to Congress and it became urgent to seek apportionment methods not subject to the paradox. Huntington [6] discovered that his pairwise comparison approach yields the so-called “five historical methods” which avoid the paradox. They are the methods of Adams, Dean, Hill, Webster and Jefferson which already appeared in the history of apportionment. Because long years of experience showed that these methods favor small states in this order, he claimed that the Hill method stands in the middle of these five methods with respect to bias, namely, it does not favor small states or large states. His claim was reinforced by a National Academy of Sciences committee in 1929, and later again in 1948. On the other hand, around the same time as Huntington, Willcox [14] paid careful attention to empirical data and claimed that the Webster method best secures the end of holding balance between small and large states.

After several decades Balinski and Young [3] showed that there are an infinity number of methods of apportionment, called the divisor methods which avoid the population paradox† and they claimed that the Webster method exists in the middle of all divisor methods with respect to bias, while the Hill method is consistently biased toward small states. Note that any method not subject to the population paradox avoids the Alabama paradox.

*A peculiar phenomenon that an increase in the number of seats causes a state to lose a seat. After the 1880 census it was reported to Congress that 8 seats would have assigned to Alabama with a house size of 299, while 7 seats with a house size of 300.

†Another peculiar phenomenon that a state with greater growth in population can lose a seat to a state with less growth.

Regarding the case of United States Department of Commerce v. Montana, 503 U.S. 442 (1992), Ernst [5] claimed that the Webster method is biased in favor of large states and the Hill method is unbiased in his sense.

Recently Drton and Schwingenschlögl [4] have showed rather rigorously that the Webster method is asymptotically unbiased as the house size tends to infinity. Schuster et al. [10] have observed that the Webster method produces practically unbiased apportionments under the 1960–2000 censuses. On the other hand, they have observed that the Hill method also produces practically unbiased apportionments under the 1960–2000 censuses.

In light of these circumstances it would be imperative to decide which method is unbiased, Hill’s or Webster’s method. For this purpose a subclass of the divisor methods called “the relaxed divisor methods” are developed in Section 2. They are all theoretically shown to be more unbiased as the house size becomes large in Section 3. Because the methods of Hill and Webster are members of this subclass, both are practically unbiased for the house size large enough.

In Section 4 it is investigated by executing computer simulations how biased they are as the house size increases. As a result, it is confirmed that the relaxed divisor methods are more unbiased as the house size becomes large, while the methods of Adams and Jefferson which are not relaxed divisor methods are consistently biased to small states and to large states, respectively. In addition, these simulations show that Webster is the first method to practically attain a state of unbiasedness.

2. Apportionment Methods

In this section, first, the definition of the divisor methods is given. Then the relaxedly proportional methods developed by the author [8] are briefly described. The methods of Hamilton, Hill and Webster are examples of the relaxedly proportional methods. Finally, the relaxed divisor methods are proposed.

2.1. Divisor methods

Let \mathbf{N} and \mathbf{N}_+ denote the sets of non-negative integers and positive integers, respectively. Define a real valued function $d(a)$ for $a \in \mathbf{N}$, known as a rounding criterion. The function $d(a)$ is strictly increasing in a . It satisfies $a \leq d(a) \leq a + 1$ for $a \in \mathbf{N}$ and moreover satisfies $d(b) = b$ and $d(c) = c + 1$ for no pair of integers $b \in \mathbf{N}_+$ and $c \in \mathbf{N}$.

Let z be a positive real number and let $[z]$ denote an integer defined by the following rule:

1. If $z < d(0)$, then $[z] = 0$.
2. If $d(a) < z < d(a + 1)$ for some $a \in \mathbf{N}$, then $[z] = a + 1$.
3. If $z = d(a)$ for some $a \in \mathbf{N}$, then $[z] = a$ or $a + 1$.

Next, let M denote a divisor method and $x > 0$ be a divisor. A divisor x means that each seat is given an approximate constituency of x persons. Let s denote the number of states and $h \geq s$ the house size, i.e., the total number of seats; the values $s = 50$ and $h = 435$ are of practical interest. Let $p_i > 0$ denote the population of state i . If the equality $\sum_{i=1}^s [p_i/x] = h$ is achieved for some divisor $x > 0$, then the number of seats to which state i is entitled is $a_i = [p_i/x]$; p_i/x is referred to as the “quotient” of state i and the s -vector $\mathbf{a} = (a_1, a_2, \dots, a_s)$ an “apportionment” of h . Given $\mathbf{p} = (p_1, p_2, \dots, p_s)$, $h \geq s$, and $d(a)$, $a \in \mathbf{N}$, a divisor method M is defined as the set of apportionments [2]:

$$\left\{ \mathbf{a} \mid a_i = \left[\frac{p_i}{x} \right] \text{ and } \sum_{i=1}^s \left[\frac{p_i}{x} \right] = h \text{ for some } x > 0 \right\}.$$

Although there can be an infinite numbers of divisor methods, the following five historical methods have received special treatment for a long time:

- the Adams method defined with $d(a) = a$,
- the Dean method defined with $d(a) = a(a + 1)/(a + 0.5)$,
- the Hill method defined with $d(a) = \sqrt{a(a + 1)}$,
- the Webster method defined with $d(a) = a + 0.5$,
- the Jefferson method defined with $d(a + 1) = a + 1$.

2.2. Relaxedly proportional methods

Let the set of states be denoted by $S = \{1, 2, \dots, s\}$ and define $v(S) = \sum_{i=1}^s v_i$ for any s -tuple (v_1, v_2, \dots, v_s) . Let \mathbf{R}_+ denote the set of positive real numbers and let “s.t.” be an abbreviation for “subject to” throughout this article.

Most of reasonable apportionment methods have their equivalent discrete optimization problems. For example, the Webster method has the following equivalent discrete optimization problem:

$$\mathbf{P}_W \quad \min \sum_{i=1}^s \frac{a_i^2}{p_i} \quad \text{s.t. } a(S) = h \text{ and } a_i \in \mathbf{N} \text{ for } i \in S.$$

Let W denote the set of all optimal solutions $\mathbf{a} = (a_1, \dots, a_s)$ to the above problem \mathbf{P}_W , then it is well known that W defines the Webster method, see [3]. In other words, any apportionment of h produced by the Webster method minimizes $\sum_i a_i^2/p_i$ subject to the above constraints, while any optimal solution $\mathbf{a} = (a_1, \dots, a_s)$ to the above problem \mathbf{P}_W is an apportionment of h under the Webster method.

Next consider its continuous relaxation:

$$\min \sum_{i=1}^s \frac{a_i^2}{p_i} \quad \text{s.t. } a(S) = h \text{ and } a_i \in \mathbf{R}_+ \text{ for } i \in S.$$

Then it is clear that there can exist some positive $\lambda > 0$ such that $d(a_i^2/p_i)/da_i = 2(a_i/p_i) = \lambda$ for any $i \in S$ at optimality, which means that a_i is completely proportional to p_i for any i at optimality. In other words, $a_i = (\lambda/2)p_i = (h/p)p_i = q_i$ at optimality, where p denotes the total population, i.e., $p = \sum p_i$ and $q_i = (h/p)p_i$ is called the “quota” of state i , i.e., the ideal share of state i . Then, the Webster method is said to be “relaxedly proportional.” In general, if an apportionment method has its equivalent discrete optimization problem and its continuous relaxation has an optimal solution identical to the vector of quotas, i.e., $\mathbf{a} = \mathbf{q}$, then the method is defined to be relaxedly proportional.

Similarly, the Hill method has the following equivalent discrete optimization:

$$\min \sum_{i=1}^s \frac{p_i^2}{a_i} \quad \text{s.t. } a(S) = h \text{ and } a_i \in \mathbf{N}_+ \text{ for } i \in S.$$

Since its continuous relaxation is:

$$\min \sum_{i=1}^s \frac{p_i^2}{a_i} \quad \text{s.t. } a(S) = h \text{ and } a_i \in \mathbf{R}_+ \text{ for } i \in S,$$

it is obvious that $\mathbf{a} = \mathbf{q}$ at optimality. Hence the Hill method is also relaxedly proportional.

On the other hand, the methods of Adams, Dean and Jefferson are not relaxedly proportional. For example, the Adams method has the following equivalent discrete optimization [7]:

$$\min \sum_{i=1}^s \frac{a_i^2 - a_i}{p_i} \quad \text{s.t. } a(S) = h \text{ and } a_i \in \mathbf{N} \text{ for } i \in S.$$

Its continuous relaxation has an optimal solution $a_i = \lambda p_i + 0.5$ where $\lambda = (h - s/2)/p$. This means that a_i is not proportional to p_i . Since the term 0.5 has a great influence on small states, the Adams method favors small states in this sense. Similarly, the Jefferson method minimizes $\sum_i (a_i^2 + a_i)/p_i$ and an optimal solution to its continuous relaxation is $a_i = \lambda p_i - 0.5$ where $\lambda = (h + s/2)/p$. Hence it favors large states in this sense.

Although there are an infinite numbers of the relaxedly proportional methods, the following three methods along with Hill's and Webster's methods are considered here. The first one was developed by Theil and Schrage [13]. The second one was proposed by Theil [12]. The last one was suggested by this author [8]. In fact, the first two have been rediscovered by Agnew [1] recently.

- The Theil-Schrage method defined with $d(0) = 0$ and $d(a) = 1/\log((a+1)/a)$ for $a \in \mathbf{N}_+$ has its equivalent optimization:

$$\min \sum_{i=1}^s (-p_i \log a_i) \quad \text{s.t. } a(S) = h \text{ and } a_i \in \mathbf{N}_+ \text{ for } i \in S.$$

- The Theil method defined with $d(0) = 1/e \approx 0.37$ and $d(a) = (1/e)(a+1)^{a+1}/a^a$ for $a \in \mathbf{N}_+$ has its equivalent optimization:

$$\min \sum_{i=1}^s a_i \log \frac{a_i}{p_i} \quad \text{s.t. } a(S) = h \text{ and } a_i \in \mathbf{N} \text{ for } i \in S.$$

- The "1/3" method defined with $d(a) = \sqrt{a^2 + a + 1/3}$ for $a \in \mathbf{N}$ has its equivalent optimization:

$$\min \sum_{i=1}^s \frac{a_i^3}{p_i^2} \quad \text{s.t. } a(S) = h \text{ and } a_i \in \mathbf{N} \text{ for } i \in S.$$

It is easy to show that these three are relaxedly proportional. Instead of the five historical methods, define the "new five," namely, the methods of Hill, Theil-Schrage (T&S for short), Theil, Webster and "1/3" in order to compare mainly Hill's and Webster's methods.

2.3. Relaxed divisor methods

The relationship between the new five methods and their respective equivalent discrete optimization problems can be easily generalized. For convenience, define $1/0 = +\infty$, $\log 0 = -\infty$ and $\mathbf{R} = \{x \mid -\infty < x < +\infty\}$, where it should be noted that infinities $\pm\infty$ are not included in \mathbf{R} in a customary way.

Consider the following discrete optimization for a finite real number $n \in \mathbf{R}$:

$$\mathbf{P}: \quad \min_a \sum_{i=1}^s f(a_i, p_i) \quad \text{s.t. } a(S) = h \text{ and } a_i \in \mathbf{N} \text{ for } i \in S,$$

where

$$f(a_i, p_i) = \begin{cases} a_i \log \frac{a_i}{p_i} & \text{if } n = 0 \\ -p_i \log a_i & \text{if } n = -1 \\ \frac{a_i^{n+1}}{n(n+1)p_i^n} & \text{otherwise.} \end{cases} \quad (2.1)$$

Although $f(a_i, p_i)$ seems to be a function of “two variables” a_i and p_i , consider $f(a_i, p_i)$ a function of one variable a_i because p_i is a constant. Notice that $n = 0$ corresponds to Theil, $n = -1$ to T&S, $n = 2$ to “1/3,” $n = -2$ to Hill and $n = 1$ to Webster.

Consider the difference $f(a_i + 1, p_i) - f(a_i, p_i)$, then it can be written as follows:

$$f(a_i + 1, p_i) - f(a_i, p_i) = \begin{cases} \log \frac{(a_i + 1)^{a_i+1}}{a_i^{a_i} p_i} & \text{if } n = 0 \\ p_i \log \frac{a_i}{a_i + 1} & \text{if } n = -1 \\ \frac{(a_i + 1)^{n+1} - a_i^{n+1}}{n(n+1)p_i^n} & \text{otherwise.} \end{cases} \quad (2.2)$$

Next, define the function $d(a), a \in \mathbf{N}_+$ (as a rounding criterion[‡]) as follows:

$$d(a) = \begin{cases} \lim_{n \rightarrow 0} \left(\frac{(a+1)^{n+1} - a^{n+1}}{n+1} \right)^{\frac{1}{n}} = \frac{1}{e} \frac{(a+1)^{a+1}}{a^a} & \text{if } n = 0 \\ \lim_{n \rightarrow -1} \left(\frac{(a+1)^{n+1} - a^{n+1}}{n+1} \right)^{\frac{1}{n}} = \frac{1}{\log \frac{a+1}{a}} & \text{if } n = -1 \\ \left(\frac{(a+1)^{n+1} - a^{n+1}}{n+1} \right)^{\frac{1}{n}} & \text{otherwise.} \end{cases} \quad (2.3)$$

And define $d(0)$ as follows:

$$d(0) = \begin{cases} 0 & \text{if } n \leq -1 \\ \frac{1}{e} \approx 0.37 & \text{if } n = 0 \\ \frac{1}{\sqrt[n]{n+1}} & \text{otherwise.} \end{cases} \quad (2.4)$$

For $d(a)$ defined in (2.3) it is shown in [11] that $a < d(a) < a + 1$ when n is finite. Because the values of $d(a)$ relate intimately with the (relative but not absolute) biases of apportionment methods (see [9] for details), it is important to note that the function $d(a)$ in (2.3) is strictly increasing in n for each fixed positive $a \in \mathbf{N}_+$. In addition, $d(0)$ is also strictly increasing in $n > -1$ while $d(0) = 0$ for $n \leq -1$.

Recall here the definition of a divisor method M . It is shown in [3] that M is equivalent to

$$\left\{ \mathbf{a} \mid \max_{a_i > 0} \frac{d(a_i - 1)}{p_i} \leq \min_{a_j \geq 0} \frac{d(a_j)}{p_j} \text{ and } a(S) = h \right\}. \quad (2.5)$$

[‡]Appearing in (2.3), the first $(1/e)(a+1)^{a+1}/a^a$ is the identric mean of two consecutive integers $a \geq 1$ and $a + 1$, and the second $1/\log\{(a+1)/a\}$ is the logarithmic mean of them.

Moreover, it can be easily shown that the following set:

$$\left\{ \mathbf{a} \mid \mathbf{a} \text{ is optimal for problem } \mathbf{P} \right\} \quad (2.6)$$

is identical to

$$\left\{ \mathbf{a} \mid \max_{a_i > 0} \{f(a_i, p_i) - f(a_i - 1, p_i)\} \leq \min_{a_j \geq 0} \{f(a_j + 1, p_j) - f(a_j, p_j)\} \text{ and } a(S) = h \right\}, \quad (2.7)$$

see Ibaraki and Katoh [7] for details.

Compare (2.2) and (2.3), then it follows from (2.5) and (2.7) that the set (2.6) defines a class of divisor methods with $d(a)$ in (2.3). Let these divisor methods be referred to as the “relaxed divisor methods.” In fact, the relaxed divisor methods are relaxedly proportional, which will be shown below.

Theorem 2.1. The relaxed divisor methods are relaxedly proportional.

Proof. The continuous relaxation of the problem \mathbf{P} is

$$\min_{\mathbf{a}} \sum_{i=1}^s f(a_i, p_i) \quad \text{s.t. } a(S) = h \text{ and } a_i \in \mathbf{R}_+ \text{ for } i \in S.$$

It is easy to find that an optimal solution is $\mathbf{a} = \mathbf{q}$ because $f''(a_i, p_i) > 0$ means that there exist a λ satisfying $f'(a_i, p_i) = \lambda$ for any $i \in S$. Hence the theorem. \square

3. Absolute Biases of Relaxed Divisor Methods

Here absolute biases[§] of the relaxed divisor methods are considered. For relative biases of methods of apportionment, see [3, 10].

Define the set $B = \{k/L \mid k \in \mathbf{N}\}$ for a positive integer L . Then consider the following “weak relaxation” of the problem \mathbf{P} :

$$\min_{\mathbf{a}} \sum_{i=1}^s f(a_i, p_i) \quad \text{s.t. } a(S) = h \text{ and } a_i \in B \text{ for } i \in S.$$

In fact, it may be assumed that L is a rational if Lh is a positive integer. Let $b_i = La_i$. Then the relation $a(S) = h$ is equivalent to $b(S) = Lh$.

It is easy to obtain the followings:

$$\sum_{i=1}^s f(b_i, p_i) = \begin{cases} L \sum_{i=1}^s a_i \log \frac{a_i}{p_i} + (L \log L)h & \text{if } n = 0 \\ -p(S) \log L - \sum_{i=1}^s p_i \log a_i & \text{if } n = -1 \\ L^{n+1} \sum_{i=1}^s \frac{a_i^{n+1}}{n(n+1)p_i^n} & \text{otherwise.} \end{cases} \quad (3.1)$$

Namely, the objective function $\sum_{i=1}^s f(b_i, p_i)$ is expressed as

$$\sum_{i=1}^s f(b_i, p_i) = C \sum_{i=1}^s f(a_i, p_i) + D$$

[§]Although virtually impossible, a method is absolutely unbiased if $\sum_{i \in T} |a_i - q_i| = 0$ for any $T \subset S$.

where C is a positive constant and D is a constant. Since C is positive, this means that minimizing $\sum_{i=1}^s f(b_i, p_i)$ is equivalent to minimizing $\sum_{i=1}^s f(a_i, p_i)$. Hence the above weak relaxation can be rewritten as:

$$\min_{\mathbf{b}} \sum_{i=1}^s f(b_i, p_i) \quad \text{s.t. } b(S) = Lh \text{ and } b_i \in \mathbf{N} \text{ for } i \in S.$$

Theorem 3.1. The relaxed divisor methods are more unbiased as the house size becomes large.

Proof. The relaxed divisor methods yield the apportionment completely proportional to the population of states if the constraint $a_i \in \mathbf{N}$ is relaxed to $a_i \in \mathbf{R}_+$ for every $i \in S$. As the integer L of the weak relaxation of \mathbf{P} becomes large, the weak relaxation becomes closer to the continuous relaxation problem. Moreover, increasing of L implies the increase of the house size. Hence the theorem. \square

4. Simulation

In order to verify the theoretical result in the preceding section, some computer simulations are performed for the new five methods. There are a number of empirical tests for bias proposed by Balinski and Young [3], by Ernst [5], by Schuster et al. [10] and others. Although most of them depend on the differences $|a_i - q_i|$, the differences tend to be smaller as the house size increases. In fact, if $h = p$ where h is the house size and p is the total population, then most of reasonable methods produce unbiased apportionments, i.e., $a_i = p_i$ for any $i \in S$. Therefore the differences $|a_i - q_i|$ become zero. Because the simulations here use the large house sizes, such tests seem not to fulfill our purpose.

Our bias test is as follows. Based on the population of states (p_1, p_2, \dots, p_s) , let an apportionment (a_1, a_2, \dots, a_s) be produced by any method of apportionment. Define the following sets:

$$F = \left\{ (i, j) \mid 1 \leq a_i < a_j \text{ and } \frac{a_i}{p_i} > \frac{a_j}{p_j} \quad i, j \in S \right\}$$

and

$$A = \left\{ (i, j) \mid 1 \leq a_i < a_j \quad i, j \in S \right\}.$$

Note that $(i, j) \in F$ means that the smaller state i is favored over the larger state j under any divisor method. Then the value $|F|/|A|$ represents the bias ratio[¶] that small states are favored [3].

First, according to the 2000 census fix an apportionment of the 435 seats among the 50 states produced by each of the new five methods. In fact, the Hill method produces the current apportionment. For each method M , let the fixed apportionment be denoted by $\mathbf{a}(M)$ and its divisor by $x(M)$, i.e., $a_i(M) = \lfloor p_i/x(M) \rfloor$ where p_i is the population of state i for the year 2000.

Let random P_i be uniformly distributed on the interval

$$d(a_i(M) - 1)x(M) \leq P_i \leq d(a_i(M))x(M),$$

then the apportionment produced by M based on the random populations P_1, \dots, P_s is the same as that based on the actual population according to the 2000 census.

[¶]If a method is almost unbiased, the bias ratio could be expected to be nearly 0.5.

To avoid the unrealistic assumption of very small states, assume in estimating the bias of M that no state's quotient is less than 0.5. In other words, the random population of each state is assumed to be uniformly distributed on the interval

$$\max\{0.5, d(a_i(M))\}x(M) \leq P_i \leq d(a_i(M))x(M).$$

One million tuples of populations (P_1, P_2, \dots, P_s) 's are generated for each of the new five methods and for each value of the house size h , where h takes values 200, 435, 1,000, 2,000, 4,350, 10,000, 20,000, and 43,500. Then the bias is estimated for each method and each h , see Table 1 and Figure 1. In addition, the same simulation is run for the 1990 census, see Table 2. It can be understood that the bias ratios of the new five methods converge to 50%, and that Webster is the first method to practically attain the 50% bias ratio. In fact, it keeps about 50% for any house size.

On the other hand, the methods of Adams and Jefferson show the consistent strong biases according to the same simulations as for the new five methods, see Table 3.

Table 1: Each entry denotes the average bias ratio (%) for one million tests applying one of the new five for each of various house sizes on the basis of the 2000 apportionment

house size	Hill	T&S	Theil	Webster	"1/3"
200	53.40	52.24	51.10	50.00	48.94
435	53.09	52.04	51.01	50.00	49.02
1,000	52.24	51.50	50.74	50.00	49.25
2,000	51.08	50.73	50.35	50.01	49.64
4,350	50.49	50.33	50.17	50.00	49.84
10,000	50.20	50.14	50.07	49.99	49.93
20,000	50.11	50.07	50.04	49.99	49.96
43,500	50.05	50.04	50.06	50.01	49.94

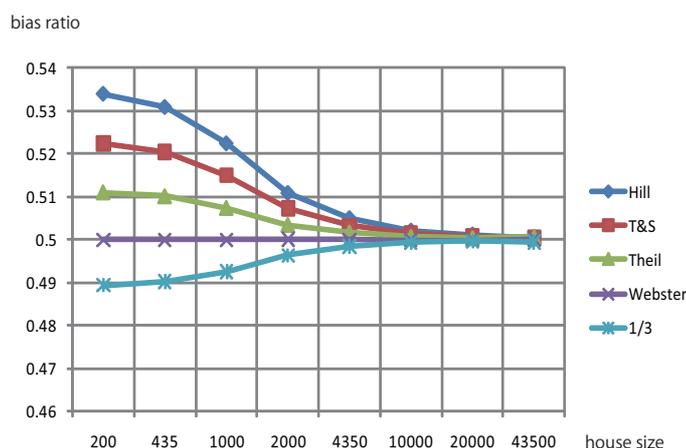


Figure 1: The house size h is placed on the horizontal axis and the bias ratio is on the vertical axis

5. Concluding Remarks

We showed in this paper that there can exist a subclass of the divisor methods whose biases decrease and eventually approach zero as the house size increases. They are called

Table 2: Each entry denotes the average bias ratio (%) for one million tests applying one of the new five for each of various house sizes on the basis of the 1990 apportionment

house size	Hill	T&S	Theil	Webster	“1/3”
200	53.36	52.21	51.10	50.01	48.99
435	53.11	52.08	51.01	49.99	48.99
1,000	52.15	51.46	50.73	49.99	49.26
2,000	51.05	50.71	50.35	50.00	49.65
4,350	50.47	50.32	50.16	50.00	49.85
10,000	50.21	50.15	50.06	50.00	49.93
20,000	50.11	50.05	50.03	50.00	49.96
43,500	50.05	50.05	50.06	50.00	49.94

Table 3: Each entry denotes the average bias ratio (%) for one million tests applying one of the Adams and Jefferson methods for each of various house sizes on the basis of the 2000 apportionment

house size	Adams	Jefferson
200	73.20	18.46
435	78.36	17.86
1,000	79.35	18.91
2,000	79.44	19.80
4,350	79.52	20.14
10,000	79.54	20.37
20,000	79.41	20.48
43,500	79.41	20.56

the relaxed divisor methods and the methods of Hill and Webster are examples of them. Hence both of the two methods are practically unbiased when the house size is large enough. In other words, Huntington and Willcox are both right if the house size is large enough. However, if the house size is the current number of representatives, i.e., 435, then the simulations performed here strongly suggest that Willcox is right.

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Tetsuo Ichimori
Osaka Institute of Technology
1-79-1 Kitayama, Hirakata City
Osaka 573-0196, Japan
E-mail: ichimori@is.oit.ac.jp