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NECESSITY AND SUFFICIENCY FOR THE EXISTENCE OF A PURE-STRATEGY NASH EQUILIBRIUM

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Abstract In this paper, we consider a non-cooperative n-person game in the strategic form. As is well known, the game has a mixed-strategy Nash equilibrium. However, it does not always have a pure-strategy Nash equilibrium. Wherein, Topkis (1979), Iimura (2003), and Sato and Kawasaki (2009) provided a sufficient condition for the game to have a pure-strategy Nash equilibrium. However, they did not consider necessary conditions.

This paper has two aims. The first is to extend the authors' sufficient condition, which is based on monotonicity of the best responses. The second is to show that the existence of a pure-strategy Nash equilibrium implies the monotonicity of the best responses or an isolation of the equilibrium.

Keywords: Game theory, pure-strategy, Nash equilibrium, non-cooperative n-person game, bimatrix game

1. Introduction

In this paper, we consider the non-cooperative n-person game $G = \{N, \{S_i\}_{i \in N}, \{p_i\}_{i \in N}\}$, where

- $N := \{1, \ldots, n\}$ is the set of players.
- For any $i \in N$, S_i denotes the finite set, with a total order $\leqq_i$, of player i's pure strategies. An element of this set is denoted by s_i.
- $p_i : S := \prod_{j=1}^{n} S_j \to \mathbb{R}$ denotes the payoff function of player i.

It is well known that we can prove the existence of a mixed-strategy Nash equilibrium, originally introduced by Nash [5, 6], applying Kakutani's fixed point theorem [3] to the best response correspondence.

On the other hand, there are a couple of unified results proving the existence of a pure-strategy Nash equilibrium. Iimura [2] provided a discrete fixed point theorem using integrally convex sets [4], and Brouwer's fixed point theorem [1]. As an application thereof, he defined a class of non-cooperative n-person games that certainly have a pure-strategy Nash equilibrium. Sato and Kawasaki [8] have provided a discrete fixed point theorem based on the monotonicity of the mapping, and have given a class of non-cooperative n-person games that also certainly have a pure-strategy Nash equilibrium. Their idea is similar to Topkis's work [10]. He introduced the so-called supermodular games. In the paper, he first got the monotonicity of the greatest and least element of each player's best response, by assuming the increasing differences for each player's payoff function. Next, relying on Tarski's fixed point theorem [9], he showed the existence of a pure-strategy Nash equilibrium in supermodular games. However, these results are concerned with only sufficiency for the existence of a pure-strategy Nash equilibrium. Hence, in this paper, we shall consider not only sufficiency but necessity for the existence of it.

This paper has two aims. The first is to extend the class of non-cooperative n-person games provided in [8], which certainly have a pure-strategy Nash equilibrium. We introduce "partial monotonicity" in Section 3. The second is to show that the partial monotonicity is necessary for the existence of a pure-strategy Nash equilibrium in a bimatrix game. This is discussed in Section 4. Here we emphasize that the extension in Section 3 not only is an extension of the result in [8] but has a crucial role in the discussion on necessity of the existence of a pure-strategy Nash equilibrium in Section 4. In order to achieve our goal, we use a directed graphic representation of set-valued mappings.

2. Preliminaries

Since S is the product of finite sets S_i's, it is also finite, say, $S = \{s^1, \dots, s^m\}$. For any non-empty set-valued mapping F from S to itself, we define a directed graph $D_F = (S, A_F)$ by $A_F = \{(s^i, s^j)\colon s^j \in F(s^i),\ s^i, s^j \in S\}$. For any selection f of F, that is, $f(s) \in F(s)$ for all $s \in S$, we similarly define a directed graph D_f. For any $s \in S$, we denote by $\mathrm{od}(s)$ and $\mathrm{id}(s)$ the outdegree and indegree of s, respectively. Then, $\mathrm{od}(s) \geq 1$ for D_F, and $\mathrm{od}(s) = 1$ for D_f.

Definition 2.1. (Cycle of length l) We say a set-valued mapping F has a directed cycle of length l if there exist l distinct points $\{s^{i_1}, s^{i_2}, \dots, s^{i_l}\}$ of S such that $s^{i_1} \in F(s^{i_l})$ and $s^{i_{k+1}} \in F(s^{i_k})$ for all $k \in \{1, \dots, l-1\}$.

Example 2.1. Take $S = \{s^1, \dots, s^9\}$ and define a non-empty set-valued mapping F as in Figure 1. For example, $F(s^6) = \{s^2, s^3, s^5, s^8\}$, $F(s^7) = \{s^7, s^8\}$ and etc. It is clear

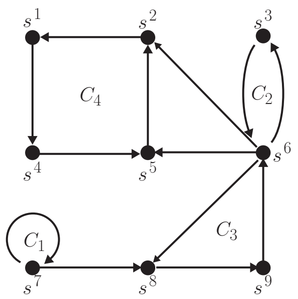

Figure 1: The graph D_F has directed cycles of length 1, 2, 3 and 4.

that $\{s^7\}$, $\{s^3, s^6\}$, $\{s^6, s^8, s^9\}$ and $\{s^1, s^4, s^5, s^2\}$ are directed cycles of length 1, 2, 3 and 4, respectively.

We now prove the following lemma required later:

Lemma 2.1. If D_f is connected in the sense of the undirected graph, then f has only one directed cycle.

Proof. We start with an arbitrary $s \in S$. Since S is finite, there exists $0 \leq k < l$ such that $f^k(s) = f^l(s)$, where f^k is the k-time composition of f. Hence, $\{f^k(s), f^{k+1}(s), \dots, f^{l-1}(s)\}$ is a directed cycle. Next, suppose that there are two distinct directed cycles C_1 and C_2; see Figure 2. Since D_f is connected, there exists a path $\pi = \{s^{i_1}, \dots, s^{i_j}\}$ joining C_1 and C_2, where $s^{i_1} \in C_1$ and $s^{i_j} \in C_2$. Further, since any directed cycle has no outward arc, we obtain $s^{i_1} = f(s^{i_2})$, $s^{i_2} = f(s^{i_3}), \dots, s^{i_{j-1}} = f(s^{i_j})$, which contradicts that C_2 has no outward arc. $\square$

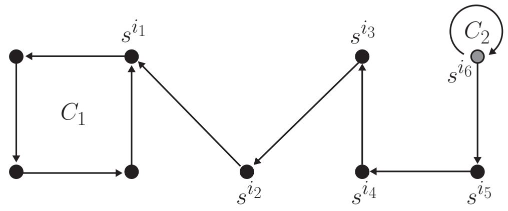

Figure 2: The graph D_f has two cycles, and $\mathrm{od}(s^{i_6}) = 2$.

3. A Sufficient Condition for the Existence of a Pure-Strategy Equilibrium

In this section, we present a class of non-cooperative n-person games that have a pure-strategy Nash equilibrium. We use the following notation:

For any $s \in S$, we set $s_{-i} := (s_1, \ldots, s_{i-1}, s_{i+1}, \ldots, s_n)$, and $S_{-i} := \prod_{j \neq i}^n S_j$. For any given $s_{-i} \in S_{-i}$, we denote by $F_i(s_{-i})$ the set of best responses of player i, that is,

$$F_i(s_{-i}) := \left\{ s_i \in S_i \colon p_i(s_i, s_{-i}) = \max_{t_i \in S_i} p_i(t_i, s_{-i}) \right\}. \tag{3.1}$$

We set $F(s) := \prod_{j=1}^n F_j(s_{-j})$ and $f(s) := (f_1(s_{-1}), \ldots, f_n(s_{-n}))$, where f_i is a selection of F_i.

An element s^* of S is called a pure-strategy Nash equilibrium if

$$p_i(s_i^*, s_{-i}^*) \geq p_i(s_i, s_{-i}^*) \quad \forall s_i \in S_i \quad (\forall i \in N).$$

Therefore, any pure-strategy Nash equilibrium is characterized by a fixed point of the best response correspondence F, that is, $s^* \in F(s^*)$. In other words, D_F has a cycle of length 1.

Our sufficient condition is based on monotonicity of a selection f. In order to define monotonicity, we need several kinds of orders.

Let T_i be a non-empty subset of S_i. For any bijection $\sigma_i : T_i \to T_i$, we define a total order $s_i \leqq_{\sigma_i} t_i$ on T_i by $\sigma_i(s_i) \leqq_i \sigma_i(t_i)$, where $\leqq_i$ is the total order on S_i. We denote by T_{σ_i} the ordered set $(T_i, \leqq_{\sigma_i})$. Further, $s_i <_{\sigma_i} t_i$ means $s_i \leqq_{\sigma_i} t_i$ and $s_i \neq t_i$.

We set $T := \prod_{i=1}^n T_i$ and $T_{-i} := \prod_{j \neq i}^n T_j$. For any $\sigma := (\sigma_1, \ldots, \sigma_n)$, T_σ denotes the partially ordered set $(T, \trianglelefteq_\sigma)$ such that $s \trianglelefteq_\sigma t$ if $s_i \leqq_{\sigma_i} t_i$ for all $i \in N$. The symbol $s \preceq_\sigma t$ means $s \trianglelefteq_\sigma t$ and $s \neq t$. T_{-i} is also equipped with the component-wise order $\trianglelefteq_{\sigma_{-i}}$, and the partially ordered set is denoted by $T_{\sigma_{-i}}$.

Definition 3.1. We say G is a partially monotone game if there exist a selection f of F, non-empty subsets $T_i \subset S_i$, and bijections σ_i from T_i into itself ($i \in N$) such that at least one of T_i's has two or more elements, $f(T) \subset T$, and

$$s_{-i} \preceq_{\sigma_{-i}} t_{-i} \;\Rightarrow\; f_i(s_{-i}) \leqq_{\sigma_i} f_i(t_{-i}) \tag{3.2}$$

for any $i \in N$.

Theorem 3.1. Any partially monotone non-cooperative n-person game has a pure-strategy Nash equilibrium.

Proof. Since T_σ is the product of totally ordered sets, it has a minimum element, say t^0. Then $t^0 \trianglelefteq_\sigma f(t^0) =: t^1$. If $t^0 = t^1$, t^0 is a fixed point. If $t^0 \neq t^1$, set

$$N_1 := \{i \in N \colon t_{-i}^0 = t_{-i}^1\}, \quad N_2 := \{i \in N \colon t_{-i}^0 \preceq_{\sigma_{-i}} t_{-i}^1\}.$$

Then $t^0 \trianglelefteq_\sigma t^1$, $0 \leq |N_1| \leq 1$, and N is a disjoint union of N_1 and N_2. Next, take

$$t_i^2 := \begin{cases} t_i^1, & i \in N_1 \\ f_i(t_{-i}^1), & i \in N_2. \end{cases}$$

Then, by partial monotonicity, we have $t_i^1 = f_i(t_{-i}^0) \leqq_{\sigma_i} f_i(t_{-i}^1) = t_i^2$ for any $i \in N_2$. Therefore, $t^1 \trianglelefteq_\sigma t^2$. Since T is finite, this procedure stops in finite steps, and we get a fixed point, which is a pure-strategy Nash equilibrium. □

Here we recall the term "monotonicity" introduced by Sato and Kawasaki [8].

Definition 3.2. ([8, Definition 3.1]) We say G is a monotone game if $\varepsilon_i = 1$ or -1 is allocated to each $i \in N$, and

$$s_{-i}^0 \preceq s_{-i}^1,\ t_i^1 \in F_i(s_{-i}^0) \ \Rightarrow\ \exists t_i^2 \in F_i(s_{-i}^1) \text{ such that } \varepsilon_i t_i^1 \leqq \varepsilon_i t_i^2$$

for any $i \in N$, where $s_{-i}^0 \preceq s_{-i}^1$ means that $\varepsilon_j s_j^0 \leqq \varepsilon_j s_j^1$ for all $j \neq i$.

Proposition 3.1. ([8, Theorem 3.1]) Any monotone non-cooperative n-person game G has a Nash equilibrium of pure strategies.

When G is a monotone game, by taking $T_i = S_i$, $\sigma_i = id$ and

$$f_i(s_{-i}) := \begin{cases} \text{maximum element of } F_i(s_{-i}), & \text{if } \varepsilon_i = 1 \\ \text{minimum element of } F_i(s_{-i}), & \text{if } \varepsilon_i = -1, \end{cases}$$

we see that G is a partially monotone game.

As a specific example, let us consider the following bimatrix game:

- $A = (a_{ij})$ is a payoff matrix of player 1 (P1), that is, $p_1(i,j) = a_{ij}$.
- $B = (b_{ij})$ is a payoff matrix of player 2 (P2), that is, $p_2(i,j) = b_{ij}$.
- $S_1 := \{1, \ldots, m_1\}$ is the set of pure strategies of P1, where $m_1 \in \mathbb{N}$.
- $S_2 := \{1, \ldots, m_2\}$ is the set of pure strategies of P2, where $m_2 \in \mathbb{N}$.
- For any $j \in S_2$, $F_1(j) := \{i^* \in S_1 : a_{i^*j} = \max_{i \in S_1} a_{ij}\}$ is the set of best responses of P1.
- For any $i \in S_1$, $F_2(i) := \{j^* \in S_2 : b_{ij^*} = \max_{j \in S_2} b_{ij}\}$ is the set of best responses of P2.
- $F(i,j) := F_1(j) \times F_2(i)$ denotes the set of best responses of $(i,j) \in S_1 \times S_2$.
- A pair (i^*, j^*) is a pure-strategy Nash equilibrium if $(i^*, j^*) \in F(i^*, j^*)$.

Example 3.1. Let $S_1 = S_2 = \{1, 2, 3\}$. The following is not a monotone bimatrix game.

$$A = \left(\begin{array}{c|c|c} \textcircled{4} & 2 & 3 \\ 2 & \textcircled{5} & \textcircled{4} \\ 3 & 1 & \textcircled{4} \end{array}\right), \quad B = \left(\begin{array}{ccc} 2 & 1 & \textcircled{3} \\ \hline 1 & \textcircled{4} & 2 \\ \hline \textcircled{3} & \textcircled{3} & 2 \end{array}\right).$$

Now we exchange the second and third columns. A and B are transformed into A' and B', respectively, as given below:

$$A' = \left(\begin{array}{c|c|c} \textcircled{4} & 3 & 2 \\ 2 & \textcircled{4} & \textcircled{5} \\ 3 & \textcircled{4} & 1 \end{array}\right), \quad B' = \left(\begin{array}{ccc} 2 & \textcircled{3} & 1 \\ \hline 1 & 2 & \textcircled{4} \\ \hline \textcircled{3} & 2 & \textcircled{3} \end{array}\right).$$

However, the bimatrix game defined by A' and B' is not a monotone game. Next, we remove the third row. Then A' and B' are transformed into A'' and B'', respectively:

$$A'' = \left(\begin{array}{c|c|c} \textcircled{4} & 3 & 2 \\ 2 & \textcircled{4} & \textcircled{5} \end{array}\right), \quad B'' = \left(\begin{array}{ccc} 2 & \textcircled{3} & 1 \\ \hline 1 & 2 & \textcircled{4} \end{array}\right).$$

The bimatrix game defined by A'' and B'' is now a monotone game for $(\varepsilon_1, \varepsilon_2) = (1, 1)$, and have a pure-strategy Nash equilibrium $(2, 3)$. In the original bimatrix game, the equilibrium is $(2, 2)$.

The procedure above is equivalent to taking $T_1 := \{1, 2\} \subset S_1$, $\sigma_1 := id$, $T_2 := S_2$ and σ_2 permutation $(2, 3)$ in Definition 3.1. Therefore, the original game is a partially monotone game.

In Figure 3 left, we plot the directed graph D_F corresponding to the best responses F of the original bimatrix game. Figure 3 right is the directed graph corresponding to the best responses of the bimatrix game after the above procedure. It is clear that the directed graph has only one cycle of length 1.

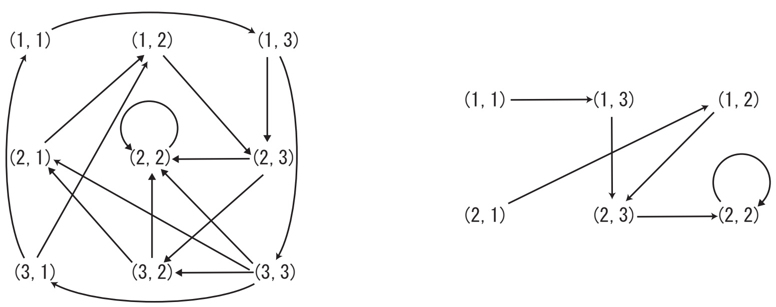

Figure 3: Left: The directed graph defined by A and B; Right: The directed graph defined by A'' and B''

4. A Necessary Condition for the Existence of a Pure-Strategy Equilibrium

In this section, we consider the bimatrix game, and show that partial monotonicity is a part of necessary for the existence of a pure-strategy Nash equilibrium.

Before we show the main theorem in this section, we recall several definitions and propositions requested later from Sato and Ito [7].

Definition 4.1. ([7, Definition 3.1]) A Nash equilibrium (i^*, j^*) is said to be *isolated* if $(i^*, j^*) \in F(i, j)$ implies $(i, j) = (i^*, j^*)$.

Proposition 4.1. ([7, Theorem 3.5]) A two-person game G is a partially monotone game if and only if there exist a selection f of F, $R_1 \subset S_1$ and $R_2 \subset S_2$ such that one of the following holds:

(i) $\#R_1 = 1$, $\#R_2 = 2$ and $f(R_1 \times R_2) \subset R_1 \times R_2$;

(ii) $\#R_1 = 2$, $\#R_2 = 1$ and $f(R_1 \times R_2) \subset R_1 \times R_2$;

(iii) there exists a permutation σ_2 on R_2 such that $\#R_1 = \#R_2 = 2$, $f(R_1 \times R_2) \subset R_1 \times R_2$,

$$j <_{\sigma_2} j' \;\Rightarrow\; f_1(j) \leqq_{id} f_1(j') \text{ for any } j, j' \in R_2, \text{ and}$$
$$i <_{id} i' \;\Rightarrow\; f_2(i) \leqq_{\sigma_2} f_2(i') \text{ for any } i, i' \in R_1.$$

We are ready to present the main theorem in this section.

Theorem 4.1. Assume that a two-person game G has a pure-strategy Nash equilibrium, say, s^*. Then either (i) or (ii) below holds:

(i) The game G is a partially monotone game.

(ii) s^* is an isolated Nash equilibrium.

Proof. Suppose that $s^* := (i^*, j^*)$ is not isolated, that is, there exists $(i, j) \neq (i^*, j^*)$ such that $(i^*, j^*) \in F(i, j)$. In other words, there exist $(i, j) \in S_1 \times S_2$ and a selection f of F such that $(i, j) \neq (i^*, j^*)$ and $(i^*, j^*) = f(i, j)$.

Case 1: When $i = i^*$ and $j \neq j^*$, we take $R_1 = \{i^*\}$ and $R_2 = \{j^*, j\}$. Then we get (i) of Proposition 4.1 because of $f_1(j) = i^*$.

Case 2: When $i \neq i^*$ and $j = j^*$, we easily obtain (ii) of Proposition 4.1 as well as Case 1.

Case 3: When $i \neq i^*$ and $j \neq j^*$. $f(i, j) = (i^*, j^*)$, that is, $f_1(j) = i^*$ and $f_2(i) = j^*$, so by taking $R_1 := \{i^*\}$ and $R_2 := \{j^*, j\}$, we have $f(R_1 \times R_2) \subset R_1 \times R_2$. Hence we get (i) of Proposition 4.1.

Therefore, we conclude that the game G is a partially monotone game from the proposition. □

If the number of players is three or more, then Theorem 4.1 fails.

Example 4.1. Let P1, P2 and P3 be players; let the player's strategies be $i \in \{1, 2\}$, $j \in \{1, 2\}$ and $k \in \{1, 2\}$, respectively; and let each player's best responses be the following:

P1	$k = 1$	$k = 2$
$j = 1$	$i = 2$	$i = 1$
$j = 2$	$i = 2$	$i = 2$

P2	$k = 1$	$k = 2$
$i = 1$	$j = 2$	$j = 2$
$i = 2$	$j = 1$	$j = 2$

P3	$j = 1$	$j = 2$
$i = 1$	$k = 2$	$k = 1$
$i = 2$	$k = 1$	$k = 2$

Then this game is not a partially monotone game. Indeed, there are only four combinations of two bijections on S_1 and S_2. The above table on P3 corresponds to $(\sigma_1, \sigma_2) = (id, id)$. Three tables below correspond to $((1, 2), id)$, $(id, (1, 2))$, and $((1, 2), (1, 2))$, respectively. In any case, the best response does not satisfy (3.2).

P3	$j = 1$	$j = 2$
$i = 2$	$k = 1$	$k = 2$
$i = 1$	$k = 2$	$k = 1$

P3	$j = 2$	$j = 1$
$i = 1$	$k = 1$	$k = 2$
$i = 2$	$k = 2$	$k = 1$

P3	$j = 2$	$j = 1$
$i = 2$	$k = 2$	$k = 1$
$i = 1$	$k = 1$	$k = 2$

On the other hand, since $(f_1(2, 2), f_2(2, 2), f_3(2, 2)) = (2, 2, 2)$, $(i, j, k) = (2, 2, 2)$ is a pure-strategy Nash equilibrium, which is not isolated, see Figure 4.

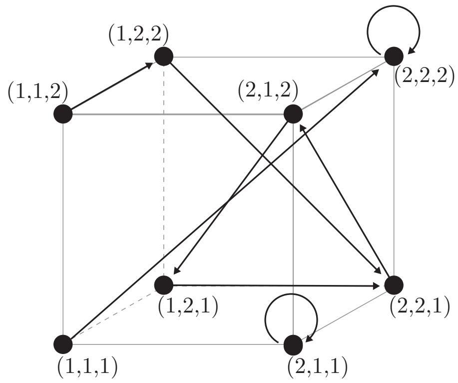

Figure 4: Point $(2, 2, 2)$ is a pure-strategy Nash equilibrium, which is not isolated.

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