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A TOPSIS-BASED ENTROPIC REGULARIZATION APPROACH FOR SOLVING FUZZY MULTI-OBJECTIVE NONLINEAR PROGRAMMING PROBLEMS

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Abstract In this work, a version of the technique for order preference by similarity ideal solution (TOPSIS) with entropic regularization approach is developed for solving the fuzzy multi-objective nonlinear programming (MONLP) problems. Applying the basic principle of compromise of TOPSIS, the fuzzy MONLP problem can be reduced into a fuzzy bi-objective nonlinear programming problem. Moreover, following the "tolerance approach," the solution of the fuzzy bi-objective nonlinear programming problem can be obtained by solving a min-max problem. An entropic regularization approach is then applied for solving such a problem. Computational results are provided to illustrate the validity and efficiency of the proposed method.

Keywords: Fuzzy sets, multiple objective decision making, TOPSIS, entropic regularization

1. Introduction

In the last two decades, one of the most rapidly developing methodologies is multi-criteria decision analysis of which multi-objective decision makings (MODM) plays a major role. It is well known that most decision making problems have multiple objectives which cannot be optimized simultaneously due to the inherent incommensurability and conflict between these objectives. Thus making a trade off between these objectives becomes a major subject to get the best compromise solution. A variety of methodologies for solving MODM problems have been proposed [11, 13, 15–17, 19, 20]. Among them, the goal programming and global criterion method are some of the most popular approaches. The methods only consider the criterion of the shortest distance from the given goal or the positive ideal solution. However, in practice, we might like to have a decision which not only makes as much profit as possible, but also avoids as much risk as possible. The single criterion of the shortest distance from the positive ideal solution is then not enough to satisfy decision maker(s) [9]. The technique for order preference by similarity ideal solution (TOPSIS) was first developed by Hwang and Yoon [10] to solve a multiple attribute decision making (MADM) problem. It provides the principle of compromise that the chosen alternative should have "the shortest distance from the positive ideal solution (PIS)" and "the farthest distance from the negative ideal solution (NIS)." Lately, this principle of compromise has also been suggested by Hwang et al. [9] for solving MODM problems. Recently, Abo-Sinna [1] extends the TOPSIS to solve multi-objective dynamics programming problems. Deng et al. [5] formulate the inter-company comparison process as a multicriteria analysis model, and presents an effective approach by modifying TOPSIS for solving such problem. Since vague concepts frequently represented in decision data for modeling real-life situations, multi-objective decision makings in a fuzzy environment becomes an important problem both in theory and practice. Chen [4] extends the concept of TOPSIS to develop a methodology for solving multi-person multi-criteria decision-making problems in fuzzy environment. Inspired and motivated by the recent research, in this study, a version of the TOPSIS with entropic regularization approaches is developed for solving the fuzzy MODM model in view of the following fuzzy multi-objective nonlinear programming problem (MONLP):

$$\begin{array}{ll} \text{Max/Min} & F(\mathbf{x}) = [f_1(\mathbf{x}), \cdots, f_k(\mathbf{x}), \cdots, f_K(\mathbf{x})] \\ \text{s.t.} & (\mathbf{x}, \mu_{\tilde{X}}(\mathbf{x})) \in \tilde{X} \triangleq \{(\mathbf{x}, \mu_{\tilde{X}}(\mathbf{x})) \mid g_s(\mathbf{x}) \stackrel{<}{\sim} 0, \ s = 1, \cdots, m\}, \end{array} \tag{1.1}$$

where $\mu_{\tilde{X}}(\cdot)$ is the membership function of the fuzzy set $\tilde{X}$, $g_s(\mathbf{x}) \stackrel{<}{\sim} 0, \ s = 1, 2, \cdots, m$, are fuzzy inequalities and "$\stackrel{<}{\sim}$" denotes the fuzzified version of "$\leq$" with the linguistic interpretation "approximately less than or equal to." Each fuzzy inequality $g_s(\mathbf{x}) \stackrel{<}{\sim} 0$ actually determines a fuzzy set $\tilde{C}_s$, whose membership function is denoted by μ_{g_s}. The membership grade $\mu_{g_s}(\mathbf{x})$ can be interpreted as the degree to which the regular inequality $g_s(\mathbf{x}) \leq 0, \ s = 1, 2, \cdots, m$, is satisfied. To specify the membership functions μ_{g_s}, it is commonly assumed that $\mu_{g_s}(\mathbf{x})$ should be 0 if the regular inequality $g_s(\mathbf{x}) \leq 0$ is strongly violated, and 1 if it is satisfied. This leads to a membership function in the following form:

$$\mu_{g_s}(\mathbf{x}) = \begin{cases} 1, & \text{if } \ g_s(\mathbf{x}) \leq 0, \\ \mu_s(g_s(\mathbf{x})), & \text{if } \ 0 < g_s(\mathbf{x}) \leq t_s, \\ 0, & \text{if } \ g_s(\mathbf{x}) > t_s, \end{cases} \tag{1.2}$$

for $s = 1, 2, \cdots, m$, where $t_s \geq 0$ is the tolerance level which a decision maker can tolerate in the accomplishment of the fuzzy inequality $g_s(\mathbf{x}) \stackrel{<}{\sim} 0$. We usually assume that $\mu_s(g_s(\mathbf{x})) \in [0, 1]$ and it is continuous and strictly decreasing over $[0, t_s]$.

Applying the basic principle of compromise of TOPSIS, the fuzzy MONLP problem (1.1) can be reduced into a fuzzy bi-objective nonlinear programming problem. Moreover, following the "tolerance approach" [12, 21], the solution of the fuzzy bi-objective problem can be obtained by solving a min-max problem. An entropic regularization approach is then applied for solving such a problem. The rest of the paper is organized as follows. In Section 2, the TOPSIS with entropic regularization approach for solving the fuzzy MONLP will be presented. A solution algorithm and the implementation issue on the algorithm will be discussed in section 3. Some numerical results are provided in section 4. The paper is concluded in section 5.

2. TOPSIS for Solving the Fuzzy Multi-Objective Nonlinear Programming Problem

To solve the fuzzy MONLP problem (1), the principle of compromise that the chosen solution should have "the shortest distance from the positive ideal solution (PIS)" and "the farthest distance from the negative ideal solution (NIS)" is adopted. Since conflict between the distance from PIS and the distance from NIS is usually existing, a compromise cannot be simultaneously obtained with the shortest distance from PIS and the farthest distance from NIS. And, unlike MADM, MODM always has an infinite number of alternatives. It becomes impossible to find the solution with the shortest distance from PIS and the longest distance from NIS. Thus both criteria of "the shortest distance from PIS" and "the farthest distance from NIS" are substituted by "as close to PIS as possible" and "as far away from NIS as possible" [9].

To mathematically formulate this principle of compromise, we first define the reference points of PIS and NIS for the fuzzy MONLP problem as follows:

$$\begin{aligned} f_k^* &= \max_{(\mathbf{x},\mu_{\tilde{X}}(\mathbf{x}))\in\tilde{X}} f_j(\mathbf{x}) \quad \forall\, j \in J, \\ &= \min_{(\mathbf{x},\mu_{\tilde{X}}(\mathbf{x}))\in\tilde{X}} f_i(\mathbf{x}) \quad \forall\, i \in I, \end{aligned} \qquad \forall\, k = 1, 2, \cdots, K, \tag{2.1}$$

$$\begin{aligned} f_k^- &= \min_{(\mathbf{x},\mu_{\tilde{X}}(\mathbf{x}))\in\tilde{X}} f_j(\mathbf{x}) \quad \forall\, j \in J, \\ &= \max_{(\mathbf{x},\mu_{\tilde{X}}(\mathbf{x}))\in\tilde{X}} f_i(\mathbf{x}) \quad \forall\, i \in I, \end{aligned} \qquad \forall\, k = 1, 2, \cdots, K, \tag{2.2}$$

where J denotes the index of objectives for maximization, $f_j(\mathbf{x})$ denotes the benefit objective for maximization, $\forall\, j \in J$, I denotes the index of objectives for minimization, $f_i(\mathbf{x})$ denotes the cost objective for minimization, $\forall\, i \in I$. Let $f^* \triangleq \{f_1^*, f_2^*, \cdots, f_K^*\}$ be the solution vector of equation (2.1) which consists of individual best feasible solutions for all objectives. f^* is called the positive ideal solution (PIS). Similarly, let $f^- \triangleq \{f_1^-, f_2^-, \cdots, f_K^-\}$ be the solution vector of equation (2.2) which consists of individual worst feasible solutions for all objectives. f^- is called the negative ideal solution (NIS).

To measure the distance from PIS and the distance from NIS for each objective, the Minkowski's L_p-metric is employed. The L_p-metric defines the distance between two points $f_k(\mathbf{x})$ and f_k^* (or f_k^-), $k = 1, 2, \cdots, K$. Moreover, because of the incommensurability among objectives, the component distance from PIS or NIS for each objective is normalized. The following distance functions are then considered.

$$d_p^{PIS}(\mathbf{x}) = \left\{ \sum_{j\in J} w_j^p \left[\frac{f_j^* - f_j(\mathbf{x})}{f_j^* - f_j^-} \right]^p + \sum_{i\in I} w_i^p \left[\frac{f_i(\mathbf{x}) - f_i^*}{f_i^- - f_i^*} \right]^p \right\}^{1/p} \tag{2.3}$$

and

$$d_p^{NIS}(\mathbf{x}) = \left\{ \sum_{j\in J} w_j^p \left[\frac{f_j(\mathbf{x}) - f_j^-}{f_j^* - f_j^-} \right]^p + \sum_{i\in I} w_i^p \left[\frac{f_i^- - f_i(\mathbf{x})}{f_i^- - f_i^*} \right]^p \right\}^{1/p}, \tag{2.4}$$

where d_p^{PIS} and d_p^{NIS} are the distances from the PIS and NIS, respectively, $w_k,\ k = 1, 2, \cdots, K$, are the relative importance (weights) of objectives, i.e., w_k indicates the degree of importance of the k-th objective, and $p = 1, 2, \cdots, \infty$ is the parameter of distance functions. The parameter p plays the role of the "balancing factor" between the distance d_p and the objects; as p increases, the distance d_p decreases, i.e., $d_1 \geq d_2 \geq \cdots \geq d_\infty$, and greater emphasis is given to the largest deviation in forming the total. Specifically, $p = 1$ implies an equal importance (weights) for all these deviations, while $p = 2$ implies that these deviations are weighted proportionately with the largest deviation having the largest weight. Finally, for $p = \infty$, the largest deviation completely dominates the distance determination [10, 17, 19, 20].

To approximately minimize d_p^{PIS}, and approximately maximize d_p^{NIS}, the fuzzy MONLP problem can be converted to the following fuzzy bi-objective nonlinear programming problem.

$$\begin{aligned} &\tilde{\min}\ d_p^{PIS}(\mathbf{x}) \\ &\tilde{\max}\ d_p^{NIS}(\mathbf{x}) \\ &\text{s.t.}\ \ g_s(\mathbf{x}) \mathrel{\substack{<\\ \approx}} 0,\ s = 1, \cdots, m, \end{aligned} \tag{2.5}$$

where $p = 1, 2, \cdots, \infty$. The membership functions $\mu_1(\mathbf{x})$ and $\mu_2(\mathbf{x})$ of the two distance objective function are assumed to be non-increasing/non-decreasing monotonous functions and can be described as follows.

$$\mu_1(\mathbf{x}) = \begin{cases} 1, & \text{if } d_p^{PIS}(\mathbf{x}) < (d_p^{PIS})^*, \\ \dfrac{(d_p^{PIS})' - d_p^{PIS}(\mathbf{x})}{(d_p^{PIS})' - (d_p^{PIS})^*}, & \text{if } (d_p^{PIS})^* \le d_p^{PIS}(\mathbf{x}) \le (d_p^{PIS})', \\ 0, & \text{if } d_p^{PIS}(\mathbf{x}) > (d_p^{PIS})', \end{cases} \quad (2.6)$$

$$\mu_2(\mathbf{x}) = \begin{cases} 1, & \text{if } d_p^{NIS}(\mathbf{x}) > (d_p^{NIS})^*, \\ \dfrac{d_p^{NIS}(\mathbf{x}) - (d_p^{NIS})'}{(d_p^{NIS})^* - (d_p^{NIS})'}, & \text{if } (d_p^{NIS})' \le d_p^{NIS}(\mathbf{x}) \le (d_p^{NIS})^*, \\ 0, & \text{if } d_p^{NIS}(\mathbf{x}) < (d_p^{NIS})', \end{cases} \quad (2.7)$$

where

$$(d_p^{PIS})^* = \min_{\mathbf{x} \in X} d_p^{PIS}(\mathbf{x}) \text{ and the solution is } \mathbf{x}^{PIS}, \quad (2.8)$$

$$(d_p^{NIS})^* = \max_{\mathbf{x} \in X} d_p^{NIS}(\mathbf{x}) \text{ and the solution is } \mathbf{x}^{NIS}, \quad (2.9)$$

$$(d_p^{PIS})' = d_p^{PIS}(\mathbf{x}^{NIS}), \quad (2.10)$$

$$(d_p^{NIS})' = d_p^{NIS}(\mathbf{x}^{PIS}). \quad (2.11)$$

Let a fuzzy decision $\tilde{D}$ of the fuzzy MONLP problem be defined as the fuzzy set resulting from the intersection of fuzzy objectives and fuzzy constrains $\tilde{C}_s,\ s = 1, 2, \cdots, m$, with a corresponding membership function

$$\mu_{\tilde{D}}(\mathbf{x}) = \min_{s=1,2,\cdots,m} \{\mu_1(\mathbf{x}), \mu_2(\mathbf{x}), \mu_{g_s}(\mathbf{x})\}. \quad (2.12)$$

According to reference [12, 21], a solution, say $\mathbf{x}^*$, of the fuzzy MONLP problem with a degree of satisfaction, $\mu_{\tilde{D}}(\mathbf{x}^*)$, can be taken as the solution with the highest membership in the fuzzy decision set $\tilde{D}$ and obtained by solving the following problem:

$$\max \min_{s=1,2,\cdots,m} \{\mu_1(\mathbf{x}), \mu_2(\mathbf{x}), \mu_{g_s}(\mathbf{x})\} \quad (2.13)$$

or

$$-\min \mu'_{\tilde{D}}(\mathbf{x}) \triangleq \max_{s=1,2,\cdots,m} \{-\mu_1(\mathbf{x}), -\mu_2(\mathbf{x}), -\mu_{g_s}(\mathbf{x})\}. \quad (2.14)$$

One major difficulty encountered in developing solution methods for solving the min-max problem (2.14) is the non-differentiability of the max function $\mu'_{\tilde{D}}(\mathbf{x})$. A distinct feature of the recent development centers around the idea of developing "smooth algorithms" [6, 8]. Among them, a class called "regularization methods" has been developed based on approximating the max function $\mu'_{\tilde{D}}(\mathbf{x})$ by certain smooth function [2, 3, 8]. Here we adopt the newly proposed "entropic regularization procedure" [7, 14]. This procedure guarantees that, for an arbitrarily small $\varepsilon > 0$, an ε-optimal solution of the min-max problem (2.14) can be obtained by solving the following problem:

$$-\min \quad \mu_q(\mathbf{x}) = \frac{1}{q} \ln\{\exp[q(-\mu_1(\mathbf{x}))] + \exp[q(-\mu_2(\mathbf{x}))] + \sum_{s=1}^{m} \exp[q(-\mu_{g_s}(\mathbf{x}))]\}, \quad (2.15)$$

with a sufficiently large q.

It should be noted that in practice a sufficiently accurate approximation can be obtained by using a moderately large q [14]. Also because of the special "log-exponential" form of $\mu_q(\mathbf{x})$, most over-flow problems in computation can be avoided.

3. An Algorithm and the Implementation Issue on the Algorithm

Based on the discussion in Section 2, a TOPSIS-based entropic regularization algorithm for solving the fuzzy MONLP is developed in this section.

Algorithm:

Step 0.1 Determine the distance parameter p. If the decision maker emphasizes the sum of individual distances (regrets for d_p^{PIS} and rewards for d_p^{NIS}), he should choose $p = 1$. On the other hand, if dominating criteria are the maximum of individual regrets and the minimum of individual rewards, $p = \infty$ should be chosen. Beyond both extreme cases, $p = 2$ will be chosen. $p = 2$ is similar to the popular least-square approach, and provides an approximation for the case $2 < p < \cdots < \infty$. Note that $p = 1, 2$, and ∞ are well-known standard values in the fields of control theory and multi-criteria decision making [19].

Step 0.2 Determine the relative importance w_k of the K objective functions. There are various methods for assessing w_k, such as eigenvector, weighted least square, entropy and LINMAP methods [10]. In this study, we assume that w_k is given from decision makers.

Step 1. Determine the positive ideal solution (f^*) and the negative ideal solution (f^-) by solving equations (2.1) and (2.2).

Step 2. Solve equation (2.5).

Step 2.1 Obtain $(d_p^{PIS})^*$, $(d_p^{NIS})^*$, $(d_p^{PIS})'$ and $(d_p^{NIS})'$ by solving equations (2.8)-(2.11). Go to Step 2.2.

Step 2.2 Obtain membership functions $\mu_1(\mathbf{x})$ and $\mu_2(\mathbf{x})$ by calculating equations (2.6) and (2.7). Go to Step 2.3.

Step 2.3 Solve equation (2.15). Go to Step 3.

Step 3 If the solution obtained by the TOPSIS-based entropic regularization approach is satisfied, i.e., the degree of satisfaction obtained in Step 2.3 reaches a preference value, say $\bar{\alpha}$, stop. Otherwise, decision makers may like to change $p,\ w_k,\ q,\ \bar{\alpha}$ and/or membership functions. Then, we will go back to Step 0.1, Step 0.2, or change the tolerance levels $t_s,\ s = 1, 2, \cdots, m$, in (1.2). The solution procedure is then repeated.

3.1. Implementation issue on the algorithm

In step 1 and step 2.1 of the proposed algorithm, we face the challenge of solving problems (2.1), (2.2), and (2.8)-(2.11). Consider the problem in (2.1).

$$\begin{aligned} \max\ & f_k(\mathbf{x}) \\ \text{s.t.}\ & (\mathbf{x}, \mu_{\tilde{X}}(\mathbf{x})) \in \tilde{X}, \end{aligned} \tag{3.1}$$

or equivalently,

$$\begin{aligned} \max\ & f_k(\mathbf{x}) \\ \text{s.t.}\ & g_s(\mathbf{x}) \stackrel{<}{\approx} 0,\ s = 1, 2, \cdots, m. \end{aligned} \tag{3.2}$$

For $\alpha \in [0, 1]$, consider the following problem:

$$\begin{aligned} \max\ & f_k(\mathbf{x}) \\ \text{s.t.}\ & \mathbf{x} \in X_\alpha, \end{aligned} \tag{3.3}$$

where $X_\alpha = \{\mathbf{x} \mid \mu_{g_s}(\mathbf{x}) \geq \alpha\}$ is the α-level set of $\tilde{X}$. Since $\mu_{g_s}(\mathbf{x}) \in [0, 1]$ and it is continuous and strictly decreasing over $[0, t_s]$, problem (3.3) is equivalent to the following problem:

$$\begin{aligned} \max\ & f_k(\mathbf{x}) \\ \text{s.t.}\ & g_s(\mathbf{x}) \leq \mu_{g_s}^{-1}(\alpha),\ s = 1, 2, \cdots, m,\ \alpha \in [0, 1]. \end{aligned} \tag{3.4}$$

This is a parametric programming and can be solved with continuous changes of α within $[0,1]$. Alternatively, we may present a list of the results to the decision maker by incremental changes of $\alpha = 0, 0.1, \cdots, 0.9, 1$.

A Similar argument holds for solving the problems (2.2), and (2.8)-(2.11). In this study, we assume that the value of $\alpha \in [0,1]$ is subjectively given from decision makers for obtaining the values of $f_k^*,\ f_k^-,\ (d_p^{PIS})',\ (d_p^{PIS})^*,\ (d_p^{NIS})',$ and $(d_p^{NIS})^*$.

4. A Numerical Example

In this section, a numerical example is provided to illustrate the validity and efficiency of the proposed method.

Consider the following fuzzy multi-objective nonlinear programming problem:

$$\begin{aligned} \max\ & f_1(\mathbf{x}) = 10x_1 - x_1^2 + 6x_2 - x_2^2 - 2x_3 - 2x_3^2 + 0.5x_2x_3 \\ \min\ & f_2(\mathbf{x}) = 3x_1 + 2x_2 - 6x_1x_3 \\ \text{s.t.}\ & g_1(\mathbf{x}) = 4x_1 + 2x_2 + x_3 - 10 \lesssim 0, \\ & g_2(\mathbf{x}) = 2x_1 + 4x_2 + x_3 - 20 \lesssim 0, x_1, x_2, x_3 \geq 1, \end{aligned} \tag{4.1}$$

with the membership functions $\mu_{g_s}(\mathbf{x}),\ s = 1, 2$, are specified as follows:

$$\mu_{g_1}(\mathbf{x}) = \begin{cases} 1, & \text{if } g_1(\mathbf{x}) \leq 0, \\ 1 - \left(\dfrac{g_1(\mathbf{x})}{7}\right)^2, & \text{if } 0 < g_1(\mathbf{x}) \leq 7, \\ 0, & \text{if } g_1(\mathbf{x}) > 7, \end{cases} \quad \mu_{g_2}(\mathbf{x}) = \begin{cases} 1, & \text{if } g_2(\mathbf{x}) \leq 0, \\ 1 - \left(\dfrac{g_2(\mathbf{x})}{10}\right)^2, & \text{if } 0 < g_2(\mathbf{x}) \leq 10, \\ 0, & \text{if } g_2(\mathbf{x}) > 10. \end{cases}$$

Step 1. Let $\tilde{X} \triangleq \{g_i(\mathbf{x}) \lesssim 0,\ i = 1, 2,$ and $x_1, x_2, x_3 \geq 1\}$. Determine the positive ideal solution (f^*) and the negative ideal solution (f^-) by solving the following problems:

$$f_1^* = \max_{\mathbf{x} \in \tilde{X}} f_1(\mathbf{x}) = 10x_1 - x_1^2 + 6x_2 - x_2^2 - 2x_3 - 2x_3^2 + 0.5x_2x_3, \tag{4.2}$$

$$f_2^* = \min_{\mathbf{x} \in \tilde{X}} f_2(\mathbf{x}) = 3x_1 + 2x_2 - 6x_1x_3, \tag{4.3}$$

$$f_1^- = \min_{\mathbf{x} \in \tilde{X}} f_1(\mathbf{x}) = 10x_1 - x_1^2 + 6x_2 - x_2^2 - 2x_3 - 2x_3^2 + 0.5x_2x_3, \tag{4.4}$$

$$f_2^- = \max_{\mathbf{x} \in \tilde{X}} f_2(\mathbf{x}) = 3x_1 + 2x_2 - 6x_1x_3. \tag{4.5}$$

To solve the problem (4.2), for $\alpha \in [0,1]$, we consider the following parametric programming problem:

$$\begin{aligned} \max\ & f_1(\mathbf{x}) = 10x_1 - x_1^2 + 6x_2 - x_2^2 - 2x_3 - 2x_3^2 + 0.5x_2x_3 \\ \text{s.t.}\ & 7\sqrt{1-\alpha} - 4x_1 - 2x_2 - x_3 + 10 \geq 0, \\ & 10\sqrt{1-\alpha} - 2x_1 - 4x_2 - x_3 + 20 \geq 0, x_1, x_2, x_3 \geq 1. \end{aligned} \tag{4.6}$$

By incremental changes of $\alpha = 0, 0.1, \cdots, 0.9, 1$, the solution of the problem (4.6) at α can be obtained as shown in Table 1. Taking $\alpha = 0.9$, the results for solving problems (4.3), (4.4), and (4.5) are shown in Table 2. Next, we compute d_p^{PIS} and d_p^{NIS} and obtain the following equations:

$$d_p^{PIS} = \left(w_1^p \left(\frac{19.8788 - f_1(\mathbf{x})}{19.8788 - (-72.5379)}\right)^p + w_2^p \left(\frac{f_2(\mathbf{x}) - (-33.3827)}{4.2136 - (-33.3827)}\right)^p\right)^{1/p},$$

$$d_p^{NIS} = \left(w_1^p \left(\frac{f_1(\mathbf{x}) - (-72.5379)}{19.8788 - (-72.5379)}\right)^p + w_2^p \left(\frac{4.2136 - f_2(\mathbf{x})}{4.2136 - (-33.3827)}\right)^p\right)^{1/p}.$$

Table 1: The solution, $\mathbf{x}^*$, of the problem (4.6) at different α^*

α	$\mathbf{x}$	f_1^*
0	(2.9000, 2.2000, 1.0000)	26.0500
0.1	(2.8282, 2.1641, 1.0000)	25.6640
0.2	(2.7522, 2.1261, 1.0000)	25.2467
0.3	(2.6713, 2.0857, 1.0000)	24.7841
0.4	(2.5844, 2.0422, 1.0000)	24.2688
0.5	(2.4899, 1.9950, 1.0000)	23.6871
0.6	(2.3854, 1.9427, 1.0000)	23.0176
0.7	(2.2688, 1.8834, 1.0000)	22.2246
0.8	(2.1261, 1.8130, 1.0000)	21.2384
0.9	(1.9427, 1.7214, 1.0000)	19.8788
1	(1.5000, 1.5000, 1.0000)	16.2500

Table 2: The results for solving problems (4.3), (4.4), and (4.5)

Problem	$\mathbf{x}$	objective value
Problem (4.3)	(1.2142, 1.0000, 5.3568)	$f_2^* = -33.3827$
Problem (4.4)	(1.0000, 1.0000, 6.2136)	$f_1^- = -72.5379$
Problem (4.5)	(1.0000, 3.6068, 1.0000)	$f_2^- = 4.2136$

Step 2. For $w_1 = w_2 = \frac{1}{2}$ and $p = 2$, solve the following problem:

$$\begin{array}{ll} \tilde{\min} & d_2^{PIS}(\mathbf{x}) \\ \tilde{\max} & d_2^{NIS}(\mathbf{x}) \\ \text{s.t.} & \mathbf{x} \in \tilde{X}. \end{array} \tag{4.7}$$

Step 2.1. Obtain $(d_2^{PIS})^*$, $(d_2^{NIS})^*$, $(d_2^{PIS})'$ and $(d_2^{NIS})'$ by solving the following problems:

$$(d_2^{PIS})^* = \min_{\mathbf{x}\in\tilde{X}} d_2^{PIS}(\mathbf{x}), \tag{4.8}$$

and the solution is $\mathbf{x}^{PIS} = (1.7987, 1, 3.0187)$.

$$(d_2^{NIS})^* = \max_{\mathbf{x}\in\tilde{X}} d_2^{NIS}(\mathbf{x}), \tag{4.9}$$

and the solution is $\mathbf{x}^{NIS} = (1.6570, 1, 3.5857)$. Then, we have

$$(d_2^{PIS})^* = d_2^{PIS}(\mathbf{x}^{PIS}) = 0.1650, \tag{4.10}$$

$$(d_2^{NIS})^* = d_2^{NIS}(\mathbf{x}^{NIS}) = 0.5456, \tag{4.11}$$

$$(d_2^{PIS})' = d_2^{PIS}(\mathbf{x}^{NIS}) = 0.1848, \tag{4.12}$$

$$(d_2^{NIS})' = d_2^{NIS}(\mathbf{x}^{PIS}) = 0.5426. \tag{4.13}$$

Step 2.2. The membership functions $\mu_1(\mathbf{x})$ and $\mu_2(\mathbf{x})$ of the two distance objective function can be described as follows.

$$\mu_1(\mathbf{x}) = \begin{cases} 1, & \text{if } d_2^{PIS}(\mathbf{x}) < 0.1650, \\ \dfrac{0.1848 - d_2^{PIS}(\mathbf{x})}{0.1848 - 0.1650}, & \text{if } 0.1650 \le d_2^{PIS}(\mathbf{x}) \le 0.1848, \\ 0, & \text{if } d_2^{PIS}(\mathbf{x}) > 0.1848, \end{cases} \tag{4.14}$$

Table 3: The results for solving problem (4.1) with equal weights $w_1 = w_2 = \frac{1}{2}$

Case	f_1	f_2	x_1	x_2	x_3	
$p = 1$,						
min d_1^{PIS}	−5.7952	−26.3948	1.7538	1.0000	3.1984	
max d_1^{NIS}	−5.7952	−26.3948	1.7538	1.0000	3.1984	
$p = 2$,						
min d_2^{PIS}	−3.0015	−25.1823	1.7987	1.0000	3.0187	
max d_2^{NIS}	−12.2687	−28.6780	1.6570	1.0000	3.5857	
max $\mu_{\tilde{D}}(\mathbf{x})$	−6.7392	−26.8424	1.6889	1.8701	3.5180	$\mu_{\tilde{D}}(\mathbf{x}) = 0.6712$
$p = 10$,						
min d_{10}^{PIS}	−1.9400	−24.6806	1.8164	1.0000	2.9481	
max d_{10}^{NIS}	19.8785	−2.3855	1.9427	1.7213	1.0000	
max $\mu_{\tilde{D}}(\mathbf{x})$	13.6874	−15.9142	2.0990	1.9077	2.0666	$\mu_{\tilde{D}}(\mathbf{x}) = 0.6252$

$$\mu_2(\mathbf{x}) = \begin{cases} 1, & \text{if } d_2^{NIS}(\mathbf{x}) > 0.5456, \\ \dfrac{d_2^{NIS}(\mathbf{x}) - 0.5426}{0.5456 - 0.5426}, & \text{if } 0.5426 \le d_2^{NIS}(\mathbf{x}) \le 0.5456, \\ 0, & \text{if } d_2^{NIS}(\mathbf{x}) < 0.5426. \end{cases} \tag{4.15}$$

Step 2.3. The maximizing solution, $\mathbf{x}^*$, of the fuzzy MONLP problem can be obtained by solving the following problem:

$$\max \min_{x_1, x_2, x_3 \ge 1} \{\mu_1, \mu_2, \mu_{g_1}, \mu_{g_2}\}. \tag{4.16}$$

This problem is equivalent to the following min-max problem:

$$-\min \max_{x_1, x_2, x_3 \ge 1} \{-\mu_1, -\mu_2, -\mu_{g_1}, -\mu_{g_2}\}. \tag{4.17}$$

A near-optimal solution of the "min-max" problem can be obtained by solving the following problem:

$$-\min_{x_1, x_2, x_3 \ge 1} \frac{1}{q} \ln\{\exp[q(-\mu_1(\mathbf{x}))] + \exp[q(-\mu_2(\mathbf{x}))] + \exp[q(-\mu_{g_1}(\mathbf{x}))] + \exp[q(-\mu_{g_2}(\mathbf{x}))]\},$$

with a sufficiently large q. In our implementation, we use a fixed $q = 100$, the compromise solution of the fuzzy MONLP problem (4.1) is $(1.6889, 1.8701, 3.5180)$ with the degree of satisfaction being equal to 0.6712. Tables 3 and 4 provide results for solving the fuzzy MONLP problem (4.1) with different cases of p and weights between objectives. From the results of Tables 3 and 4, we can see that solutions of "min d_1^{PIS}" and "max d_1^{NIS}" are the same in both cases of equal weight ($w_1 = w_2 = \frac{1}{2}$) and unequal weight ($w_1 = \frac{3}{4}, w_2 = \frac{1}{4}$). It is because that $d_1^{PIS} = 1 - d_1^{NIS}$. In this case, we don't need to solve Equations (2.13) and (2.15). When $p \ge 2$ ($p = 2, 10$), solutions of "min d_p^{PIS}" and "max d_p^{NIS}" are always different. For those cases of different solutions, we need to solve Equation (2.15) to obtain a compromise solution which has the shortest distance from the positive ideal solution and the farthest distance from the negative ideal solution.

5. Conclusions

In this paper, the solution of a fuzzy multi-objective nonlinear programming problem is studied. A TOPSIS-based entropic regularization approach is developed for solving the

Table 4: The results for solving problem (4.1) with unequal weights $w_1 = \frac{3}{4}, w_2 = \frac{1}{4}$

Case	f_1	f_2	x_1	x_2	x_3	
$p = 1$,						
min d_1^{PIS}	14.5020	−12.0605	2.1568	1.0000	1.5865	
max d_1^{NIS}	14.5020	−12.0605	2.1568	1.0000	1.5865	
$p = 2$,						
min d_2^{PIS}	9.8985	−16.8599	2.0439	1.0000	2.0379	
max d_2^{NIS}	19.8728	−2.6289	1.9775	1.6518	1.0000	
max $\mu_{\tilde{D}}(\mathbf{x})$	16.6543	−9.9613	1.9625	2.1846	1.7170	$\mu_{\tilde{D}}(\mathbf{x}) = 0.6838$
$p = 10$,						
min d_{10}^{PIS}	8.1138	−18.3764	2.0050	1.0000	2.1938	
max d_{10}^{NIS}	19.8785	−2.3855	1.9427	1.7213	1.0000	
max $\mu_{\tilde{D}}(\mathbf{x})$	15.8120	−12.3463	2.0467	2.0468	1.8387	$\mu_{\tilde{D}}(\mathbf{x}) = 0.6527$

fuzzy MONLP. Computational results are provided to illustrate the validity and efficiency of the proposed method. Also, because $\mu_q(\mathbf{x})$ appears in a special "log-exponential" form, it is highly smooth and avoids most over-flow problems in computation. Compared to other approaches to solving fuzzy multiobjective programming problems, our work essentially reduces the problem to minimizing an infinitely smooth function without any constraints for efficient computation.

It should be noted that in the crisp MODM theory, only the feasible solutions are considered as candidates for efficient solutions (Pareto optimal solutions). For the fuzzy multi-objective nonlinear programming problem (1.1), however, solutions do not only differ with respect to the associated vales of the objectives, but also with respect to their degree of feasibility. In this case, the satisfactory (compromise) solution obtained by the proposed algorithm need not be nondominated.

Moreover, the max-min operator employed here is not compensatory. In this case, the satisfactory level of "as close to PIS as possible" cannot be increased by decreasing the satisfactory level of "as far away from NIS as possible." This may be inconsistent with the basic principle of compromise of TOPSIS. Future studies should apply compensatory operators (such as min-bounded sum operator, fuzzy "and", weighted mean operator, etc.) to obtain the compromise solution of the fuzzy MONLP problem.

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