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COMPARISONS OF REPLACEMENT POLICIES WITH CONSTANT AND RANDOM TIMES

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Abstract This paper proposes random age, periodic and block replacement policies which are made at random variable times, and optimal policies that minimize their expected cost rates are discussed analytically and computed numerically. We compare such random replacements with their standard policies that are made at constant times. Comparison results show that when costs for random and constant replacements are the same, the standard policies are better than the random ones. Furthermore, it is computed numerically that if how much the random replacement cost is lower than that for the constant one, then the standard and random replacements have the same optimal cost rates. That is, the modified random replacement costs and their optimal times are discussed and computed when the random replacements would be better than the standard policies.

Keywords: Maintenance, random replacement, age replacement, periodic replacement, block replacement

1. Introduction

It has been well-known that the random replacement is not better theoretically than the standard policy with a constant replacement time T [1, p. 72], and is not used in practice at all. However, if the random replacement would be easier and more economical than the standard policy, this should be applied to actual systems.

As systems have become more complex and higher reliable, their failures might occur randomly with time and have a kind of exponential distributions [1, p. 18]. For such systems, it would not be suitable to determine maintenances at constant planned times theoretically and economically due to several unpredictable factors. On the other hand, in place of the traditional maintenances with constant time T, we propose random age, periodic and block replacement policies to spread widely in practical fields. Such policies without operational suspensions for jobs would be performed more easily to maintenance policies for machines with random working times [2, 3] and computers with random processing times [9]. That is, when a job has a variable working time or processing time, it would be better to do maintenance or replacement after the job is just completed even though the maintenance time has arrived [6, p. 245]. A representative practical example for such random policies is to maintain a database or to perform a backup of data when a transaction is processing its sequences of operations, because it is necessary to guarantee ACID (atomicity, consistency, isolation, and durability) properties of database transactions, especially for a distributed transaction across a distributed database. In other words, if any part of transaction fails, the entire transaction fails and the database state is left unchanged [4, 5].

By considering the factors of random working times in operations, the reliability quantities of the random age replacement policy were obtained [1, p. 72]. Several schedules of jobs that have random processing times were summarized [12]. The properties of replacement policies between two successive failed units, where the unit is replaced at random times, were investigated [13]. A summary of discussions for inspections with random policies were made [8]. Under the assumptions of random failure and maintenance, replacement and inspection with planned and random policies were considered, and their comparisons were made [11, 14]. Combining a planned replacement with working times, the age and periodic replacement policies, where the unit is replaced at a planned time T and at the Nth random working time, were discussed [2, 3]. From the viewpoint of unnecessary replacement in [2, 3], the notion of "whichever occurs last" between planned and random replacements was explored and discussed [15]. Such a notion of random maintenance was applied to a parallel system with random number of units to satisfy the random jobs [10].

This paper firstly takes up the combined constant and random age replacement policy in which the unit is replaced at a constant time T or at a random time Y, where Y is a random variable, and obtains the expected cost rate. We discuss optimal constant and random replacement times for the standard and random policies, respectively. It has already known that such a random replacement is not better than the standard policy that is made at constant time T, when costs for constant and random replacements are the same. Under the assumption of random replacement cost might be more economical than that at constant time, we compare two expected costs and discuss numerically that if how much the random replacement cost is lower than that for the standard one, they have the same optimal cost rates. That is, the modified random replacement cost and its optimal time are discussed and computed when the random replacement would be better than the standard policy. Furthermore, we make similar discussions for the periodic replacement [1, 6] and block replacement [1, 6]. Expected cost rates of each replacement are obtained and optimal random policies which minimize them are derived analytically. Furthermore, we compare the random replacements with their standard policies and compute modified costs for random replacements when two expected costs of random and standard policies are the same.

2. Age Replacement

It is assumed that every operating unit has an identical failure distribution $F(t)$ with a finite mean μ $(0 < \mu < \infty)$, failure rate $h(t) \equiv f(t)/\overline{F}(t)$ and cumulative hazard rate $H(t) \equiv \int_0^t h(u)\mathrm{d}u$, where $f(t)$ is a density function of $F(t)$, i.e., $F(t) \equiv \int_0^t f(u)\mathrm{d}u$, and $\overline{\Phi}(t) \equiv 1 - \Phi(t)$ for any function $\Phi(t)$, and $h(t)$ increases strictly with t to $h(\infty) \equiv \lim_{t\to\infty} h(t)$, which might be infinity. Furthermore, $M(t) \equiv \sum_{n=1}^{\infty} F^{(n)}(t)$ which is called a renewal function in stochastic processes [7, p. 50], where $F^{(n)}(t)$ $(n = 0, 1, 2, \cdots)$ denotes the n-fold Stieltjes convolution of $F(t)$ with itself and $F^{(0)}(t) \equiv 1$ for $t \geq 0$, and $m(t) \equiv M'(t)$ which is called a renewal density. In addition, a random variable Y has a general distribution $G(t) \equiv \Pr\{Y \leq t\}$ with a finite mean $\theta \equiv \int_0^\infty \overline{G}(t)\mathrm{d}t$ $(0 < \theta < \infty)$.

Suppose that the unit is replaced before failure at a planned time T or at a random time Y, whichever occurs first, where T $(0 < T \leq \infty)$ is constant and Y is a positive random variable with a general distribution $G(t)$ with a finite mean θ $(0 < \theta \leq \infty)$. Then, the expected cost rate is [6, p. 247]

$$C_A(T, G) = \frac{c_T + (c_F - c_T)\int_0^T \overline{G}(t)\mathrm{d}F(t) + (c_R - c_T)\int_0^T \overline{F}(t)\mathrm{d}G(t)}{\int_0^T \overline{G}(t)\overline{F}(t)\mathrm{d}t}, \tag{2.1}$$

where c_F = replacement cost at failure, c_T = replacement cost at time T, c_R = replacement cost at time Y, and $c_F > c_T$ and $c_F > c_R$. An optimal replacement time T^* which minimizes

$C_A(T,G)$ was derived analytically [1, 2].

Clearly, when $G(t) = 0$ for any $t \geq 0$, i.e., the unit is replaced only at time T, the expected cost rate is [1, 6]

$$C_A(T) = \frac{c_T + (c_F - c_T)F(T)}{\int_0^T \overline{F}(t)\mathrm{d}t}, \tag{2.2}$$

which is called the *standard age replacement.* If $h(\infty) > c_F/[\mu(c_F - c_T)]$, then an optimal time T_A^* $(0 < T_A^* < \infty)$ which minimizes $C_A(T)$ satisfies uniquely

$$h(T)\int_0^T \overline{F}(t)\mathrm{d}t - F(T) = \frac{c_T}{c_F - c_T}, \tag{2.3}$$

and the resulting cost rate is

$$C_A(T_A^*) = (c_F - c_T)h(T_A^*). \tag{2.4}$$

On the other hand, when $T = \infty$, i.e., the unit is replaced only at time Y is

$$C_A(G) \equiv \lim_{T\to\infty} C_A(T,G) = \frac{c_R + (c_F - c_R)\int_0^\infty F(t)\mathrm{d}G(t)}{\int_0^\infty \overline{G}(t)\overline{F}(t)\mathrm{d}t}, \tag{2.5}$$

which is called the *random age replacement.*

It has been already shown [1, p. 87] that when $c_T = c_R$, (2.5) can be written as

$$C_A(G) = \frac{\int_0^\infty Q(t)\mathrm{d}G(t)}{\int_0^\infty S(t)\mathrm{d}G(t)},$$

where

$$Q(t) \equiv c_T + (c_F - c_T)F(t), \quad S(t) \equiv \int_0^t \overline{F}(u)\mathrm{d}u.$$

Suppose that there exists a minimum value T $(0 < T \leq \infty)$ which minimizes $Q(t)/S(t)$. Because

$$\frac{Q(t)}{S(t)} \geq \frac{Q(T)}{S(T)},$$

which follows that

$$\int_0^\infty Q(t)\mathrm{d}G(t) \geq \frac{Q(T)}{S(T)}\int_0^\infty S(t)\mathrm{d}G(t).$$

So that,

$$C_A(G) \geq \frac{Q(T)}{S(T)} = C_A(G_T) = C_A(T),$$

where $G_T(t)$ is the degenerate distribution placing unit mass at T, i.e., $G_T(t) \equiv 0$ for $t < T$ and 1 for $t \geq T$. If $T = \infty$, then the units is replaced only at failure and the expected cost rate is

$$C_A \equiv \lim_{T\to\infty} C_A(T) = \frac{c_F}{\mu}. \tag{2.6}$$

Therefore, the optimal replacement policy is nonrandom and the expected cost rate is given in (2.2).

Next, when $G(t) = 1 - \mathrm{e}^{-t/\theta}$, we find an optimal θ_A^* which minimizes the expected cost rate

$$C_A(\theta) = \frac{c_R + (c_F - c_R)\int_0^\infty \mathrm{e}^{-t/\theta}\mathrm{d}F(t)}{\int_0^\infty \mathrm{e}^{-t/\theta}\overline{F}(t)\mathrm{d}t}. \tag{2.7}$$

Differentiating $C_A(\theta)$ with respect to θ and setting it equal to zero,

$$r(\theta)\int_0^\infty \mathrm{e}^{-t/\theta}\overline{F}(t)\mathrm{d}t - \int_0^\infty \mathrm{e}^{-t/\theta}\mathrm{d}F(t) = \frac{c_R}{c_F - c_R}, \tag{2.8}$$

where $r(\theta) \equiv \lim_{T\to\infty} r(T,\theta)$, and for $0 < T \leq \infty$,

$$r(T,\theta) \equiv \frac{\int_0^T t\mathrm{e}^{-t/\theta}\mathrm{d}F(t)}{\int_0^T t\mathrm{e}^{-t/\theta}\overline{F}(t)\mathrm{d}t}. \tag{2.9}$$

First, we investigate the properties of $r(T,\theta)$: It can be easily seen that because $h(t)$ increases strictly with t, $r(T,\theta) \leq h(T)$ and increases strictly with T from $h(0)$ to $r(\theta)$. Furthermore, differentiating $r(T,\theta)$ with θ,

$$\begin{aligned}\frac{\mathrm{d}r(T,\theta)}{\mathrm{d}\theta} &= \frac{1}{[\int_0^T \theta t\mathrm{e}^{-t/\theta}\overline{F}(t)\mathrm{d}t]^2}\left[\int_0^T t^2\mathrm{e}^{-t/\theta}\mathrm{d}F(t)\int_0^T t\mathrm{e}^{-t/\theta}\overline{F}(t)\mathrm{d}t\right.\\ &\quad \left. - \int_0^T t\mathrm{e}^{-t/\theta}\mathrm{d}F(t)\int_0^T t^2\mathrm{e}^{-t/\theta}\overline{F}(t)\mathrm{d}t\right].\end{aligned}$$

Denoting $L(T)$ be the numerator of the right-hand side, $L(0) = 0$ and

$$L'(T) = T\mathrm{e}^{-T/\theta}\overline{F}(T)\int_0^T t\mathrm{e}^{-t/\theta}\overline{F}(t)[h(T) - h(t)](T-t)\mathrm{d}t > 0,$$

i.e., $L(T) > 0$ for $0 < T < \infty$. So that, $r(T,\theta)$ increases strictly with θ to

$$r(T) \equiv \frac{\int_0^T t\mathrm{d}F(t)}{\int_0^T t\overline{F}(t)\mathrm{d}t}.$$

From the above result, the left-hand side of (2.8) also increases with θ from 0 to $r(\infty)\mu - 1$, where $r(\infty) \equiv \int_0^\infty t\mathrm{d}F(t)/\int_0^\infty t\overline{F}(t)\mathrm{d}t$. Therefore, if $r(\infty) > c_F/[\mu(c_F - c_R)]$, then there exists an optimal θ_A^* $(0 < \theta_A^* < \infty)$ which satisfies (2.8), and the resulting cost rate is

$$C_A(\theta_A^*) = (c_F - c_R)r(\theta_A^*). \tag{2.10}$$

Note that $r(\theta)$ plays the same role as the failure rate $h(t)$ in the standard age replacement.

We have already known that if $c_T \leq c_R$, then the standard age replacement is better than the random policy. When $c_T > c_R$, $r(\infty) > c_F/[\mu(c_F - c_R)]$ and $G(t) = 1 - \mathrm{e}^{-t/\theta}$, we compare the expected cost rates $C_A(T)$ in (2.2) and $C_A(\theta)$ in (2.7). We derive a modified optimal policy $\widehat{\theta}$ and its modified replacement cost $\widehat{c}_R$ $(\widehat{c}_R < c_T)$ in which two optimal cost rates $C_A(T_A^*)$ and $C_A(\widehat{\theta})$ are the same. First, we compute T_A^* $(0 < T_A^* < \infty)$ which satisfies (2.3) for c_T and c_F, and $C_A(T_A^*)$ in (2.4). Next, we compute modified $\widehat{c}_R$ which satisfies

$$r(\theta)\int_0^\infty \mathrm{e}^{-t/\theta}\overline{F}(t)\mathrm{d}t + \int_0^\infty (1-\mathrm{e}^{-t/\theta})\mathrm{d}F(t) = \frac{c_F}{c_F-\widehat{c}_R},$$
$$(c_F - c_T)h(T_A^*) = (c_F - \widehat{c}_R)r(\theta),$$

i.e., we firstly obtain $\widehat{\theta}$ for T_A^* which satisfies

$$\frac{1}{r(\theta)}\int_0^\infty (1-\mathrm{e}^{-t/\theta})\mathrm{d}F(t) + \int_0^\infty \mathrm{e}^{-t/\theta}\overline{F}(t)\mathrm{d}t = \frac{1}{h(T_A^*)}\frac{c_F}{c_F-c_T}, \tag{2.11}$$

and using $\widehat{\theta}$, we compute $\widehat{c}_R$ which satisfies

$$\frac{c_F-\widehat{c}_R}{c_F-c_T} = \frac{h(T_A^*)}{r(\widehat{\theta})}. \tag{2.12}$$

It is assumed as a numerical example that the failure time has a gamma distribution with order k, i.e., $F(t) = \sum_{j=k}^\infty[(\lambda t)^j/j!]\mathrm{e}^{-\lambda t}$ $(k = 2, 3, \cdots)$, $f(t) = [\lambda(\lambda t)^{k-1}/(k-1)!]\mathrm{e}^{-\lambda t}$, and $h(t) = [\lambda(\lambda t)^{k-1}/(k-1)!]/\sum_{j=0}^{k-1}[(\lambda t)^j/j!]$, which increases strictly with t from 0 to λ. Then, if $k > c_F/(c_F - c_T)$, then an optimal T_A^* $(0 < T_A^* < \infty)$ satisfies uniquely

$$\frac{\lambda(\lambda T)^{k-1}/(k-1)!}{\sum_{j=0}^{k-1}[(\lambda T)^j/j!]}\sum_{j=0}^{k-1}\int_0^T \frac{(\lambda t)^j}{j!}\mathrm{e}^{-\lambda t}\mathrm{d}t - \sum_{j=k}^\infty \frac{(\lambda T)^j}{j!}\mathrm{e}^{-\lambda T} = \frac{c_T}{c_F-c_T},$$

and the resulting cost rate is

$$C_A(T_A^*) = (c_F-c_T)\frac{\lambda(\lambda T_A^*)^{k-1}/(k-1)!}{\sum_{j=0}^{k-1}[(\lambda T_A^*)^j/j!]}.$$

On the other hand, when $F(t)$ is a gamma distribution,

$$\int_0^\infty t\mathrm{e}^{-t/\theta}\mathrm{d}F(t) = \frac{k\theta}{1+\lambda\theta}\left(\frac{\lambda\theta}{1+\lambda\theta}\right)^k,$$
$$\int_0^\infty t\mathrm{e}^{-t/\theta}\overline{F}(t)\mathrm{d}t = \frac{1}{\lambda^2}\sum_{j=1}^k j\left(\frac{\lambda\theta}{1+\lambda\theta}\right)^{j+1},$$
$$r(\theta) = \frac{k\lambda[\lambda\theta/(1+\lambda\theta)]^{k-1}}{\sum_{j=1}^k j[\lambda\theta/(1+\lambda\theta)]^{j-1}},$$

which increases strictly with θ from 0 to λ. Then, if $k > c_F/(c_F - c_R)$, then an optimal θ_A^* $(0 < \theta_A^* < \infty)$ satisfies uniquely

$$\frac{kX^{k-1}}{\sum_{j=1}^k jX^{j-1}}\sum_{j=1}^k X^j - X^k = \frac{c_R}{c_F-c_R},$$

where $X \equiv \lambda\theta/(1+\lambda\theta)$, and the resulting cost rate is

$$C_A(\theta_A^*) = (c_F-c_R)\frac{k\lambda[\lambda\theta_A^*/(1+\lambda\theta_A^*)]^{k-1}}{\sum_{j=1}^k j[\lambda\theta_A^*/(1+\lambda\theta_A^*)]^{j-1}}.$$

Table 1 presents T_A^* for c_T/c_F, θ_A^* for c_R/c_F which satisfies (2.3) and (2.8) when $c_T = c_R$, and modified $\widehat{\theta}$, $\widehat{c}_R/c_F$, and $\widehat{c}_R/c_T$ which satisfy (2.11) and (2.12) when $F(t) = \sum_{j=k}^{\infty}(t^j/j!)\mathrm{e}^{-t}$ ($k = 2, 3, 4$). From the comparison results among T_A^*, θ_A^* and $\widehat{\theta}$, it is shown that $T_A^* > \theta_A^*$ for small c_T/c_F or c_R/c_F, however, $T_A^* < \theta_A^*$ for large ones. Note that $\widehat{\theta} < \theta_A^*$, i.e., we should replace the unit earlier for the random policy in order to have the same expected cost rate with the standard policy. It also could be show clearly that $\widehat{c}_R/c_F$ decreases with k. From the numerical value of $\widehat{c}_R/c_T$, we can find that if how much the modified $\widehat{c}_R$ is less than c_T, the expected costs for the standard and random age replacements are almost the same. Taking $k = 2$ for an example, when $\widehat{c}_R$ is a little more than about 60% of c_T, we can adopt the random replacement.

Table 1: Optimal T_A^*, θ_A^*, $\widehat{\theta}$, $\widehat{c}_R/c_F$, and $\widehat{c}_R/c_T$ when $F(t) = \sum_{j=k}^{\infty}(t^j/j!)\mathrm{e}^{-t}$.

c_T/c_F or	$k=2$					$k=3$					$k=4$				
c_R/c_F	T_A^*	θ_A^*	$\widehat{\theta}$	$\widehat{c}_R/c_F$	$\widehat{c}_R/c_T$	T_A^*	θ_A^*	$\widehat{\theta}$	$\widehat{c}_R/c_F$	$\widehat{c}_R/c_T$	T_A^*	θ_A^*	$\widehat{\theta}$	$\widehat{c}_R/c_F$	$\widehat{c}_R/c_T$
0.01	0.157	0.051	0.030	0.006	0.600	0.357	0.195	0.137	0.005	0.500	0.631	0.348	0.248	0.004	0.400
0.02	0.233	0.062	0.044	0.012	0.600	0.468	0.299	0.204	0.010	0.500	0.784	0.471	0.329	0.008	0.400
0.05	0.412	0.335	0.207	0.031	0.620	0.697	0.540	0.348	0.026	0.520	1.069	0.757	0.496	0.022	0.440
0.1	0.680	0.793	0.430	0.064	0.640	0.984	0.955	0.557	0.053	0.530	1.400	1.220	0.725	0.046	0.460
0.2	1.306	4.017	1.104	0.129	0.645	1.512	2.462	1.042	0.109	0.545	1.957	2.649	1.216	0.099	0.495

3. Periodic Replacement

A new unit begins to operate at time 0 and undergoes only minimal repair at each failure. Suppose that the unit is replaced at time T or at time Y, whichever occurs first. Then, the expected cost rate is [6, p. 250]

$$C_P(T, G) = \frac{c_M \int_0^T \overline{G}(t)h(t)\mathrm{d}t + c_T + (c_R - c_T)G(T)}{\int_0^T \overline{G}(t)\mathrm{d}t}, \tag{3.1}$$

where c_M = minimal repair cost at each failure, and c_T and c_R are given in (2.1). The optimal replacement time T_P^* which minimizes $C_P(T, G)$ was derived analytically [3, 6].

Clearly, when $G(t) = 0$ for any $t \geq 0$, i.e., the unit is replaced only at time T, the expected cost rate is [1, 6]

$$C_P(T) = \frac{c_M H(T) + c_T}{T}, \tag{3.2}$$

which is called the *standard periodic replacement.* If $\int_0^\infty t\mathrm{d}h(t) > c_T/c_M$, then an optimal T_P^* ($0 < T_P^* < \infty$) which minimizes $C_P(T)$ satisfies

$$Th(T) - H(T) = \frac{c_T}{c_M}, \quad \text{i.e.,} \quad \int_0^T t\mathrm{d}h(t) = \frac{c_T}{c_M}, \tag{3.3}$$

and the resulting cost rate is

$$C_P(T_P^*) = c_M h(T_P^*). \tag{3.4}$$

On the other hand, when $T = \infty$, i.e., the unit is replaced only at time Y is

$$C_P(G) \equiv \lim_{T\to\infty} C_P(T, G) = \frac{c_M \int_0^\infty \overline{G}(t)h(t)\mathrm{d}t + c_R}{\int_0^\infty \overline{G}(t)\mathrm{d}t}, \tag{3.5}$$

which is called the *random periodic replacement.*

By the method similar to Section 2, when $c_T = c_R$, (3.5) can be written as

$$C_P(G) = \frac{\int_0^\infty Q(t)\mathrm{d}G(t)}{\int_0^\infty S(t)\mathrm{d}G(t)},$$

where

$$Q(t) \equiv c_M H(t) + c_R, \quad S(t) = t.$$

Suppose that there exists a minimum value T $(0 < T \leq \infty)$ which minimizes $Q(t)/S(t)$. Because

$$\frac{Q(t)}{S(t)} \geq \frac{Q(T)}{S(T)},$$

it follows that

$$C_P(G) \geq \frac{Q(T)}{S(T)} = C_P(G_T) = C_P(T).$$

If $T = \infty$, then the unit is replaced only at failure and the expected cost rate is

$$C_P \equiv \lim_{T\to\infty} C_P(T) = c_M h(\infty),$$

where $h(\infty)$ might be infinity. Therefore, the optimal replacement policy is nonrandom and the expected cost rate is given in (3.2).

Next, when $G(t) = 1 - \mathrm{e}^{-t/\theta}$ and $H(t) = \lambda t^\alpha$ $(\alpha > 1)$, we find an optimal θ_P^* which minimizes the expected cost rate

$$C_P(\theta) = \frac{c_M \int_0^\infty \mathrm{e}^{-t/\theta}\lambda\alpha t^{\alpha-1}\mathrm{d}t + c_R}{\theta} = \frac{c_M\lambda\Gamma(\alpha+1)\theta^\alpha + c_R}{\theta}, \tag{3.6}$$

where $\Gamma(\alpha) \equiv \int_0^\infty x^{\alpha-1}\mathrm{e}^{-x}\mathrm{d}x$ for $\alpha > 0$.

An optimal θ_P^* which minimizes $C_P(\theta)$ is easily given by

$$\theta_P^* = \left[\frac{c_R}{c_M}\frac{1}{\lambda(\alpha-1)\Gamma(\alpha+1)}\right]^{1/\alpha}, \tag{3.7}$$

and the resulting cost rate is

$$C_P(\theta_P^*) = c_M\lambda\alpha\Gamma(\alpha+1)(\theta_P^*)^{\alpha-1}. \tag{3.8}$$

On the other hand, an optimal T_P^* which satisfies (3.3) is

$$T_P^* = \left[\frac{c_T}{c_M}\frac{1}{\lambda(\alpha-1)}\right]^{1/\alpha}, \tag{3.9}$$

and the resulting cost rate is

$$C_P(T_P^*) = c_M\lambda\alpha(T_P^*)^{\alpha-1}. \tag{3.10}$$

It can be easily seen that when $c_T = c_R$, $T_P^* = [\Gamma(\alpha+1)]^{1/\alpha}\theta_P^*$, and hence, $\theta_P^* < T_P^*$ and $C_P(T_P^*) < C_P(\theta_P^*)$. So that, the standard periodic replacement is better than the random policy, as shown already. Furthermore, we derive a modified optimal policy $\widehat{\theta}$ and its modified replacement cost $\widehat{c}_R$ ($\widehat{c}_R < c_T$) in which two optimal cost rates $C_P(T_P^*)$ and $C_P(\widehat{\theta})$ are the same. That is, compute $\widehat{\theta}$ satisfies

$$\widehat{\theta} = \frac{T_P^*}{[\Gamma(\alpha+1)]^{1/(\alpha-1)}}.$$

Using $\widehat{\theta}$, we compute

$$\frac{\widehat{c}_R}{c_M} = \lambda(\alpha-1)\Gamma(\alpha+1)(\widehat{\theta})^{\alpha} = \frac{c_T}{c_M}\frac{1}{[\Gamma(\alpha+1)]^{1/(\alpha-1)}}. \tag{3.11}$$

Table 2 presents optimal T_P^* for c_T/c_M, θ_P^* for c_R/c_M which satisfies (3.9) and (3.7) when $c_T = c_R$, and modified $\widehat{\theta}$, $\widehat{c}_R/c_M$, and $\widehat{c}_R/c_T$ when $F(t) = 1 - \mathrm{e}^{-t^{\alpha}}$ ($\alpha = 2, 3, 4$). It indicates that $T_P^* > \theta_P^* > \widehat{\theta}$ and $\widehat{c}_R/c_F$ decreases with α. It is of great interest that $\widehat{c}_R/c_T$ depends only on α, because from (3.11), $\widehat{c}_R/c_T = \widehat{\theta}/T_P^* = \Gamma(\alpha+1)^{-1/(\alpha-1)}$. For example, when $\alpha = 2$, $\widehat{c}_R/c_T$ is about 0.5, i.e., when the random replacement cost is 50% of the periodic one, the two expected costs $C_P(T_P^*)$ and $C_P(\widehat{\theta})$ are the same.

Table 2: Optimal T_P^*, θ_P^*, $\widehat{\theta}$, $\widehat{c}_R/c_M$, and $\widehat{c}_R/c_T$ when $F(t) = 1 - \mathrm{e}^{-t^{\alpha}}$.

c_T/c_M or c_R/c_M	$\alpha=2$					$\alpha=3$					$\alpha=4$				
	T_P^*	θ_P^*	$\widehat{\theta}$	$\widehat{c}_R/c_M$	$\widehat{c}_R/c_T$	T_P^*	θ_P^*	$\widehat{\theta}$	$\widehat{c}_R/c_M$	$\widehat{c}_R/c_T$	T_P^*	θ_P^*	$\widehat{\theta}$	$\widehat{c}_R/c_M$	$\widehat{c}_R/c_T$
0.1	0.316	0.224	0.158	0.050	0.500	0.368	0.203	0.150	0.041	0.410	0.427	0.193	0.148	0.035	0.350
0.2	0.477	0.316	0.224	0.100	0.500	0.464	0.255	0.189	0.082	0.410	0.508	0.230	0.176	0.069	0.350
0.5	0.707	0.500	0.354	0.250	0.500	0.630	0.347	0.257	0.204	0.410	0.639	0.289	0.222	0.173	0.350
1.0	1.000	0.707	0.500	0.500	0.500	0.794	0.437	0.324	0.408	0.410	0.760	0.343	0.263	0.347	0.350
2.0	1.414	1.000	0.707	1.000	0.500	1.000	0.550	0.408	0.816	0.410	0.904	0.408	0.313	0.693	0.350
5.0	2.236	1.581	1.118	2.500	0.500	1.357	0.747	0.554	2.041	0.410	1.136	0.513	0.394	1.733	0.350
10.0	3.162	2.236	1.581	5.000	0.500	1.710	0.941	0.698	4.082	0.410	1.351	0.610	0.468	3.467	0.350

4. Block Replacement

A new unit begins to operate at time 0 and failed units are replaced at each failure. Suppose that the unit is replaced at time T or at time Y, whichever occurs first. Then, replacing $H(t)$ in (3.1) with $M(t)$, the expected cost rate is

$$C_B(T, G) = \frac{c_F\int_0^T \overline{G}(t)m(t)\mathrm{d}t + c_T + (c_R - c_T)G(T)}{\int_0^T \overline{G}(t)\mathrm{d}t}, \tag{4.1}$$

where c_F = replacement cost at each failure, and c_T and c_R are given in (2.1).

Clearly, when $G(t) = 0$ for any $t \geq 0$, i.e., the unit is replaced only at time T, the expected cost rate is [1, 6]

$$C_B(T) = \frac{c_F M(T) + c_T}{T}, \tag{4.2}$$

which is called the *standard block replacement*. An optimal T_B^* which minimizes $C_B(T)$ satisfies

$$Tm(T) - M(T) = \frac{c_T}{c_F}, \tag{4.3}$$

and the resulting cost rate is

$$C_B(T_B^*) = c_F m(T_B^*). \tag{4.4}$$

On the other hand, when $T = \infty$, i.e., the unit is replaced only at time Y is

$$C_B(G) \equiv \lim_{T\to\infty} C_B(T,G) = \frac{c_F \int_0^\infty \overline{G}(t)m(t)\mathrm{d}t + c_R}{\int_0^\infty \overline{G}(t)\mathrm{d}t}, \tag{4.5}$$

which is called the *random block replacement*. By the method similar to Sections 2 and 3, when $c_T = c_R$, the optimal policy which minimizes $C_B(G)$ is nonrandom, and the expected cost rate is given in (4.2).

Next, when $G(t) = 1 - \mathrm{e}^{-t/\theta}$, the expected cost rate in (4.5) is rewritten as

$$C_B(\theta) = \frac{c_F \int_0^\infty \mathrm{e}^{-t/\theta} m(t)\mathrm{d}t + c_R}{\theta} = \frac{c_F M^*(1/\theta) + c_R}{\theta}, \tag{4.6}$$

where $M^*(1/\theta) \equiv \int_0^\infty \mathrm{e}^{-t/\theta}\mathrm{d}M(t)$ which is the LS transform of $M(t)$ for any $\theta > 0$.

In particular, when $F(t)$ is a gamma distribution with order k ($k = 2, 3, \cdots$), i.e., $F(t) = \sum_{j=k}^\infty [(\lambda t)^j/j!]\mathrm{e}^{-\lambda t}$ and $M(t) = \sum_{n=1}^\infty \sum_{j=kn}^\infty [(\lambda t)^j/j!]\mathrm{e}^{-\lambda t}$ [1, 7], the expected cost rate is

$$C_B(\theta) = \frac{1}{\theta}\left[\frac{c_F(\lambda\theta)^k}{(1+\lambda\theta)^k - (\lambda\theta)^k} + c_R\right]. \tag{4.7}$$

Differentiating $C_B(\theta)$ with respect to θ and setting it equal to zero,

$$\frac{(\lambda\theta)^k}{[(1+\lambda\theta)^k - (\lambda\theta)^k]^2}[(k-1-\lambda\theta)(1+\lambda\theta)^{k-1} + (\lambda\theta)^k] = \frac{c_R}{c_F}, \tag{4.8}$$

and the resulting cost rate is

$$C_B(\theta_B^*) = \frac{c_F k\lambda[\lambda\theta_B^*(1+\lambda\theta_B^*)]^{k-1}}{[(1+\lambda\theta_B^*)^k - (\lambda\theta_B^*)^k]^2}. \tag{4.9}$$

For example, an optimal θ_B^* is, when $k = 2$,

$$\left(\frac{\lambda\theta}{1+2\lambda\theta}\right)^2 = \frac{c_R}{c_F},$$

whose left-hand side increases strictly with θ from 0 to 1/4, and when $k = 3$,

$$\frac{(\lambda\theta)^3(2+3\lambda\theta)}{[1+3\lambda\theta+3(\lambda\theta)^2]^2} = \frac{c_R}{c_F},$$

whose left-hand side increases strictly from 0 to 1/3.

When $c_T > c_R$, $G(t) = 1 - \mathrm{e}^{-t/\theta}$ and $F(t) = \sum_{j=k}^\infty [(\lambda t)^j/j!]\mathrm{e}^{-\lambda t}$ ($k = 2, 3, \cdots$), we compute a modified optimal policy $\widehat{\theta}$ and its modified replacement cost $\widehat{c}_R$ ($\widehat{c}_R < c_T$) in which two optimal cost rates $C_B(T_B^*)$ and $C_B(\widehat{\theta})$ are the same. First, from (4.3), we compute T_B^* which satisfies

$$\sum_{j=1}^\infty \frac{(\lambda T)^{kj}}{(kj-1)!}\mathrm{e}^{-\lambda T} - \sum_{n=1}^\infty \sum_{j=kn}^\infty \frac{(\lambda T)^j}{j!}\mathrm{e}^{-\lambda T} = \frac{c_T}{c_F}. \tag{4.10}$$

Using T_B^*, we compute $\widehat{\theta}$ which satisfies

$$\frac{k[\lambda\theta(1+\lambda\theta)]^{k-1}}{[(1+\lambda\theta)^k-(\lambda\theta)^k]^2}=\sum_{j=1}^{\infty}\frac{(\lambda T_B^*)^{kj-1}}{(kj-1)!}\mathrm{e}^{-\lambda T_B^*}. \tag{4.11}$$

Using $\widehat{\theta}$, from (4.8), we compute

$$\frac{\widehat{c}_R}{c_F}=\frac{(\lambda\widehat{\theta})^k}{[(1+\lambda\widehat{\theta})^k-(\lambda\widehat{\theta})^k]^2}[(k-1-\lambda\widehat{\theta})(1+\lambda\widehat{\theta})^{k-1}+(\lambda\widehat{\theta})^k]. \tag{4.12}$$

For example, when $k=2$, T_B^* is given by a solution of the equation

$$\left(\frac{1}{4}+\frac{\lambda T}{2}\right)(1-\mathrm{e}^{-2\lambda T})-\frac{\lambda T}{2}=\frac{c_T}{c_F},$$

$\widehat{\theta}$ is given by

$$\frac{\lambda\theta(1+\lambda\theta)}{(1+2\lambda\theta)^2}=\frac{\lambda}{4}(1-\mathrm{e}^{-2\lambda T_B^*}),$$

and using $\widehat{\theta}$,

$$\frac{\widehat{c}_R}{c_F}=\left(\frac{\lambda\widehat{\theta}}{1+2\lambda\widehat{\theta}}\right)^2.$$

Table 3 presents optimal T_B^* for c_T/c_F, θ_B^* for c_R/c_F which satisfies (4.3) and (4.8) when $c_T=c_R$, modified $\widehat{\theta}$, $\widehat{c}_R/c_F$, and $\widehat{c}_R/c_T$ when $F(t)=\sum_{j=k}^{\infty}(t^j/j!)\,\mathrm{e}^{-t}$ $(k=2,3,4)$. It has the similar comparison results with Table 1, i.e., $\theta_B^*>\widehat{\theta}$, and $T_B^*>\theta_B^*$ for small c_T/c_F or c_R/c_F and $T_B^*<\theta_B^*$ for large ones, and both $\widehat{c}_R/c_F$ and $\widehat{c}_R/c_T$ decrease with k.

Table 3: Optimal T_B^*, θ_B^*, $\widehat{\theta}$, $\widehat{c}_R/c_F$, and $\widehat{c}_R/c_T$ when $F(t)=\sum_{j=k}^{\infty}(t^j/j!)\mathrm{e}^{-t}$.

c_T/c_F or	$k=2$					$k=3$					$k=4$				
c_R/c_F	T_B^*	θ_B^*	$\widehat{\theta}$	$\widehat{c}_R/c_F$	$\widehat{c}_R/c_T$	T_B^*	θ_B^*	$\widehat{\theta}$	$\widehat{c}_R/c_F$	$\widehat{c}_R/c_T$	T_B^*	θ_B^*	$\widehat{\theta}$	$\widehat{c}_R/c_F$	$\widehat{c}_R/c_T$
0.01	0.157	0.125	0.085	0.005	0.500	0.355	0.235	0.164	0.004	0.400	0.630	0.365	0.260	0.004	0.400
0.02	0.233	0.197	0.131	0.011	0.550	0.467	0.330	0.226	0.009	0.450	0.781	0.487	0.338	0.008	0.400
0.05	0.412	0.405	0.255	0.029	0.580	0.691	0.565	0.367	0.024	0.480	1.059	0.771	0.507	0.022	0.440
0.1	0.688	0.860	0.495	0.062	0.620	0.969	0.981	0.587	0.053	0.530	1.374	1.234	0.748	0.047	0.470
0.2	1.497	4.236	1.734	0.151	0.755	1.487	2.503	1.213	0.123	0.615	1.881	2.669	1.331	0.109	0.545

Finally, when the unit fails, it is not replaced and remains in a failure state for the time interval from a failure to its detection [6, p. 120]. Suppose that the unit is replaced at time T or at time Y, whichever occurs first. Then, the mean time from failure to its detection is

$$\overline{G}(T)\int_0^T(T-t)\mathrm{d}F(t)+\int_0^T\left[\int_0^t(t-u)\mathrm{d}F(u)\right]\mathrm{d}G(t)=\int_0^T F(t)\overline{G}(t)\mathrm{d}t.$$

Thus, the expected cost rate is

$$C_D(G)=\frac{c_D\int_0^T F(t)\overline{G}(t)\mathrm{d}t+c_T+(c_R-c_T)G(T)}{\int_0^T\overline{G}(t)\mathrm{d}t}, \tag{4.13}$$

where c_D = downtime cost per unit of time from failure to its detection, and c_T and c_R are given in (2.1).

Clearly, when $G(t) = 0$ for any $t \geq 0$, i.e., the unit is replaced only at time T, the expected cost rate is [1, p. 120]

$$C_D(T) = \frac{c_D \int_0^T F(t)\mathrm{d}t + c_T}{T}. \tag{4.14}$$

If $\mu c_D > c_T$, then an optimal T_D^* which minimizes $C_D(T)$ satisfies uniquely

$$TF(T) - \int_0^T F(t)\mathrm{d}t = \frac{c_T}{c_D}, \quad \text{i.e.,} \quad \int_0^T t\mathrm{d}F(t) = \frac{c_T}{c_D}, \tag{4.15}$$

and the resulting cost rate is

$$C_D(T_D^*) = c_D F(T_D^*). \tag{4.16}$$

On the other hand, when $T = \infty$, i.e., the unit is replaced only at time Y is

$$C_D(G) \equiv \lim_{T\to\infty} C_D(T, G) = \frac{c_D \int_0^\infty F(t)\overline{G}(t)\mathrm{d}t + c_R}{\int_0^\infty \overline{G}(t)\mathrm{d}t}. \tag{4.17}$$

By the method similar to Sections 2–4, when $c_T = c_R$, the optimal policy which minimizes $C_D(G)$ is nonrandom, and the expected cost rate is given in (4.16).

Next, when $G(t) = 1 - \mathrm{e}^{-t/\theta}$, the expected cost rate in (4.17) is written as

$$C_D(\theta) = c_D \int_0^\infty \mathrm{e}^{-t/\theta}\mathrm{d}F(t) + \frac{c_R}{\theta}. \tag{4.18}$$

Differentiating $C_D(\theta)$ with respect to θ and setting it equal to zero,

$$\int_0^\infty t\mathrm{e}^{-t/\theta}\mathrm{d}F(t) = \frac{c_R}{c_D}, \tag{4.19}$$

whose left-hand increases strictly with θ from 0 to μ. Thus, if $\mu c_D > c_R$, then there exists a finite and unique θ_D^* $(0 < \theta_D^* < \infty)$ which satisfies (4.19), and the resulting cost rate is

$$C_D(\theta_D^*) = c_D \int_0^\infty \frac{t}{\theta_D^*}\mathrm{e}^{-t/\theta_D^*} F(t)\mathrm{d}t. \tag{4.20}$$

In particular, when $\overline{F}(t) = \sum_{j=0}^{k-1}[(\lambda t)^j/j!]\mathrm{e}^{-\lambda t}$ $(k = 1, 2, \cdots)$, (4.19) becomes

$$k\left(\frac{\lambda\theta}{1+\lambda\theta}\right)^{k+1} = \frac{c_R}{c_D/\lambda}, \tag{4.21}$$

whose left-hand side increases strictly with θ from 0 to k. Therefore, if $k(c_D/\lambda) > c_R$, then there exists a finite and unique θ_D^* $(0 < \theta_D^* < \infty)$ which satisfies (4.21), and the resulting cost rate is

$$C_D(\theta_D^*) = c_D\left(\frac{\lambda\theta_D^*}{1+\lambda\theta_D^*}\right)^k \left(1 + \frac{k}{1+\lambda\theta_D^*}\right). \tag{4.22}$$

Next, when $c_T > c_R$, $G(t) = 1 - \mathrm{e}^{-t/\theta}$, $F(t) = \sum_{j=k}^{\infty}[(\lambda t)^j/j!]\mathrm{e}^{-\lambda t}$ and $k(c_D/\lambda) > c_T$, we compute a modified optimal policy $\widehat{\theta}$ and its modified replacement cost $\widehat{c}_R$ ($\widehat{c}_R < c_T$) in which two optimal cost rates $C_D(T_D^*)$ and $C_D(\widehat{\theta})$ are the same. First, from (4.15), we compute T_D^* which satisfies

$$k \sum_{j=k+1}^{\infty} \frac{(\lambda T)^j}{j!} \mathrm{e}^{-\lambda T} = \frac{c_T}{c_D/\lambda}. \tag{4.23}$$

Using T_D^*, we compute $\widehat{\theta}$ which satisfies

$$\left(\frac{\lambda\widehat{\theta}}{1+\lambda\widehat{\theta}}\right)^k \left(1 + \frac{k}{1+\lambda\widehat{\theta}}\right) = \sum_{j=k}^{\infty} \frac{(\lambda T_D^*)^j}{j!} \mathrm{e}^{-\lambda T_D^*}. \tag{4.24}$$

Using $\widehat{\theta}$, from (4.21),

$$\frac{\widehat{c}_R}{c_D/\lambda} = k\left(\frac{\lambda\widehat{\theta}}{1+\lambda\widehat{\theta}}\right)^{k+1}. \tag{4.25}$$

Table 4 presents optimal T_D^* for c_T/c_D, θ_D^* for c_R/c_D, which satisfies (4.15) and (4.19) when $c_T = c_R$, modified $\widehat{\theta}$, $\widehat{c}_R/c_D$, and $\widehat{c}_R/c_T$ when $F(t) = \sum_{j=k}^{\infty}(t^j/j!)\mathrm{e}^{-t}$ ($k = 2, 3, 4$), which has similar variation trends with Tables 1 and 3.

Table 4: Optimal T_D^*, θ_D^*, $\widehat{\theta}$, $\widehat{c}_R/c_D$ and $\widehat{c}_R/c_T$ when $F(t) = \sum_{j=k}^{\infty}(t^j/j!)\mathrm{e}^{-t}$.

c_T/c_D or c_R/c_D	$k=2$					$k=3$					$k=4$				
	T_D^*	θ_D^*	$\widehat{\theta}$	$\widehat{c}_R/c_D$	$\widehat{c}_R/c_T$	T_D^*	θ_D^*	$\widehat{\theta}$	$\widehat{c}_R/c_D$	$\widehat{c}_R/c_T$	T_D^*	θ_D^*	$\widehat{\theta}$	$\widehat{c}_R/c_D$	$\widehat{c}_R/c_T$
0.1	0.818	0.583	0.399	0.046	0.460	1.195	0.746	0.521	0.041	0.410	1.624	0.916	0.649	0.038	0.380
0.2	1.102	0.866	0.574	0.097	0.485	1.508	1.033	0.700	0.086	0.430	1.970	1.219	0.839	0.079	0.395
0.3	1.331	1.134	0.730	0.150	0.500	1.745	1.285	0.851	0.134	0.446	2.233	1.473	0.993	0.123	0.410
0.4	1.535	1.408	0.884	0.207	0.518	1.946	1.527	0.990	0.184	0.446	2.433	1.710	1.131	0.168	0.420
0.5	1.727	1.702	1.041	0.266	0.532	2.128	1.770	1.125	0.236	0.472	2.617	1.939	1.260	0.216	0.432
0.6	1.914	2.025	1.207	0.327	0.545	2.297	2.091	1.259	0.290	0.483	2.785	2.167	1.386	0.265	0.442
0.7	2.099	2.382	1.385	0.397	0.567	2.457	2.279	1.395	0.345	0.493	2.941	2.398	1.510	0.315	0.450
0.8	2.285	2.800	1.580	0.459	0.574	2.612	2.554	1.535	0.403	0.505	3.090	2.633	1.633	0.367	0.459
0.9	2.476	3.279	1.796	0.530	0.589	2.764	2.847	1.679	0.463	0.514	3.231	2.877	1.758	0.421	0.468
1.0	2.674	3.847	2.040	0.604	0.604	2.913	3.164	1.830	0.525	0.525	3.369	3.130	1.884	0.476	0.476

In general, the results of three periodic replacements in Sections 3 and 4 are summarized as follows [6, p. 12]: The expected cost rate of random and periodic replacement is

$$C(T, G) = \frac{c_i \int_0^T \overline{G}(t)\varphi(t)\mathrm{d}t + c_T + (c_R - c_T)G(T)}{\int_0^T \overline{G}(t)\mathrm{d}t}, \tag{4.26}$$

where $\varphi(t) = h(t)$, $m(t)$, $F(t)$ and $i = c_M, c_F, c_D$. The expected cost rate of periodic replacement is

$$C(T) = \frac{c_i \int_0^T \varphi(t)\mathrm{d}t + c_T}{T}. \tag{4.27}$$

Differentiating $C(T)$ with respect to T and setting it equal to zero,

$$T\varphi(T) - \int_0^T \varphi(t)\mathrm{d}t = \frac{c_T}{c_i} \quad \text{or} \quad \int_0^T t\mathrm{d}\varphi(t) = \frac{c_T}{c_i}. \tag{4.28}$$

If there exists T^* which satisfies (4.28), then the resulting cost rate is

$$C(T^*) = c_i \varphi(T^*). \tag{4.29}$$

The expected cost rate of random replacement is

$$C(G) = \frac{c_i \int_0^\infty \overline{G}(t)\varphi(t)\mathrm{d}t + c_R}{\int_0^\infty \overline{G}(t)\mathrm{d}t}. \tag{4.30}$$

5. Conclusions

We have taken up the random age, periodic and block replacements and compared them with their standard policies. It has been already known that the random replacement is not better than the standard policy with constant time T. However, if the random replacement cost would be lower than that for constant time, the random replacement should be adopted in practical fields. It would be much useful to has been shown that if how much the random replacement cost is lower than that for the standard one, the random replacement is better than the standard policy. From these results, the random replacement should be applied actual systems from economical and environmental viewpoints.

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