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TAIL ASYMPTOTICS FOR CUMULATIVE PROCESSES SAMPLED AT HEAVY-TAILED RANDOM TIMES WITH APPLICATIONS TO QUEUEING MODELS IN MARKOVIAN ENVIRONMENTS

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Abstract This paper considers the tail asymptotics for a cumulative process $\{B(t); t \ge 0\}$ sampled at a heavy-tailed random time T. The main contribution of this paper is to establish several sufficient conditions for the asymptotic equality $\mathsf{P}(B(T) > bx) \sim \mathsf{P}(M(T) > bx) \sim \mathsf{P}(T > x)$ as $x \to \infty$, where $M(t) = \sup_{0 \le u \le t} B(u)$ and b is a certain positive constant. The main results of this paper can be used to obtain the subexponential asymptotics for various queueing models in Markovian environments. As an example, using the main results, we derive subexponential asymptotic formulas for the loss probability of a single-server finite-buffer queue with an on/off arrival process in a Markovian environment.

Keywords: Queue, Markovian environment, heavy-tailed, tail asymptotics, sampling, cumulative process

1. Introduction

The main purpose of this paper is to provide mathematical tools for obtaining the heavy-tailed asymptotic behavior of queueing models in Markovian environments. Many researchers have studied the heavy-tailed asymptotics of the random sum of random variables (r.v.s), and several interesting results have been reported in the literature. However, those results cannot be applied directly to queueing models in Markovian environments, such as queues with batch Markovian arrival processes (BMAPs) [30] and general semi-Markovian arrival processes. Therefore in this paper, we construct a framework to study the heavy-tailed asymptotics for such queueing models.

Let $\{B(t); t \ge 0\}$ denote a (possibly delayed) cumulative process on $\mathbb{R} := (-\infty, \infty)$, where $|B(0)| < \infty$ with probability one (w.p.1) (see, e.g., [46, Section 2.11]). By definition, there exist regenerative points $0 \le \tau_0 < \tau_1 < \tau_2 < \cdots$ such that $\{B(t + \tau_n) - B(\tau_n); t \ge 0\}$ $(n = 0, 1, \dots)$ is stochastically equivalent to $\{B(t + \tau_0) - B(\tau_0); t \ge 0\}$ and is independent of $\{B(u); 0 \le u < \tau_n\}$. Let

$$\Delta B_n = \begin{cases} B(\tau_0), & n = 0, \\ B(\tau_n) - B(\tau_{n-1}), & n = 1, 2, \dots, \end{cases} \qquad \Delta\tau_n = \begin{cases} \tau_0, & n = 0, \\ \tau_n - \tau_{n-1}, & n = 1, 2, \dots, \end{cases} \tag{1.1}$$

$$\Delta B_n^* = \begin{cases} \displaystyle\sup_{0 \le t \le \tau_0} \max(B(t), 0), & n = 0, \\ \displaystyle\sup_{\tau_{n-1} \le t \le \tau_n} B(t) - B(\tau_{n-1}), & n = 1, 2, \dots. \end{cases}$$

Clearly, $\Delta B_n^* \ge \Delta B_n$ for $n = 0, 1, \dots$. Further $\{\Delta\tau_n; n = 1, 2, \dots\}$ (resp. $\{\Delta B_n; n = 1, 2, \dots\}$ and $\{\Delta B_n^*; n = 1, 2, \dots\}$) is a sequence of independent and identically distributed (i.i.d.) r.v.s, which is independent of $\Delta\tau_0$ (resp. ΔB_0 and ΔB_0^*).

Throughout this paper, we assume that

$$\begin{aligned} &\mathsf{P}(0 \le \Delta\tau_n < \infty) = \mathsf{P}(0 \le \Delta B_n^* < \infty) = 1 \quad (n = 0, 1), \\ &\mathsf{E}[|\Delta B_1|] < \infty, \quad 0 < \mathsf{E}[\Delta\tau_1] < \infty, \quad b := \mathsf{E}[\Delta B_1]/\mathsf{E}[\Delta\tau_1] > 0. \end{aligned} \tag{1.2}$$

Under these basic conditions, we study the heavy-tailed asymptotics of $B(T)$, where T is a nonnegative r.v. representing the sampling time of $\{B(t)\}$. More specifically, we establish sufficient conditions for a simple asymptotic formula:

$$\mathsf{P}(B(T) > bx) \stackrel{x}{\sim} \mathsf{P}(M(T) > bx) \stackrel{x}{\sim} \mathsf{P}(T > x), \tag{1.3}$$

where $M(t) = \sup_{0 \le u \le t} B(u)$ for $t \ge 0$, and where for any functions f and g, $f(x) \stackrel{x}{\sim} g(x)$ represents $\lim_{x\to\infty} f(x)/g(x) = 1$ (if the limit holds).

We now give a brief discussion of the conditions for (1.3) to hold. Note that if $\{B(t)\}$ has no deviation, i.e., $B(t) = bt$ for all $t \ge 0$, then $\mathsf{P}(B(T) > bx) = \mathsf{P}(T > x)$. In general, however, $B(t) - bt$ has a deviation from zero, which is caused by the distributions of ΔB_n and $\Delta\tau_n$ $(n = 0, 1)$. Thus $\mathsf{P}(B(T) > bx)$ may be decomposed in an intuitive way:

$$\mathsf{P}(B(T) > bx) \approx \mathsf{P}(T > x) + \big(\text{remainder term associated with } \Delta B_n \text{ and } \Delta\tau_n\big). \tag{1.4}$$

If the remainder term of (1.4) is negligible compared with $\mathsf{P}(T > x)$ as $x \to \infty$, then (1.3) holds. Asmussen et al. [4] show that if T is independent of $\{B(t)\}$, then an important necessary condition for (1.3) is that $\sqrt{T}$ is heavy-tailed, i.e., $\mathsf{P}(T > x) = e^{-o(\sqrt{x})}$, where for any functions f and g, $f(x) = o(g(x))$ represents $\lim_{x\to\infty} f(x)/g(x) = 0$ (if the limit holds). On the other hand, if the remainder term of (1.4) is not negligible, then it is likely that the asymptotic behavior of $\mathsf{P}(B(T) > bx)$ is complicated. Indeed, Asmussen et al. [4] and Foss and Korshunov [15] consider such cases, and they present some asymptotic formulas with implicit functions for two special cumulative processes: the Poisson counting process [4] and the sum of nonnegative r.v.s [15]. Although it is challenging to generalize those results, we leave it for future work. In this paper, we focus on the case where (1.3) holds.

As mentioned at the beginning, this study is motivated by the heavy-tailed asymptotics for queueing models in Markovian environments. A typical example of the application of this study is as follows. Consider a stationary BMAP/GI/1 queue. Suppose that $B(t)$ is the total number of stationary BMAP arrivals in the interval $(0, t]$, which is a cumulative process. Further suppose that T is the service time of one customer and is independent of $\{B(t)\}$. In this setting, b is the arrival rate and $b\mathsf{E}[T]$ is the traffic intensity. Note here (see, e.g., Proposition 3.1 in Masuyama et al. [34]) that the subexponential asymptotics of the stationary queue length L is connected to that of $B(T)$ as follows:

$$\mathsf{P}(L > x) \stackrel{x}{\sim} \frac{1}{1 - b\mathsf{E}[T]} \int_x^\infty \mathsf{P}(B(T) > y)\mathrm{d}y.$$

Therefore, if the subexponential asymptotics of $\mathsf{P}(B(T) > x)$ is given, we can obtain an asymptotic formula for the stationary queue length L. Especially, when (1.3) holds, we have the following simple and explicit formula:

$$\mathsf{P}(L > x) \stackrel{x}{\sim} \frac{b\mathsf{E}[T]}{1 - b\mathsf{E}[T]} \cdot \mathsf{P}(T_\mathrm{e} > x/b),$$

where T_e denotes the equilibrium r.v. of T, i.e., $\mathsf{P}(T_\mathrm{e} \le x) = (1/\mathsf{E}[T]) \int_0^x \mathsf{P}(T > y)\mathrm{d}y$ for $x \ge 0$.

Next we review related work. For this purpose, we introduce two classes of distributions (for details, see Appendix A.1).

Definition 1.1 A nonnegative r.v. X and its distribution function (d.f.) F_X belong to the pth-order long-tailed class $\mathcal{L}^p$ $(p \ge 1)$ if $X^{1/p} \in \mathcal{L}$, i.e., $\mathsf{P}(X^{1/p} > x) > 0$ for all $x \ge 0$ and $\mathsf{P}(X^{1/p} > x + y) \overset{x}{\sim} \mathsf{P}(X^{1/p} > x)$ for some (thus all) $y > 0$. Further if $X \in \mathcal{L}^{1/\theta}$ (resp. $F_X \in \mathcal{L}^{1/\theta}$) for any $0 < \theta \le 1$, we write $X \in \mathcal{L}^\infty$ (resp. $F_X \in \mathcal{L}^\infty$) and call X (resp. F_X) infinite-order long-tailed.

Definition 1.2 A nonnegative r.v. X and its d.f. F_X belong to the consistent variation class $\mathcal{C}$ if $\overline{F}_X(x) > 0$ for all $x \ge 0$ and

$$\lim_{v\downarrow 1}\liminf_{x\to\infty}\frac{\overline{F}_X(vx)}{\overline{F}_X(x)} = 1 \quad \text{or equivalently,} \quad \lim_{v\uparrow 1}\limsup_{x\to\infty}\frac{\overline{F}_X(vx)}{\overline{F}_X(x)} = 1,$$

where $\overline{F}_X(x) = 1 - F_X(x)$ for all $x \in \mathbb{R}$. Note that every distribution with a consistently varying tail is infinite-order long-tailed (i.e., $\mathcal{C} \subset \mathcal{L}^\infty$; see Lemma A.4).

The related work is classified into two cases: (i) T is independent of $\{B(t)\}$; and (ii) T may depend on $\{B(t)\}$. The former is called *independent-sampling case*, and the latter is called *dependent-sampling case*. The dependent-sampling case includes a case where T is a stopping time with respect to $\{B(t)\}$.

To the best of our knowledge, there are a few results for the dependent-sampling case. Robert and Segers [39] consider a special case where

$$B(t) = \sum_{n=1}^{\lfloor t\rfloor} X_n \quad \text{with the } X_n\text{'s being i.i.d. nonnegative r.v.s.} \tag{1.5}$$

Note here that the summation over the empty set is defined as zero, e.g., $\sum_{n=k}^{l} \cdot = 0$ for $k > l$. Thus if (1.5) holds, then

$$\Delta\tau_0 = 0, \quad \Delta B_0 = \Delta B_0^* = 0, \quad \Delta\tau_n = 1, \quad \Delta B_n = \Delta B_n^* = X_n \ (n = 1, 2, \dots). \tag{1.6}$$

For this special case, Robert and Segers [39] present the following:

Proposition 1.1 (Theorem 4.1 in Robert and Segers [39]) *Suppose that* $X, X_1, X_2, \dots$ *are i.i.d. nonnegative r.v.s. Further suppose that (i)* T *satisfies*

$$\lim_{x\to\infty}\frac{\mathsf{P}(T > x + ya(x))}{\mathsf{P}(T > x)} = e^{-y}, \qquad y \in \mathbb{R}, \tag{1.7}$$

for some function $a(x)$ $(x \ge 0)$ *such that* $x^{2/3} = o(a(x))$*; and (ii)* $\mathsf{E}[e^{\gamma X}] < \infty$ *for some* $\gamma > 0$*. Under these conditions, we have*

$$\mathsf{P}(X_1 + \cdots + X_{\lfloor T\rfloor} > \mathsf{E}[X]x) \overset{x}{\sim} \mathsf{P}(T > x). \tag{1.8}$$

Proposition 1.2 (Theorem 3.1 in Robert and Segers [39]) *Suppose that* $X, X_1, X_2, \dots$ *are i.i.d. nonnegative r.v.s. Further suppose that (i)* $T \in \mathcal{C}$*; (ii)* $\mathsf{E}[X^\gamma] < \infty$ *for some* $\gamma > 1$*; and (iii)* $x\mathsf{P}(X > x) = o(\mathsf{P}(T > x))$*. Under these conditions, (1.8) holds.*

Compared with Proposition 1.1, Proposition 1.2 requires a heavier tail of T but relaxes the condition on X, which is implied by (1.4).

For the independent-sampling case, several results have been reported. However, as far as we know, only Jelenković et al. [25] consider the general cumulative process $\{B(t)\}$:

Proposition 1.3 (Proposition 3 in Jelenković et al. [25]) *Suppose that T is independent of $\{B(t); t \ge 0\}$. Further suppose that (i) $T \in \mathcal{L}^2$ (i.e., $\sqrt{T} \in \mathcal{L}$); (ii) $\mathsf{E}[(\Delta\tau_1)^2] < \infty$ and $\Delta B_n \ge 0$ ($n = 0, 1$) w.p.1; and (iii) $\mathsf{E}[\exp\{\eta\sqrt{\Delta B_n^*}\}] < \infty$ ($n = 0, 1$) for some $\eta > 0$. Under these conditions, (1.3) holds.*

According to Jelenković et al. [25]'s result, the condition on ΔB_n^* and thus ΔB_n is insensitive to the tail of T, given that $T \in \mathcal{L}^2$. On the other hand, (1.4) implies that the conditions on ΔB_n and $\Delta\tau_n$ for (1.3) to hold are weaker as the tail of T is heavier. In fact, as with the dependent-sampling case, such a result has been reported by Aleškevičienė et al. [1].

Proposition 1.4 (Theorem 1.2 in Aleškevičienė et al. [1]) *Suppose that $X, X_1, X_2, \dots$ are i.i.d. nonnegative r.v.s and T is independent of $\{X_n; n = 1, 2, \dots\}$. Further suppose that (i) $T \in \mathcal{C}$; (ii) $\mathsf{E}[X] < \infty$; and (iii) $\mathsf{E}[T] < \infty$ and $\mathsf{P}(X > x) = o(\mathsf{P}(T > x))$. Under these conditions, (1.8) holds.*

Note here that Proposition 1.3 does not allow that the tail distribution of X is heavier than $e^{-\eta\sqrt{x}}$; whereas Proposition 1.4 does.

Lin and Shen [29] extend Proposition 1.4 to the case where the X_n's are asymptotically quadrant sub-independent and identically distributed (see Theorem 2.1 (I) therein). Robert and Segers [39] present a theorem result similar to Proposition 1.4 (see Theorem 3.2 therein). The theorem states that (1.8) requires $\mathsf{E}[X^r] < \infty$ for some $r > 1$, which is more restrictive than condition (ii) of Proposition 1.4. However, the theorem also presents a sufficient condition for (1.8) with $\mathsf{E}[T] = \infty$, which is described in the following:

Proposition 1.5 (Theorem 3.2 in Robert and Segers [39]) *Suppose that $X, X_1, X_2, \dots$ are i.i.d. nonnegative r.v.s and T is independent of $\{X_n; n = 1, 2, \dots\}$. Further suppose that (i) $T \in \mathcal{C}$ and $\mathsf{E}[T] = \infty$; (ii) $\mathsf{E}[X^r] < \infty$ for some $r > 1$; and (iii) for some $1 \le q < r$,*

$$\limsup_{x\to\infty} \frac{\mathsf{E}[T \cdot \mathbb{1}(T \le x)]}{x^q \mathsf{P}(T > x)} < \infty,$$

where $\mathbb{1}(\chi)$ denotes the indicator function of event (or condition) χ. Under these conditions, (1.8) holds.

In what follows, we summarize the contributions of this paper. For the dependent-sampling case, we assume that $\{B(t)\}$ is nondecreasing with t (e.g., $\{B(t)\}$ is the counting process of BMAP arrivals). Under this assumption, we present two theorems: Theorems 3.1 and 3.2, which are extensions of Propositions 1.1 and 1.2, respectively, to the general cumulative process. In addition, the two theorems are still more general than the corresponding propositions even if (1.5) holds, i.e., $B(T)$ is reduced to the random sum of i.i.d. nonnegative r.v.s.

As for the independent-sampling case, we do not necessarily assume that $\{B(t)\}$ is nondecreasing with t, which means that ΔB_n can take negative values. We first present two theorems: Theorems 3.3 and 3.4. Theorem 3.3 provides a weaker sufficient condition for (1.3) than that in Proposition 1.3. Theorem 3.4 is an extension of Propositions 1.4 and 1.5 to the general cumulative process. However, unfortunately, when $\{B(t)\}$ satisfies (1.5), one of the conditions of Theorem 3.4 is more restrictive than the corresponding ones of Propositions 1.4 and 1.5. Thus, instead of the general cumulative process, we next consider a special case where $B(t) = B(\lfloor t \rfloor)$ for all $t \ge 0$ and $\{B(n); n = 0, 1, \dots\}$ is the additive component of a discrete-time Markov additive process (see, e.g., [3, Chapter XI, Section 2]), which implies that $B(T)$ is the random sum of r.v.s with Markovian correlation. Under this assumption, we prove Theorems 3.5 and 3.6, which completely include Propositions 1.4 and 1.5 as special cases. Further the two theorems are readily extended to the case where $\{B(t)\}$ is the additive component of a continuous-time Markov additive process.

As mentioned above, our results for the independent-sampling case are more general than those in the literature and thus can be applied to derive new asymptotic formulas for queueing models in Markovian environments. Indeed, Masuyama [32] derives some new subexponential asymptotic formulas for the BMAP/GI/1 queue by using the results of this paper. Masuyama [33] also presents subexponential asymptotic formulas for the BMAP/GI/1 queue with retrials by combining the results of [32] with the subexponential tail equivalence of the queue length distributions of BMAP/GI/1 queues with and without retrials. In addition, unlike the previous studies, our results for the independent-sampling case can be applied to queues with negative customers (see, e.g., [5]) because the results do not necessarily require the monotonicity of $\{B(t)\}$.

To demonstrate the utility of our results for the dependent-sampling case as well as the independent-sampling case, we discuss their application to the subexponential asymptotics of the loss probability of a discrete-time single-server queue with a finite buffer fed by an on/off arrival process in a Markovian environment. In the on/off arrival process, the lengths of on-periods (resp. off-periods) are i.i.d. with a general distribution, and arrivals in each on-period follow a discrete-time BMAP started with some initial distribution at the beginning of the on-period. We call the arrival process *on/off batch Markovian arrival process (ON/OFF-BMAP)*, which is a generalization of the batch-on/off process [17] and is closely related to a platoon arrival process (PAP) [2, 8] (see also Remarks 4.1 and 4.2). For analytical convenience, we assume that service times are all equal to the unit of time. The queueing model is denoted by (ON/OFF-BMAP)/D/1/K in Kendall's notation. For this queue, we derive subexponential asymptotic formulas for the loss probability by combining our results with the existing one on a finite GI/GI/1 queue [21].

The rest of this paper is organized as follows. Section 2 introduce some definitions. Section 3 presents the main results of this paper, and Section 4 discusses their application to the (ON/OFF-BMAP)/D/1/K queue. Appendix A is devoted to technical lemmas. The proofs of all the lemmas and the main results are given in Appendices B and C.

2. Basic Definitions

In this section, we provide the definitions of the subexponential distribution and some related classes of distributions. For later use, we first introduce the following notations. Let C (resp. c) denote a special symbol representing a sufficiently large (resp. small) positive constant, which takes an appropriate value according to the context. Thus C (resp. c) can take different values in different places. For example, C in a place may be equal to $C+1$, $2C$ and C^2, etc. in other places. For any $x \in \mathbb{R}$, let $x^+ = \max(x, 0)$. For any r.v. U in $\mathbb{R}$, let F_U denote the d.f. of U, i.e., $F_U(x) = \mathsf{P}(U \le x)$ for $x \in \mathbb{R}$, which is assumed to be right-continuous. Further let $\overline{F}_U = 1 - F_U$ and $Q_U = -\log \overline{F}_U$. The latter is called the cumulative hazard function of U. Finally, for any nonnegative functions f and g, $f(x) = O(g(x))$, $f(x) \lesssim_x g(x)$ and $f(x) \gtrsim_x g(x)$ represent

$$\limsup_{x\to\infty} f(x)/g(x) < \infty, \qquad \limsup_{x\to\infty} f(x)/g(x) \le 1, \qquad \liminf_{x\to\infty} f(x)/g(x) \ge 1,$$

respectively.

2.1. Subexponential distributions

We begin with the definition of the subexponential class.

Definition 2.1 A nonnegative r.v. X and its d.f. F_X belong to the subexponential class $\mathcal{S}$ if $\mathsf{P}(X > x) > 0$ for all $x \ge 0$ and $\mathsf{P}(X_1 + X_2 > x) \overset{x}{\sim} 2\mathsf{P}(X > x)$, where X_1 and X_2 are independent copies of X.

Remark 2.1 The class $\mathcal{S}$ was first introduced by Chistyakov [10], and it was shown that $\mathcal{S}$ is a strictly subclass of class $\mathcal{L}$, i.e., $\mathcal{S} \subset \mathcal{L}$ (see [38]).

Next we introduce two subclasses of $\mathcal{S}$. The first one is class $\mathcal{S}^*$, which is a well-known subclass of $\mathcal{S}$.

Definition 2.2 A nonnegative r.v. X and its d.f. F_X belong to class $\mathcal{S}^*$ if $\mathsf{E}[X] < \infty$ and

$$\lim_{x\to\infty} \int_0^x \frac{\overline{F}_X(x-y)}{\overline{F}_X(x)} \overline{F}_X(y) \mathrm{d}y = 2\mathsf{E}[X].$$

Remark 2.2 An important property of $\mathcal{S}^*$ is that $F \in \mathcal{S}^*$ implies $F, F_\mathrm{e} \in \mathcal{S}$, where F_e denotes the equilibrium distribution (or integrated tail distribution) of F, i.e., $F_\mathrm{e}(x) = \int_0^x \overline{F}(y)\mathrm{d}y / \int_0^\infty \overline{F}(y)\mathrm{d}y$ for $x \ge 0$ (see [26, Theorem 3.2]).

The second one is the subexponential concave class $\mathcal{SC}$, which is a subclass of $\mathcal{S}^*$, i.e., $\mathcal{SC} \subset \mathcal{S}^*$ (see [40, Lemma 1]). The class $\mathcal{SC}$ plays a key role in establishing large deviation bounds for a cumulative process. The definition of $\mathcal{SC}$ is as follows:

Definition 2.3 A nonnegative r.v. X and its d.f. F_X and cumulative hazard function Q_X belong to the subexponential concave class $\mathcal{SC}$ if the following are satisfied: (i) Q_X is eventually concave; (ii) $\lim_{x\to\infty} Q_X(x)/\log x = \infty$; and (iii) there exist some $0 < \alpha < 1$ and $x_0 > 0$ such that $Q_X(x)/x^\alpha$ is nonincreasing for all $x \ge x_0$, i.e.,

$$\frac{Q_X(x)}{Q_X(u)} \le \left(\frac{x}{u}\right)^\alpha, \qquad x \ge u \ge x_0. \tag{2.1}$$

We may use the notation $\mathcal{SC}_\alpha$ to emphasize the parameter α.

Remark 2.3 Typical examples of the cumulative hazard function in $\mathcal{SC}$ are (i) $(\log x)^\gamma x^\alpha$ and (ii) $(\log x)^\beta$ for sufficiently large x, where $0 < \alpha < 1$, $\beta > 1$ and $\gamma \in \mathbb{R}$.

Remark 2.4 If a nonnegative r.v. X satisfies $\mathsf{E}[e^{Q(X)}] < \infty$ for some cumulative hazard function $Q \in \mathcal{SC}$, then $\mathsf{E}[X^p] < \infty$ for any $p \ge 0$ because $e^{Q(x)} \ge x^p$ for sufficiently large $x > 0$ (see condition (ii) of Definition 2.3).

Appendix A.2 provides some lemmas and further remarks on $\mathcal{SC}$.

2.2. Dominatedly varying distributions

The definition of the dominated variation class is as follows:

Definition 2.4 A nonnegative r.v. X and its d.f. F_X belong to the dominated variation class $\mathcal{D}$ if $\overline{F}_X(x) > 0$ for all $x \ge 0$ and

$$\limsup_{x\to\infty} \frac{\overline{F}_X(vx)}{\overline{F}_X(x)} < \infty,$$

for some (thus for all) $v \in (0, 1)$.

Remark 2.5 $\mathcal{C} \subset \mathcal{L} \cap \mathcal{D} \subset \mathcal{S}^* \subset \mathcal{S}$ (see [26, Theorem 3.2] and [11, 13]).

For any d.f. F, let

$$\overline{F}_*(v) = \liminf_{x\to\infty} \frac{\overline{F}(vx)}{\overline{F}(x)}, \quad \overline{F}^*(v) = \limsup_{x\to\infty} \frac{\overline{F}(vx)}{\overline{F}(x)}, \quad v > 0,$$

and let

$$r_+(F) = -\lim_{v\to\infty} \frac{\log \overline{F}_*(v)}{\log v}, \quad r_-(F) = -\lim_{v\to\infty} \frac{\log \overline{F}^*(v)}{\log v}.$$

Strictly, $r_+(F)$ and $r_-(F)$ are called the upper and lower Matuszewska indices of the function $1/\overline{F}(x)$ on $[0,\infty)$ (see, e.g., Section 2.1 in [6]). For simplicity, however, they are sometimes called the upper and lower Matuszewska indices of d.f. F.

Proposition 2.1 (Proposition 2.2.1 in [6]) *If $F \in \mathcal{D}$, then for any $\alpha_1 < r_-(F)$ and $\alpha_2 > r_+(F)$ there exist positive numbers $x_i > 0$ and $C_i > 0$ $(i = 1, 2)$ such that*

$$\frac{\overline{F}(x)}{\overline{F}(y)} \le C_1 \left(\frac{x}{y}\right)^{-\alpha_1}, \qquad \forall x \ge \forall y \ge x_1,$$
$$\frac{\overline{F}(x)}{\overline{F}(y)} \ge C_2 \left(\frac{x}{y}\right)^{-\alpha_2}, \qquad \forall x \ge \forall y \ge x_2.$$

The second inequality implies that $x^{-\alpha} = o(\overline{F}(x))$ for all $\alpha > r_+(F)$.

3. Main Results

This section consists of three subsections. In subsection 3.1, we present four sets of conditions under which (1.3) holds for the general cumulative process. Unfortunately, the last set of conditions is not completely weaker than the corresponding ones in the literature if $\{B(t)\}$ satisfies (1.5), i.e., $B(T)$ is reduced to the random sum of nonnegative r.v.s. Thus in subsection 3.2, we discuss a special case where $B(t) = B(\lfloor t \rfloor)$ for all $t \ge 0$ and $\{B(n); n = 0, 1, \dots\}$ is the additive component of a discrete-time Markov additive process. For the special case, we have two sets of conditions, which are weaker than the known ones even if $\{B(t)\}$ satisfies (1.5). Finally in subsection 3.3, we extend the results presented in subsection 3.2 to a continuous-time Markov additive process.

3.1. General case

In this subsection, we assume $b = 1$, i.e., $\mathsf{E}[\Delta B_1] = \mathsf{E}[\Delta \tau_1]$ without loss of generality. Indeed, $\{B(t)/b; t \ge 0\}$ is a cumulative process with the same regenerative points as those of $\{B(t)\}$, and the asymptotic equality (1.3) is rewritten as

$$\mathsf{P}(B(T)/b > x) \overset{x}{\sim} \mathsf{P}(M(T)/b > x) \overset{x}{\sim} \mathsf{P}(T > x).$$

In what follows, we first consider the dependent-sampling case and then the independent-sampling case.

3.1.1. Dependent-sampling case

In the dependent-sampling case, we assume that $\{B(t); t \ge 0\}$ is nondecreasing with t. In this case, $M(t) = B(t)$ for all $t \ge 0$ and thus (1.3) is reduced to

$$\mathsf{P}(B(T) > bx) \overset{x}{\sim} \mathsf{P}(T > x).$$

Theorem 3.1 *Suppose that* $\{B(t); t \ge 0\}$ *is nondecreasing with* t. *Further suppose that (i)* $T \in \mathcal{L}^{1/\theta}$ *for some* $0 < \theta \le 1/3$*; and (ii)* $\mathsf{E}[\exp\{Q((-B(0))^+ + \Delta\tau_0)\}] < \infty$, $\mathsf{E}[\exp\{Q(\Delta\tau_1)\}] < \infty$, $\mathsf{E}[\exp\{Q((\Delta B_0)^+)\}] < \infty$ *and* $\mathsf{E}[\exp\{Q(\Delta B_1)\}] < \infty$ *(*$n = 0, 1$*) for some* $Q \in \mathcal{SC}$ *such that*

$$x^{3\theta/2} = O(Q(x)). \tag{3.1}$$

Under these conditions, $\mathsf{P}(B(T) > x) \overset{x}{\sim} \mathsf{P}(T > x)$.

Proof: See Appendix C.1. □

Remark 3.1 We prove Theorem 3.1 by using Lemma A.7 (i) and (ii), which require condition (ii) (see Remark A.5). In addition to the nondecreasingness of $\{B(t)\}$, we assume $B(0) \ge 0$. It then follows that $(-B(0))^+ = 0$ and $(\Delta B_0)^+ = \Delta B_0$. Therefore condition (ii) is reduced to $\mathsf{E}[\exp\{Q(\Delta\tau_n)\}] < \infty$ and $\mathsf{E}[\exp\{Q(\Delta B_n)\}] < \infty$ $(n = 0, 1)$ for some $Q \in \mathcal{SC}$ such that $x^{3\theta/2} = O(Q(x))$.

Theorem 3.1 is a generalization of Proposition 1.1. To compare the two results, we suppose that $\{B(t)\}$ satisfies (1.5). We then have (1.6) and $B(0) = 0$. Therefore conditions (i) and (ii) of Theorem 3.1 are reduced to the following (see Remark 3.1):

(I) $T \in \mathcal{L}^{1/\theta}$ for some $0 < \theta \le 1/3$; and

(II) $\mathsf{E}[\exp\{Q(X)\}] < \infty$ for some $Q \in \mathcal{SC}$ satisfying (3.1).

Condition (I) is equivalent to $T \in \mathcal{L}^3$ (see Lemma A.1 (ii)). On the other hand, condition (i) of Proposition 1.1 implies that T belongs to the maximum domain of attraction of the Gumbel distribution (see, e.g., Theorem 3.3.27 in [12]). It further follows from (1.7) and $x^{2/3} = o(a(x))$ that

$$1 \ge \lim_{x\to\infty} \frac{\mathsf{P}(T > x + x^{2/3})}{\mathsf{P}(T > x)} \ge \lim_{x\to\infty} \frac{\mathsf{P}(T > x + \varepsilon a(x))}{\mathsf{P}(T > x)} = e^{-\varepsilon} \to 1 \qquad \text{as } \varepsilon \to 0,$$

which shows that $T \in \mathcal{L}^3$. Thus condition (I) is weaker than condition (i) of Proposition 1.1. In addition, condition (II) is satisfied by condition (ii) of Proposition 1.1 due to $Q(x) = o(x)$ (see Definition 2.3). As a result, the conditions of Theorem 3.1 are weaker than those of Proposition 1.1.

Theorem 3.2 *Suppose that* $\{B(t); t \ge 0\}$ *is nondecreasing with* t. *Further suppose that (i)* $T \in \mathcal{C}$*; (ii)* $\mathsf{E}[(\Delta\tau_1)^2] < \infty$*; (iii)* $\mathsf{P}(-B(0) > x) = o(\mathsf{P}(T > x))$, $\mathsf{P}(\Delta\tau_n > x) = o(\mathsf{P}(T > x))$ *and* $\mathsf{P}(\Delta B_n > x) = o(\mathsf{P}(T > x))$ *(*$n = 0, 1$*); (iv)* $x\mathsf{P}(|\Delta B_1 - \Delta\tau_1| > x) = o(\mathsf{P}(T > x))$*; and (v) either of the following is satisfied:*

(a) $\mathsf{E}[|\Delta B_1 - \Delta\tau_1|^r] < \infty$ *for some* $r > 1$*; or*

(b) $\int_y^\infty x^{-1}\mathsf{P}(T > x)\mathrm{d}x < \infty$ *for some* $y \in (0, \infty)$.

Under these conditions, $\mathsf{P}(B(T) > x) \overset{x}{\sim} \mathsf{P}(T > x)$.

Proof: See Appendix C.2. □

Remark 3.2 The asymptotic upper bound $\mathsf{P}(B(T) > x) \lesssim_x \mathsf{P}(T > x)$ is proved under the conditions that (iii′) $\mathsf{P}(\Delta B_n > x) = o(\mathsf{P}(T > x))$ $(n = 0, 1)$, (iv′) $x\mathsf{P}(\Delta B_1 - \Delta\tau_1 > x) = o(\mathsf{P}(T > x))$ and (v′) either of the following holds:

(a) $\mathsf{E}[\{(\Delta B_1 - \Delta\tau_1)^+\}^r] < \infty$ for some $r > 1$ or

(b) $\int_y^\infty x^{-1}\mathsf{P}(T > x)\mathrm{d}x < \infty$ for some $y \in (0, \infty)$;

whereas the asymptotic lower bound $\mathsf{P}(B(T) > x) \gtrsim_x \mathsf{P}(T > x)$ is proved under the conditions that (iii″) $\mathsf{P}(-B(0) > x) = o(\mathsf{P}(T > x))$ and $\mathsf{P}(\Delta\tau_n > x) = o(\mathsf{P}(T > x))$ $(n = 0, 1)$, (iv″) $x\mathsf{P}(\Delta\tau_1 - \Delta B_1 > x) = o(\mathsf{P}(T > x))$ and (v″) either of the following holds:

(a) $\mathsf{E}[\{(\Delta\tau_1 - \Delta B_1)^+\}^r] < \infty$ for some $r > 1$ or
(b) $\int_y^\infty x^{-1}\mathsf{P}(T > x)\mathrm{d}x < \infty$ for some $y \in (0, \infty)$.

These conditions are integrated into conditions (iii), (iv) and (v). Further in the proof of the two asymptotic bounds, we use Lemmas A.9, which requires condition (ii).

Theorem 3.2 is a generalization of Proposition 1.2. We compare them, assuming that $\{B(t)\}$ satisfies (1.5). Under this assumption, conditions (i)–(v) of Theorem 3.2 are reduced to the following:

(I) $T \in \mathcal{C}$;
(II) $x\mathsf{P}(X > x) = o(\mathsf{P}(T > x))$; and
(III) either of the following is satisfied:
  (A) $\mathsf{E}[X^r] < \infty$ for some $r > 1$; or
  (B) $\int_y^\infty x^{-1}\mathsf{P}(T > x)\mathrm{d}x < \infty$ for some $y \in (0, \infty)$.

Clearly, the set of conditions (I), (II) and (III.A) is the same as that of conditions (i), (ii) and (iii) of Proposition 1.2. Further the set of conditions (I), (II) and (III.B) does not imply that of conditions (I), (II) and (III.A). In fact, suppose that $\mathsf{P}(T > x) \overset{x}{\sim} (\log x)^{-2}$ and $\mathsf{P}(X > x) \overset{x}{\sim} x^{-1}(\log x)^{-3}$. We then have $T \in \mathcal{C}$ and $x\mathsf{P}(X > x) = o(\mathsf{P}(T > x))$. It also holds that $\mathsf{E}[X] < \infty$ and $\int_y^\infty x^{-1}\mathsf{P}(T > x)\mathrm{d}x < \infty$ for some $y \in (0, \infty)$, which follow from

$$\int_y^\infty \frac{\mathrm{d}x}{x(\log x)^m} = \frac{1}{(m-1)(\log y)^{m-1}}, \quad y > 1,\ m \neq 1.$$

Thus conditions (I), (II) and (III.B) are satisfied. However, condition (III.A) does not hold, i.e., $\mathsf{E}[X^r] = \infty$ for any $r > 1$ because

$$\mathsf{P}(X^r > x) \overset{x}{\sim} \frac{r^3}{x^{1/r}(\log x)^3} \gtrsim_x \frac{r^3}{x^{(1/r)+(r-1)/(2r)}} = \frac{r^3}{x^{(r+1)/(2r)}},$$

where $0 < (r+1)/(2r) < 1$ for $r > 1$.

Consequently, the conditions of Theorem 3.2 are still weaker than those of Proposition 1.2 in the context of the random sum of nonnegative i.i.d. r.v.s.

3.1.2. Independent-sampling case

Theorem 3.3 *Suppose that T is independent of $\{B(t); t \ge 0\}$. Further suppose that (i) $T \in \mathcal{L}^{1/\theta}$ for some $0 < \theta \le 1/2$; (ii) $\mathsf{E}[(\Delta\tau_1)^2] < \infty$ and $\mathsf{E}[(\Delta B_1)^2] < \infty$; and (iii) $\mathsf{E}[\exp\{Q(\Delta B_n^*)\}] < \infty$ $(n = 0, 1)$ for some $Q \in \mathcal{SC}$ such that $x^\theta = O(Q(x))$. Under these conditions, $\mathsf{P}(B(T) > x) \overset{x}{\sim} \mathsf{P}(M(T) > x) \overset{x}{\sim} \mathsf{P}(T > x)$.*

Proof: See Appendix C.3. □

Remark 3.3 We use Lemma A.7 (i) to prove $\mathsf{P}(M(T) > x) \lesssim_x \mathsf{P}(T > x)$. For this purpose, conditions (ii) and (iii) are assumed. Further the proof of $\mathsf{P}(B(T) > x) \gtrsim_x \mathsf{P}(T > x)$ requires the central limit theorem (CLT) for $\{B(t)\}$, which holds under condition (ii) (see, e.g., [3, Chapter VI, Theorem 3.2]).

Theorem 3.3 is a generalization of Proposition 1.3. Condition (i) of Theorem 3.3 is equivalent to condition (i) of Proposition 1.3, i.e., $T \in \mathcal{L}^2$ (see Lemma A.1 (ii)). Condition (ii) of Theorem 3.3 is weaker than the corresponding condition of Proposition 1.3 because the positivity of ΔB_n and condition (iii) of Proposition 1.3 imply $\mathsf{E}[(\Delta B_1)^2] < \infty$ (see Remark 2.4). In addition, if $Q(x) = \eta\sqrt{x}$ for some $\eta > 0$, then condition (iii) of Theorem 3.3 is reduced to condition (iii) of Proposition 1.3. As a results, the conditions of Theorem 3.3 are weaker than those of Proposition 1.3.

Theorem 3.4 *Suppose that T is independent of $\{B(t); t \ge 0\}$. Further suppose that (i) $T \in \mathcal{C}$; (ii) $\mathsf{E}[\sup_{\tau_0 \le t \le \tau_1} |B(t) - B(\tau_0)|] < \infty$ and $\mathsf{E}[(\Delta\tau_1)^2] < \infty$; (iii) $\mathsf{P}(\Delta B_n^* > x) = o(\mathsf{P}(T > x))$ $(n = 0, 1)$; (iv) $x\mathsf{P}(\Delta B_1 - \Delta\tau_1 > x) = o(\mathsf{P}(T > x))$; and (v) either of the following is satisfied:*

(a) $\mathsf{E}[\{(\Delta B_1 - \Delta\tau_1)^+\}^r] < \infty$ for some $r > 1$; or
(b) $\int_y^\infty x^{-1}\mathsf{P}(T > x)\mathrm{d}x < \infty$ for some $y \in (0, \infty)$.

Under these conditions, $\mathsf{P}(B(T) > x) \overset{x}{\sim} \mathsf{P}(M(T) > x) \overset{x}{\sim} \mathsf{P}(T > x)$.

Proof: See Appendix C.4. □

Remark 3.4 We prove the asymptotic upper bound $\mathsf{P}(M(T) > x) \lesssim_x \mathsf{P}(T > x)$ of Theorem 3.4 in a similar way to that of Theorem 3.2. To do this, we require condition (i), $\mathsf{E}[(\Delta\tau_1)^2] < \infty$ and conditions (iii)–(v). On the other hand, we prove the asymptotic lower bound $\mathsf{P}(B(T) > x) \gtrsim_x \mathsf{P}(T > x)$ of Theorem 3.4 by using the strong law of large numbers (SLLN) for $\{B(t)\}$, i.e., $\lim_{t\to\infty} B(t)/t = b$ w.p.1, which requires $\mathsf{E}[\sup_{\tau_0 \le t \le \tau_1} |B(t) - B(\tau_0)|] < \infty$ in condition (ii) (see [3, Chapter VI, Theorem 3.1]).

We make a comparison of Theorem 3.4 with Propositions 1.4 and 1.5. Suppose that $\{B(t)\}$ satisfies (1.5). It then follows that conditions (i)–(v) of Theorem 3.4 are reduced to the following:

(I) $T \in \mathcal{C}$;
(II) $x\mathsf{P}(X > x) = o(\mathsf{P}(T > x))$; and
(III) $\mathsf{E}[X^r] < \infty$ for some $r > 1$ or $\int_y^\infty x^{-1}\mathsf{P}(T > x)\mathrm{d}x < \infty$ for some $y \in (0, \infty)$.

Theorem 3.4 does not necessarily require either the condition $\mathsf{E}[T] < \infty$ of Proposition 1.4 or condition (ii) of Proposition 1.5. On the other hand, Proposition 1.4 does not necessarily require condition (II) (which is obvious). Further we can confirm that condition (II) is not necessary for Proposition 1.5, as follows.

Suppose that $\mathsf{P}(T > x) \overset{x}{\sim} x^{-\alpha}$ for some $0 < \alpha < 1$. In this case, $\mathsf{E}[T] = \infty$ and $T \in \mathcal{C}$ (see Appendix A.3), which shows that condition (i) of Proposition 1.5 is satisfied. In addition, $\mathsf{E}[T \cdot 1\!\!1(T \le x)] = O(x\mathsf{P}(T > x))$ (see Remark below Theorem 3.2 in [39]). Therefore condition (iii) of Proposition 1.5 holds for $q = 1$. We now assume that $\mathsf{P}(X > x) = (x+1)^{-\beta}$ for some $1 < \beta < \alpha + 1$. We then have $\mathsf{E}[X^r] < \infty$ for all $r < \beta$, and thus condition (ii) of Proposition 1.5 is satisfied. As a result, all the conditions of Proposition 1.5 hold, whereas condition (II) does not hold.

The above discussion shows that Theorem 3.4 is not a complete generalization of Propositions 1.4 and 1.5.

3.2. Special case: discrete-time Markov additive process

In this subsection, we extend Propositions 1.4 and 1.5 to the random sum of (possibly negative) r.v.s with Markovian correlation. For this purpose, we introduce a discrete-time Markov additive process.

Let $\{J_n; n = 0, 1, \dots\}$ is a discrete-time Markov chain with a finite state space $\mathbb{D} := \{0, 1, \dots, d-1\}$. Let X_n's ($n = 0, 1, \dots$) denote r.v.s such that for all $i, j \in \mathbb{D}$ and $x \in \mathbb{R}$,

$$\mathsf{P}(X_0 \le x, J_0 = i) = \beta_i(x),$$
$$\mathsf{P}(X_{n+1} \le x, J_{n+1} = j \mid J_n = i) = H_{i,j}(x), \quad n = 0, 1, \dots,$$

where $\sum_{i\in\mathbb{D}} \beta_i(\infty) = 1$ and $\sum_{j\in\mathbb{D}} H_{i,j}(\infty) = 1$ for all $i \in \mathbb{D}$. Let $S_n = \sum_{\nu=0}^{n} X_\nu$ for $n = 0, 1, \dots$. It then follows that $\{(S_n, J_n); n = 0, 1, \dots\}$ is a Markov additive process with initial distribution $\boldsymbol{\beta}(x) = (\beta_i(x))_{i\in\mathbb{D}}$ and Markov additive kernel (called "kernel" for short) $\boldsymbol{H}(x) = (H_{i,j}(x))_{i,j\in\mathbb{D}}$ ($x \in \mathbb{R}$). Further let $\widehat{\boldsymbol{\beta}}(\xi)$ and $\widehat{\boldsymbol{H}}(\xi)$ denote the characteristic functions of $\boldsymbol{\beta}(x)$ and $\boldsymbol{H}(x)$, i.e.,

$$\widehat{\boldsymbol{\beta}}(\xi) = \int_{x\in\mathbb{R}} e^{\mathrm{i}\xi x} \mathrm{d}\boldsymbol{\beta}(x), \qquad \widehat{\boldsymbol{H}}(\xi) = \int_{x\in\mathbb{R}} e^{\mathrm{i}\xi x} \mathrm{d}\boldsymbol{H}(x),$$

respectively, where $\mathrm{i} = \sqrt{-1}$.

In what follows, we make the following assumption:

Assumption 3.1 (i) Let $B(t) = S_{\lfloor t\rfloor} = \sum_{n=0}^{\lfloor t\rfloor} X_n$ for $t \ge 0$;

(ii) the background process $\{J_n\}$ is irreducible, i.e., $\boldsymbol{H}(\infty)$ is an irreducible stochastic matrix; and

(iii) the mean drift of the additive component $\{S_n\}$ is finite and positive, i.e.,

$$h := \boldsymbol{\varpi} \int_{x\in\mathbb{R}} x \mathrm{d}\boldsymbol{H}(x)\boldsymbol{e} \in (0, \infty), \tag{3.2}$$

where $\boldsymbol{\varpi} = (\varpi_i)_{i\in\mathbb{D}}$ is the stationary probability vector of $\boldsymbol{H}(\infty)$, and where $\boldsymbol{e}$ is a column vector of ones with an appropriate dimension.

It is easy to see that $\{B(t); t \ge 0\}$ is a cumulative process because $\{(B(n), J_n); n = 0, 1, \dots\}$ is a discrete-time Markov additive process. Let $0 \le \tau_0 < \tau_1 < \cdots$ denote hitting times of $\{J_n\}$ to state zero, which are regenerative points of the cumulative process $\{B(t)\}$. Clearly, $\Delta\tau_1 \ge 1$ w.p.1. Further from (1.1), we have $\tau_0 = \Delta\tau_0$ and thus $\mathsf{P}(\Delta\tau_0 = 0) = \mathsf{P}(J_0 = 0) = \beta_0(\infty)$.

Let $\widehat{\psi}_0(z,\xi) = \mathsf{E}[z^{\Delta\tau_0} e^{\mathrm{i}\xi\Delta B_0}]$ and $\widehat{\psi}_1(z,\xi) = \mathsf{E}[z^{\Delta\tau_1} e^{\mathrm{i}\xi\Delta B_1}]$. We then have

$$\widehat{\psi}_0(z,\xi) = \widehat{\beta}_0(\xi) + \widehat{\boldsymbol{\beta}}_+(\xi) \left(\boldsymbol{I} - z\widehat{\boldsymbol{H}}_+(\xi)\right)^{-1} z\widehat{\boldsymbol{h}}_+(\xi), \tag{3.3}$$

$$\widehat{\psi}_1(z,\xi) = z\widehat{H}_{0,0}(\xi) + z\widehat{\boldsymbol{\eta}}_+(\xi) \left(\boldsymbol{I} - z\widehat{\boldsymbol{H}}_+(\xi)\right)^{-1} z\widehat{\boldsymbol{h}}_+(\xi), \tag{3.4}$$

where $\boldsymbol{I}$ denotes the identity matrix with an appropriate dimension and

$$\widehat{\boldsymbol{\beta}}(\xi) = \begin{matrix} \{0\} & \mathbb{D}\setminus\{0\} \\ \big(\ \widehat{\beta}_0(\xi) & \widehat{\boldsymbol{\beta}}_+(\xi)\ \big) \end{matrix}, \quad \widehat{\boldsymbol{H}}(\xi) = \begin{matrix} & \begin{matrix} \{0\} & \mathbb{D}\setminus\{0\} \end{matrix} \\ \begin{matrix} \{0\} \\ \mathbb{D}\setminus\{0\} \end{matrix} & \begin{pmatrix} \widehat{H}_{0,0}(\xi) & \widehat{\boldsymbol{\eta}}_+(\xi) \\ \widehat{\boldsymbol{h}}_+(\xi) & \widehat{\boldsymbol{H}}_+(\xi) \end{pmatrix} \end{matrix}.$$

The first term of (3.3) corresponds to the event where $J_0 = 0$ and thus $\Delta\tau_0 = 0$. The first term of (3.4) corresponds to the event where a regenerative cycle lasts only for a unit of time, i.e., the background process $\{J_n\}$ moves from state zero to state zero in one transition. As for the second terms of (3.3) and (3.4), they correspond to the events where $\{J_n\}$ moves from state zero to a state in $\mathbb{D} \setminus \{0\}$ and then eventually returns to state zero.

Fixing $\xi = 0$ in (3.3) and (3.4) and taking the inverse of them with respect to z, we have

$$\begin{aligned} \mathsf{P}(\Delta\tau_0 = k) &= \mathbb{1}(k = 0)\widehat{\beta}_0(0) + \mathbb{1}(k \ge 1)\widehat{\boldsymbol{\beta}}_+(0)\left(\widehat{\boldsymbol{H}}_+(0)\right)^{k-1}\widehat{\boldsymbol{h}}_+(0), \\ \mathsf{P}(\Delta\tau_1 = k) &= \mathbb{1}(k = 1)\widehat{H}_{0,0}(0) + \mathbb{1}(k \ge 2)\widehat{\boldsymbol{\eta}}_+(0)\left(\widehat{\boldsymbol{H}}_+(0)\right)^{k-2}\widehat{\boldsymbol{h}}_+(0), \end{aligned} \tag{3.5}$$

for $k = 0, 1, 2, \dots$. Therefore $\Delta\tau_0$ and $\Delta\tau_1$ follow discrete phase-type distributions [28]. Further we have the following result by combining the renewal reward theory (see, e.g., [46, Chapter 2, Theorem 2]) and the discrete-time version of the ergodic theorem (see, e.g., [7, Chapter 3, Theorem 4.1]):

Proposition 3.1 *Under Assumption 3.1,*

$$b := \frac{\mathsf{E}[\Delta B_1]}{\mathsf{E}[\Delta\tau_1]} = \boldsymbol{\varpi}\int_{x\in\mathbb{R}} x\mathrm{d}\boldsymbol{H}(x)\boldsymbol{e} = h \in (0, \infty).$$

In what follows, we present two theorems that supersede Propositions 1.4 and 1.5. Before doing this, we introduce three lemmas for the proofs of the theorems.

Lemma 3.1 *Suppose that Assumptions 3.1 holds. Further let* $\overline{\boldsymbol{\beta}}(x) = \int_x^\infty \mathrm{d}\boldsymbol{\beta}(y)$ *and* $\overline{\boldsymbol{H}}(x) = \int_x^\infty \mathrm{d}\boldsymbol{H}(y)$ *for* $x \in \mathbb{R}$ *and suppose that there exist some* $\tilde{c} \in [0, \infty)$ *and some nonnegative r.v.* $Y \in \mathcal{S}$ *such that*

$$\limsup_{x\to\infty} \frac{\overline{\boldsymbol{\beta}}(x)}{\mathsf{P}(Y > x)} \le \tilde{c}\widetilde{\boldsymbol{\beta}}, \qquad \limsup_{x\to\infty} \frac{\overline{\boldsymbol{H}}(x)}{\mathsf{P}(Y > x)} \le \tilde{c}\widetilde{\boldsymbol{H}},$$

where $\widetilde{\boldsymbol{\beta}} = (\widetilde{\beta}_i)_{i\in\mathbb{D}}$ *is a finite nonnegative vector and* $\widetilde{\boldsymbol{H}} = (\widetilde{H}_{i,j})_{i,j\in\mathbb{D}}$ *is a finite nonnegative matrix. We then have*

$$\limsup_{x\to\infty} \frac{\mathsf{P}(\Delta B_n > x)}{\mathsf{P}(Y > x)} \le \tilde{c}C, \quad n = 0, 1.$$

Proof: See Appendix C.5. □

Lemma 3.2 *If the assumptions of Lemma 3.1 are satisfied, then*

$$\limsup_{x\to\infty} \frac{\mathsf{P}(\Delta B_1 > x \mid \Delta\tau_1 = k)}{\mathsf{P}(Y > x)} \le \tilde{c}Ck, \qquad \forall k = 1, 2, \dots,$$

where C *is independent of* k.

Proof: See Appendix C.6. □

Lemma 3.3 *If the assumptions of Lemma 3.1 are satisfied, then for all* $t \ge 0$ *and* $m = 0, 1, \dots,$

$$\limsup_{x\to\infty} \frac{\mathsf{P}\left(\sum_{i=1}^m \Delta B_i > x \mid N(t) = m\right)}{\mathsf{P}(Y > x)} \le \tilde{c}Ct, \tag{3.6}$$

where $N(t) = \max\{k \ge 0; \sum_{i=1}^k \Delta\tau_i \le t\}$ *for* $t \in \mathbb{R}$.

Proof: See Appendix C.7. □

The following theorems present two sets of conditions for (1.3). Note here that under Assumption 3.1, the asymptotic equality (1.3) is reduced to $\mathsf{P}(S_{\lfloor T\rfloor} > hx) \overset{x}{\sim} \mathsf{P}(M_{\lfloor T\rfloor} > hx) \overset{x}{\sim} \mathsf{P}(T > x)$, where $M_n = \max_{0\le k\le n} S_k$.

Theorem 3.5 *Suppose that Assumption 3.1 holds and T is independent of the Markov additive process $\{(S_n, J_n)\}$. Further suppose that $T \in \mathcal{C}$, $\mathsf{E}[T] < \infty$ and*

$$\int_{|y|>x} \mathrm{d}\boldsymbol{\beta}(y) = o(\mathsf{P}(T > x)), \qquad \int_{|y|>x} \mathrm{d}\boldsymbol{H}(y) = o(\mathsf{P}(T > x)). \tag{3.7}$$

Under these conditions, $\mathsf{P}(S_{\lfloor T\rfloor} > hx) \overset{x}{\sim} \mathsf{P}(M_{\lfloor T\rfloor} > hx) \overset{x}{\sim} \mathsf{P}(T > x)$.

Proof: See Appendix C.8. □

Theorem 3.6 *Suppose that Assumption 3.1 holds and T is independent of the Markov additive process $\{(S_n, J_n)\}$. Further suppose that $T \in \mathcal{C}$ and there exists some nonnegative r.v. $Y \in \mathcal{S}$ such that*

$$\int_{|y|>x} \mathrm{d}\boldsymbol{\beta}(y) = O(\mathsf{P}(Y > x)), \qquad \int_{|y|>x} \mathrm{d}\boldsymbol{H}(y) = O(\mathsf{P}(Y > x)), \tag{3.8}$$

$$\lim_{x\to\infty} \mathsf{E}[T \cdot \mathbb{1}(T \le x, N(T) \le x/\mathsf{E}[\Delta\tau_1])] \frac{\mathsf{P}(Y > x)}{\mathsf{P}(T > x)} = 0. \tag{3.9}$$

Under these conditions, $\mathsf{P}(S_{\lfloor T\rfloor} > hx) \overset{x}{\sim} \mathsf{P}(M_{\lfloor T\rfloor} > hx) \overset{x}{\sim} \mathsf{P}(T > x)$.

Proof: See Appendix C.9. □

Remark 3.5 Equation (3.9) holds if

$$\lim_{x\to\infty} \mathsf{E}[T \cdot \mathbb{1}(T \le x)] \frac{\mathsf{P}(Y > x)}{\mathsf{P}(T > x)} = 0.$$

It is easy to see that Theorems 3.5 and 3.6 include Propositions 1.4 and 1.5, respectively, as special cases.

3.3. Special case: continuous-time Markov additive process

In this subsection, we consider a continuous-time Markov additive process $\{(B(t), J(t)); t \ge 0\}$ with state space $\mathbb{R} \times \mathbb{D}$, where $\{B(t)\}$ is the additive component and $\{J(t)\}$ is the background process. Let $\boldsymbol{D}(x) = (D_{i,j}(x))_{i,j\in\mathbb{D}}$ $(x \in \mathbb{R})$ denote the kernel of $\{(B(t), J(t))\}$ such that $\boldsymbol{D}(x) \ge \boldsymbol{O}$ for all $x < 0$ and $\boldsymbol{D}(x) - \boldsymbol{D}(0) \ge \boldsymbol{O}$ for all $x \ge 0$, where $\boldsymbol{O}$ denotes the zero matrix. Further for later use, let $\widehat{\boldsymbol{D}}(\xi) = \int_{x\in\mathbb{R}} e^{\mathrm{i}\xi x}\mathrm{d}\boldsymbol{D}(x)$ and $[\,\cdot\,]_{i,j}$ denote the (i,j)th element of the matrix between square brackets.

In what follows, we make the following assumption:

Assumption 3.2 (i) For all $t \ge 0$,

$$\mathsf{E}[\exp\{\mathrm{i}\xi B(t)\} \cdot \mathbb{1}(J(t) = j) \mid J(0) = i] = \left[\exp\{\widehat{\boldsymbol{D}}(\xi)t\}\right]_{i,j}, \quad i, j \in \mathbb{D}; \tag{3.10}$$

(ii) $\widehat{\boldsymbol{D}}(0) = \boldsymbol{D}(\infty)$ is an irreducible infinitesimal generator; and (iii) $\boldsymbol{\pi} \int_{x\in\mathbb{R}} x\mathrm{d}\boldsymbol{D}(x)\boldsymbol{e} \in (0,\infty)$, where $\boldsymbol{\pi} = (\pi_i)_{i\in\mathbb{D}}$ denotes the stationary probability vector of $\widehat{\boldsymbol{D}}(0)$.

Under Assumption 3.2, $\{B(t)\}$ is a cumulative process. It thus follows from the renewal reward theory (see, e.g., [46, Chapter 2, Theorem 2]) and the continuous-time version of the ergodic theorem (see, e.g., [7, Chapter 8, Theorem 6.2]) that

$$b := \frac{\mathsf{E}[\Delta B_1]}{\mathsf{E}[\Delta \tau_1]} = \boldsymbol{\pi} \int_{x\in\mathbb{R}} x \mathrm{d}\boldsymbol{D}(x)\boldsymbol{e} \in (0,\infty).$$

Further it follows from (3.10) that

$$\begin{aligned}
&\mathsf{E}[\exp\{\mathrm{i}\xi B(T)\} \cdot \mathbb{1}(J(T)=j) \mid J(0)=i] \\
&= \left[\int_0^\infty \exp\{\widehat{\boldsymbol{D}}(\xi)t\}\mathrm{d}\mathsf{P}(T\le t)\right]_{i,j} \\
&= \sum_{n=0}^\infty \int_0^\infty e^{-\gamma t}\frac{(\gamma t)^n}{n!}\mathrm{d}\mathsf{P}(T\le t)\cdot\left[\left\{\boldsymbol{I}+\gamma^{-1}\widehat{\boldsymbol{D}}(\xi)\right\}^n\right]_{i,j} \\
&= \sum_{n=0}^\infty p_n \cdot \left[\{\widehat{\boldsymbol{K}}(\xi)\}^n\right]_{i,j}, \qquad (3.11)
\end{aligned}$$

where

$$p_n = \int_0^\infty e^{-\gamma t}\frac{(\gamma t)^n}{n!}\mathrm{d}\mathsf{P}(T\le t) \quad (n=0,1,\dots), \quad \widehat{\boldsymbol{K}}(\xi) = \boldsymbol{I}+\gamma^{-1}\widehat{\boldsymbol{D}}(\xi), \quad \gamma = \max_{i\in\mathbb{D}}|D_{i,i}(\infty)|.$$

Note here that $\widehat{\boldsymbol{K}}(\xi)$ is the characteristic function of $\boldsymbol{K}(x) := \mathbb{1}(x\ge 0)\boldsymbol{I}+\gamma^{-1}\boldsymbol{D}(x)$ $(x\in\mathbb{R})$, which can be considered as the kernel of a discrete-time Markov additive process $\{(S_n,J_n)\}$ discussed in the previous subsection. Note also that $\{p_n; n=0,1,\dots\}$ is the distribution of the counting process of Poisson arrivals with rate γ during time interval $(0,T]$. It is easy to see that if $T\in\mathcal{C}$, then the counting process satisfies all the conditions of Theorem 3.4 and thus

$$\sum_{n=k+1}^\infty p_n \overset{k}{\sim} \mathsf{P}(T > k/\gamma).$$

We now define T' as a nonnegative integer-valued r.v. such that $\mathsf{P}(T'=n)=p_n$ $(n = 0,1,\dots)$ and T' is independent of a discrete-time Markov additive process $\{(S_n,J_n)\}$ with initial condition $S_0=X_0=0$ (i.e., $\int_{\{0\}}\mathrm{d}\boldsymbol{\beta}(x)\boldsymbol{e}=1$) and kernel $\boldsymbol{H}(x)=\boldsymbol{K}(x)$ $(x\in\mathbb{R})$. It then follows from (3.11) that $(B(T),J(T))$ is stochastically equivalent to $\{(S_{T'},J_{T'})\}$. As a result, using Theorems 3.5 and 3.6, we can readily prove the following corollaries, whose proofs are omitted.

Corollary 3.1 *Suppose that Assumption 3.2 holds and T is independent of the Markov additive process $\{(B(t),J(t))\}$. Further suppose that $T\in\mathcal{C}$, $\mathsf{E}[T]<\infty$ and*

$$\int_{|y|>x}\mathrm{d}\boldsymbol{D}(y) = o(\mathsf{P}(T>x)).$$

Under these conditions, $\mathsf{P}(B(T)>bx) \overset{x}{\sim} \mathsf{P}(M(T)>bx) \overset{x}{\sim} \mathsf{P}(T>x)$.

Corollary 3.2 *Suppose that Assumption 3.2 holds and T is independent of the Markov additive process $\{(B(t),J(t))\}$. Further suppose that $T\in\mathcal{C}$ and there exists some nonnegative r.v. $Y\in\mathcal{S}$ such that*

$$\int_{|y|>x}\mathrm{d}\boldsymbol{D}(y) = O(\mathsf{P}(Y>x)), \qquad \lim_{x\to\infty}\mathsf{E}[T\cdot\mathbb{1}(T\le x)]\frac{\mathsf{P}(Y>x)}{\mathsf{P}(T>x)} = 0.$$

Under these conditions, $\mathsf{P}(B(T) > bx) \overset{x}{\sim} \mathsf{P}(M(T) > bx) \overset{x}{\sim} \mathsf{P}(T > x)$.

4. Application

In this section, we first introduce a new (discrete-time) on/off arrival process, ON/OFF-BMAP, mentioned in Section 1. We then consider a single-server finite-buffer queue with an ON/OFF-BMAP and deterministic service times. For this queueing model, we derive some subexponential asymptotic formulas for the loss probability by using the main results presented in Section 3.

4.1. ON/OFF batch Markovian arrival process

We describe the definition of ON/OFF-BMAPs in discrete time. The time interval $[n, n+1]$ ($n \in \mathbb{Z}$) is called slot n, where $\mathbb{Z} = \{0, \pm 1, \pm 2 \dots\}$. The ON/OFF-BMAP is an on/off arrival process, where on-periods and off-periods are repeated alternately. For simplicity, slots in on-periods (resp. off-periods) are called *on-slots* (resp. *off-slots*).

The lengths of off-periods are i.i.d., and no arrivals occur in any off-slot. On the other hand, at least one arrival occurs in each on-slot w.p.1 and the number of arrivals during each on-period follows a BMAP started with some initial distribution at the beginning of the on-period. Further the lengths of on-periods are i.i.d., but the length of each on-period may depend on the BMAP in the on-period. In what follows, the BMAP in the mth ($m \in \mathbb{Z}$) on-period is called the mth BMAP.

To describe the ON/OFF-BMAP more precisely, we define some notations. Let $N_{m,n}$ ($m \in \mathbb{Z}$, $n = 0, 1, \dots$) denote the number of arrivals in the nth slot of the mth on-period. Let $J_{m,0}, J_{m,1}, J_{m,2}, \dots$ ($m \in \mathbb{Z}$) denote the background states of the mth BMAP, which belong to $\mathbb{D} = \{0, 1, \dots, d-1\}$. We then assume that

$$\mathsf{P}(N_{m,0} = k, J_{m,0} = i) = \alpha_i(k), \qquad i \in \mathbb{D},\ k = 1, 2, \dots, \tag{4.1}$$

where $\boldsymbol{\alpha}(k) = (\alpha_i(k))_{i \in \mathbb{D}}$ is a $1 \times d$ nonnegative vector such that $\underline{\boldsymbol{\alpha}} := \sum_{k=1}^{\infty} \boldsymbol{\alpha}(k)$ is a probability vector. We also assume that for $n = 1, 2, \dots$,

$$\mathsf{P}(N_{m,n} = k, J_{m,n} = j \mid J_{m,n-1} = i) = \Lambda_{i,j}(k), \qquad i, j \in \mathbb{D},\ k = 1, 2, \dots, \tag{4.2}$$

where $\boldsymbol{\Lambda}(k) = (\Lambda_{i,j}(k))_{i,j \in \mathbb{D}}$ is a $d \times d$ substochastic matrix such that $\underline{\boldsymbol{\Lambda}} := \sum_{k=1}^{\infty} \boldsymbol{\Lambda}(k)$ is an irreducible stochastic matrix.

Let I_m^{on} ($m \in \mathbb{Z}$) denote the length of the mth on-period. Let Φ_m ($m \in \mathbb{Z}$) denote

$$\Phi_m = \{I_m^{\mathrm{on}}, (N_{m,0}, J_{m,0}), (N_{m,1}, J_{m,1}), \dots, (N_{m,I_m^{\mathrm{on}}-1}, J_{m,I_m^{\mathrm{on}}-1})\}. \tag{4.3}$$

We then assume that the Φ_m's ($m \in \mathbb{Z}$) are i.i.d. Thus the I_m^{on}'s ($m \in \mathbb{Z}$) are i.i.d. r.v.s, though each I_m^{on} may depend on the mth BMAP, i.e., $\{(N_{m,n}, J_{m,n}); n = 0, 1, \dots\}$.

For later use, let λ denote the arrival rate during on periods, i.e., the time-average number of arrivals in an on-slot. It follows from the ergodic theorem (see, e.g., [7, Chapter 3, Theorem 4.1]) that

$$\lambda = \boldsymbol{\phi} \sum_{k=1}^{\infty} k \boldsymbol{\Lambda}(k) \boldsymbol{e} \ge 1, \tag{4.4}$$

where $\boldsymbol{\phi} = (\phi_i)_{i \in \mathbb{D}}$ denotes the stationary probability vector of $\underline{\boldsymbol{\Lambda}}$.

Remark 4.1 The ON/OFF-BMAP is a generalization of the batch-on/off process introduced by Galmés and Puigjaner [17]. In the batch-on/off process, the numbers of arrivals in individual on-slots are i.i.d. and independent of the lengths of on-periods. Based on the Wiener-Hopf factorization (see, e.g., [3, Chapter VIII, Section 3]), Galmés and Puigjaner [18, 19] study the response time distribution of a single-server queue with a batch-on/off process and deterministic service times.

Remark 4.2 The ON/OFF-BMAP is similar to the PAP proposed by Alfa and Neuts [2] and Breuer and Alfa [8]. The PAP can be considered as a special case of the ON/OFF-BMAP in the sense that the lengths of on-periods (resp. off-periods) follow a phase-type distribution. However, the PAP allows that no arrivals occur in an *on-slot*.

4.2. Loss probability of (ON/OFF-BMAP)/D/1/K queue

We begin with the description of our queueing model. Customers arrive at the system according to an ON/OFF-BMAP. The system has a single server and a buffer of finite capacity $K-1$ (thus the system capacity is equal to K). The service times of customers are all equal to the length of one slot. According to Kendall's notation, our queueing model is denoted by (ON/OFF-BMAP)/D/1/K.

For analytical convenience, we assume that arrivals in each on-slot occur at the same time, immediately after the beginning of the on-slot. We also assume that departure points are located immediately before the ends of slots. Under these assumptions, we observe the queue length process immediately after the ends of off-periods.

Let $L_m^{(K)}$ ($m \in \mathbb{Z}$) denote the queue length immediately after the end of the mth off-period. Let $I_m^{\rm off}$ ($m \in \mathbb{Z}$) denote the length of the mth off-period, where the $I_m^{\rm off}$'s are i.i.d. r.v.s. Further let A_m ($m \in \mathbb{Z}$) denote the increment in the queue length during the mth on-period, i.e.,

$$A_m = \sum_{n=0}^{I_m^{\rm on}-1} (N_{m,n} - 1), \tag{4.5}$$

where the A_m's are i.i.d. r.v.s because the Φ_m's in (4.3) are i.i.d. We then have

$$L_{m+1}^{(K)} = (\min(L_m^{(K)} + A_{m+1}, K) - I_{m+1}^{\rm off})^+.$$

We now define $P_{\rm loss}^{(K)}$ as the loss probability, which is the time-average of losses. Note that in the mth renewal cycle consisting of the mth on- and off-periods, the numbers of arrivals and losses are equal to $A_m + I_m^{\rm on}$ and $(L_{m-1}^{(K)} + A_m - K)^+$, respectively. It then follows from the renewal reward theory (see, e.g., [46, Chapter 2, Theorem 2]) that

$$P_{\rm loss}^{(K)} = \frac{\mathsf{E}[(L_{m-1}^{(K)} + A_m - K)^+]}{\mathsf{E}[A_m + I_m^{\rm on}]}.$$

4.3. Subexponential asymptotics of the loss probability

In this subsection, we derive some subexponential asymptotic formulas for the loss probability $P_{\rm loss}^{(K)}$. To achieve this, we combine our main results with the following proposition:

Proposition 4.1 (Theorem 5 in [21]) *Let A, $I^{\rm on}$ and $I^{\rm off}$ denote generic r.v.s for i.i.d. sequences $\{A_m\}$, $\{I_m^{\rm on}\}$ and $\{I_m^{\rm off}\}$, respectively. Suppose $0 < \mathsf{E}[A] < \infty$ and let $A_{\rm e}$ denote the equilibrium r.v. of A, i.e., $\mathsf{P}(A_{\rm e} \le x) = (1/\mathsf{E}[A]) \int_0^x \mathsf{P}(A > y) {\rm d}y$ for $x \ge 0$. If $\mathsf{E}[A] < \mathsf{E}[I^{\rm off}]$ and $A_{\rm e} \in \mathcal{S}$, then*

$$P_{\rm loss}^{(K)} \overset{K}{\sim} \frac{\mathsf{E}[(A-K)^+]}{\mathsf{E}[A] + \mathsf{E}[I^{\rm on}]} = \frac{\mathsf{E}[A]}{\mathsf{E}[A] + \mathsf{E}[I^{\rm on}]} \mathsf{P}(A_{\rm e} > K).$$

In the rest of this subsection, we set

$$T = I_m^{\rm on} - 1, \quad B(t) = \sum_{n=0}^{\lfloor t \rfloor}(N_{m,n} - 1), \quad t \ge 0, \tag{4.6}$$

where $N_{m,n} - 1$ for $m \in \mathbb{Z}$ and $n = 0, 1, \dots$ due to (4.1) and (4.2). It then follows from (4.5) and (4.6) that

$$A \overset{\rm d}{=} B(T), \tag{4.7}$$

where the symbol $\overset{\rm d}{=}$ denotes equality in distribution.

For simplicity, let $X_n = N_{m,n} - 1 \ge 0$ and $J_n = J_{m,n}$ for $n = 0, 1, \dots$. Further let $S_n = \sum_{\nu=0}^{n} X_\nu$ for $n = 0, 1, \dots$. It then follows that $B(t) = S_{\lfloor t \rfloor}$ for $t \ge 0$ and $\{(S_n, J_n); n = 0, 1, \dots\}$ is a Markov additive process with state space $\{0, 1, \dots\} \times \mathbb{D}$, initial distribution $\boldsymbol{\alpha}(k)$ and Markov additive kernel $\boldsymbol{\Lambda}(k + 1)$ $(k = 0, 1, \dots)$. Thus the stochastic process $\{B(t)\}$ defined in (4.6) is a cumulative process of the same type as that in subsection 3.2. As with subsection 3.2, let $0 \le \tau_0 < \tau_1 < \cdots$ denote hitting times of $\{J_n\}$ to state zero, which are regenerative points of $\{B(t)\}$. From (4.4) and Proposition 3.1, we have

$$b := \mathsf{E}[\Delta B_1] \big/ \mathsf{E}[\Delta \tau_1] = \lambda - 1. \tag{4.8}$$

We now assume the following:

Assumption 4.1 $\lambda > 1$ and there exists some nonnegative r.v. $Y \in \mathcal{S}$ such that

$$\sum_{l=k+1}^{\infty} \boldsymbol{\alpha}(l) = O(\mathsf{P}(Y > k)), \qquad \sum_{l=k+1}^{\infty} \boldsymbol{\Lambda}(l) = O(\mathsf{P}(Y > k)).$$

In what follows, we present four subexponential asymptotic formulas for the loss probability $P_{\rm loss}^{(K)}$. The first two formulas are obtained from the results for the dependent-sampling case, and the others are from those for the independent-sampling case.

Theorem 4.1 *Suppose that Assumption 4.1 holds and* $\mathsf{E}[A] < \mathsf{E}[I^{\rm off}]$. *Further suppose that (i)* $I^{\rm on} \in \mathcal{L}^{1/\theta}$ *for some* $0 < \theta \le 1/3$; *(ii)* $\mathsf{E}[I^{\rm on}] < \infty$ *and the equilibrium r.v.* $I_{\rm e}^{\rm on}$ *of* $I^{\rm on}$ *is subexponential (i.e.,* $I_{\rm e}^{\rm on} \in \mathcal{S}$*); and (iii)* $\mathsf{E}[\exp\{Q(Y)\}] < \infty$ *for some* $Q \in \mathcal{SC}$ *such that* $x^{3\theta/2} = O(Q(x))$. *We then have*

$$P_{\rm loss}^{(K)} \overset{K}{\sim} \frac{(\lambda - 1)\mathsf{E}[I^{\rm on}]}{\mathsf{E}[A] + \mathsf{E}[I^{\rm on}]} \mathsf{P}(I_{\rm e}^{\rm on} > K/(\lambda - 1)). \tag{4.9}$$

In addition, if (iv) each for $m \in \mathbb{Z}$, $\{I_m^{\rm on} \ge n+1\}$ *is independent of* $N_{m,n}$ *for all* $n = 0, 1, \dots$; *and (v)* $\boldsymbol{\alpha}(k) = \boldsymbol{\phi}\boldsymbol{\Lambda}(k)$ *for* $k = 1, 2, \dots$, *then*

$$P_{\rm loss}^{(K)} \overset{K}{\sim} \frac{\lambda - 1}{\lambda} \mathsf{P}(I_{\rm e}^{\rm on} > K/(\lambda - 1)). \tag{4.10}$$

Remark 4.3 Condition (v) implies that BMAPs in on-periods are stationary and thus $\mathsf{P}(J_{m,n} = j) = \phi_j$ $(j \in \mathbb{D})$ for all $m \in \mathbb{Z}$ and $n = 0, 1, \dots, I_m^{\rm on} - 1$.

Proof of Theorem 4.1. We first show that the sampling time T and the cumulative process $\{B(t)\}$ in (4.6) satisfy the conditions of Theorem 3.1. Condition (i) of Theorem 4.1 yields

$$\mathsf{P}(T > x) = \mathsf{P}(I^{\rm on} > x + 1) \overset{x}{\sim} \mathsf{P}(I^{\rm on} > x),$$

and thus $T \in \mathcal{L}^{1/\theta}$ for some $0 < \theta \le 1/3$, i.e., condition (i) of Theorem 3.1 is satisfied.

Note here that $\{B(t)\}$ in (4.6) satisfies $B(0) \ge 0$ and thus condition (ii) is reduced to $\mathsf{E}[\exp\{Q(\Delta\tau_n)\}] < \infty$ and $\mathsf{E}[\exp\{Q(\Delta B_n)\}] < \infty$ ($n = 0, 1$) for some $Q \in \mathcal{SC}$ such that $x^{3\theta/2} = O(Q(x))$ (see Remark 3.1). Further it follows from Assumption 4.1 and Lemma 3.1 that for $n = 0, 1$,

$$\mathsf{P}(\Delta B_n > x) \le C\mathsf{P}(Y > x), \qquad \forall x \ge 0. \tag{4.11}$$

Therefore condition (iii) of Theorem 4.1 implies $\mathsf{E}[\exp\{Q(\Delta B_n)\}] < \infty$ ($n = 0, 1$) for some $Q \in \mathcal{SC}$ such that $x^{3\theta/2} = O(Q(x))$. Recall that the distribution of $\Delta\tau_n$ is phase-type and $Q(x) = o(x)$ for any $Q \in \mathcal{SC}$ (see Definition 2.3). We then have $\mathsf{E}[\exp\{Q(\Delta\tau_n)\}] < \infty$ ($n = 0, 1$) for any $Q \in \mathcal{SC}$. As a result, condition (ii) of Theorem 3.1 holds.

Applying Theorem 3.1 to (4.7) and using (4.8), we obtain

$$\mathsf{P}(A > x) \overset{x}{\sim} \mathsf{P}(I^{\rm on} - 1 > x/(\lambda - 1)) \overset{x}{\sim} \mathsf{P}(I^{\rm on} > x/(\lambda - 1)), \tag{4.12}$$

from which and $\mathsf{E}[I^{\rm on}] < \infty$ we have $\mathsf{E}[A] < \infty$ and

$$\mathsf{P}(A_{\rm e} > x) \overset{x}{\sim} \frac{(\lambda - 1)\mathsf{E}[I^{\rm on}]}{\mathsf{E}[A]} \mathsf{P}(I^{\rm on}_{\rm e} > x/(\lambda - 1)). \tag{4.13}$$

Thus $A_{\rm e} \in \mathcal{S}$ due to $I^{\rm on}_{\rm e} \in \mathcal{S}$. As a result, (4.13) and Proposition 4.1 yield (4.9).

Finally, we prove (4.10). From (4.5), we have

$$\mathsf{E}[A_m + I^{\rm on}_m] = \mathsf{E}\left[\sum_{n=0}^{I^{\rm on}_m - 1} N_{m,n}\right]. \tag{4.14}$$

Note here that condition (v) of Theorem 4.1 and (4.4) yield (see Remark 4.3)

$$\mathsf{E}[N_{m,n}] = \boldsymbol{\phi} \sum_{k=1}^{\infty} k\boldsymbol{\Lambda}(k)\boldsymbol{e} = \lambda, \qquad \forall m \in \mathbb{Z},\ \forall n = 0, 1, \dots.$$

This equation and condition (iv) of Theorem 4.1 imply that Wald's lemma (see, e.g., [7, Chapter 1, Theorem 3.2]) is applicable to (4.14). We thus have

$$\mathsf{E}[A_m + I^{\rm on}_m] = \mathsf{E}[N_{m,0}]\mathsf{E}[I^{\rm on}] = \lambda\mathsf{E}[I^{\rm on}], \qquad m \in \mathbb{Z}. \tag{4.15}$$

Substituting (4.15) into (4.9) yields (4.10). □

Remark 4.4 If $I^{\rm on}_m - 1$ is a stopping time with respect to $\{N_{m,n}; n = 0, 1, \dots\}$, then condition (iv) of Theorem 4.1 is satisfied.

Theorem 4.2 *Suppose that Assumption 4.1 holds and* $\mathsf{E}[A] < \mathsf{E}[I^{\rm off}]$. *Further suppose that (i)* $I^{\rm on} \in \mathcal{C}$; *(ii)* $\mathsf{E}[I^{\rm on}] < \infty$; *and (iii)* $x\mathsf{P}(Y > x) = o(\mathsf{P}(I^{\rm on} > x))$. *We then have (4.9). In addition, if (iv) each for* $m \in \mathbb{Z}$, $\{I^{\rm on}_m \ge n + 1\}$ *is independent of* $N_{m,n}$ *for all* $n = 0, 1, \dots$; *and (v)* $\boldsymbol{\alpha}(k) = \boldsymbol{\phi}\boldsymbol{\Lambda}(k)$ *for* $k = 1, 2, \dots$, *then (4.10) holds.*

Proof: Suppose that the sampling time T and the cumulative process $\{B(t)\}$ in (4.6) satisfy the conditions of Theorem 3.2. Applying Theorem 3.2 to (4.7), we have (4.12) and thus (4.13). Note here that (4.12) and $I^{\rm on} \in \mathcal{C}$ imply $A \in \mathcal{C} \subset \mathcal{S}^*$, which leads to $A_{\rm e} \in \mathcal{S}$ (see Remarks 2.2 and 2.5). Therefore we can prove (4.9) and (4.10) by following the proof of Theorem 4.1 (and using Theorem 3.2 instead of Theorem 3.1).

In what follows, we confirm that T and $\{B(t)\}$ in (4.6) satisfy conditions (ii)–(v) of Theorem 3.2 (condition (i) is obvious due to $T \stackrel{\rm d}{=} I^{\rm on} - 1$ and $I^{\rm on} \in \mathcal{C}$). Since the distribution of $\Delta\tau_n$ ($n = 0, 1$) is phase-type,

$$\mathsf{E}[(\Delta\tau_n)^p] < \infty, \qquad \forall p > 0, \tag{4.16}$$

which implies condition (ii) of Theorem 3.2. Further since $T \in \mathcal{C} \subset \mathcal{D}$, Proposition 2.1 shows that $\mathsf{P}(T > x) = O(x^{-\gamma})$ for some $\gamma > 0$. From this and (4.16), we have

$$\limsup_{x\to\infty} \frac{x\mathsf{P}(\Delta\tau_n > x)}{\mathsf{P}(T > x)} \le C \limsup_{x\to\infty} x^{\gamma+1}\mathsf{P}(\Delta\tau_n > x) = 0. \tag{4.17}$$

Note here that (4.11) holds due to Assumption 4.1 and Lemma 3.1. It then follows from condition (iii) of Theorem 4.2 that for $n = 0, 1$,

$$x\mathsf{P}(\Delta B_n > x) = O(x\mathsf{P}(Y > x)) = o(\mathsf{P}(T > x)). \tag{4.18}$$

Note also that $\mathsf{P}(-B(0) > x > 0) = 0$ due to $B(0) \ge 0$. Therefore (4.17) and (4.18) imply that condition (iii) of Theorem 3.2 is satisfied. In addition,

$$x\mathsf{P}(|\Delta B_1 - \Delta\tau_1| > x) \le x[\mathsf{P}(\Delta B_1 > x) + \mathsf{P}(\Delta\tau_1 > x)] = o(\mathsf{P}(T > x)),$$

which shows that condition (iv) of Theorem 3.2 is satisfied. Finally, condition (v.b) of Theorem 3.2 holds due to $\mathsf{E}[T] < \infty$. □

Theorem 4.3 *Suppose that Assumption 4.1 holds,* $\mathsf{E}[A] < \mathsf{E}[I^{\rm off}]$ *and* $I_m^{\rm on}$ *is independent of the mth BMAP for all* $m \in \mathbb{Z}$. *Further suppose that (i)* $I^{\rm on} \in \mathcal{L}^{1/\theta}$ *for some* $0 < \theta \le 1/2$; *(ii)* $\mathsf{E}[I^{\rm on}] < \infty$ *and* $I_{\rm e}^{\rm on} \in \mathcal{S}$; *and (iii)* $\mathsf{E}[\exp\{Q(Y)\}] < \infty$ *for some* $Q \in \mathcal{SC}$ *such that* $x^\theta = O(Q(x))$. *Under these conditions, (4.9) holds. In addition, if* $\boldsymbol{\alpha}(k) = \boldsymbol{\phi}\boldsymbol{\Lambda}(k)$ *for* $k = 1, 2, \dots,$ *then (4.10) holds.*

Proof: According to the proofs of Theorems 4.1 and 4.2, it suffices to show that T and $\{B(t)\}$ in (4.6) satisfy conditions (i)–(iii) of Theorem 3.3.

Since $T \stackrel{\rm d}{=} I^{\rm on} - 1$, condition (i) of Theorem 4.3 implies condition (i) of Theorem 3.3. Further since $\{B(t)\}$ in (4.6) is nondecreasing with t, we have $\Delta B_n^* = \Delta B_n \ge 0$ ($n = 0, 1$). It thus follows from (4.11) and condition (iii) of Theorem 4.3 that $\mathsf{E}[\exp\{Q(\Delta B_n^*)\}] < \infty$ ($n = 0, 1$) for some $Q \in \mathcal{SC}$ such that $x^\theta = O(Q(x))$, which implies condition (iii) of Theorem 3.3. Note here that $\mathsf{E}[\exp\{Q(\Delta B_n^*)\}] < \infty$ ($n = 0, 1$) leads to $\mathsf{E}[(\Delta B_1)^2] < \infty$ (see Remark 2.4). Note also that $\mathsf{E}[(\Delta\tau_1)^2] < \infty$ due to (4.16). Therefore condition (ii) of Theorem 3.3 are satisfied. □

Theorem 4.4 *Suppose that Assumption 4.1 holds,* $\mathsf{E}[A] < \mathsf{E}[I^{\rm off}]$ *and* $I_m^{\rm on}$ *is independent of the mth BMAP for all* $m \in \mathbb{Z}$. *Further suppose that (i)* $I^{\rm on} \in \mathcal{C}$; *(ii)* $\mathsf{E}[I^{\rm on}] < \infty$; *and (iii)* $\mathsf{P}(Y > x) = o(\mathsf{P}(I^{\rm on} > x))$. *Under these conditions, (4.9) holds. In addition, if* $\boldsymbol{\alpha}(k) = \boldsymbol{\phi}\boldsymbol{\Lambda}(k)$ *for* $k = 1, 2, \dots,$ *then (4.10) holds.*

Proof: It is easy to see that the conditions of Theorem 3.5 are satisfied. Thus similarly to the other theorems in this subsection, we can prove (4.9) and (4.10). □

Finally, we mention previous studies related to the results presented in this subsection. Zwart [47] and Jelenković and Momčilović [24] study the subexponential asymptotics of the loss probabilities of finite-buffer fluid queues fed by the superposition of independent on/off sources that generate fluid at constant rates. These studies assume that the lengths of the on-periods of each on/off source follow a regularly or consistently varying distribution, and then present asymptotic formulas for the loss probability such that the decay of the loss probability is connected to the tail of the equilibrium distribution of on-period lengths.

A. Technical Lemmas

This appendix presents technical lemmas, whose proofs are all given in Appendix B.

A.1. Higher-order long-tailed distributions

In this section, we consider the class $\mathcal{L}^p$ ($p \ge 1$) of higher-order long-tailed distributions. By definition, $\mathcal{L}^1 = \mathcal{L}$ (see Definition 1.1). Further $\mathcal{L}^2$ is equivalent to the class of square-root insensitive distributions (see Lemma 1 in [25]). We can readily confirm that the following are examples of the distributions in $\mathcal{L}^p$:

(i) $\mathsf{P}(X > x) \overset{x}{\sim} \exp\{-x^{\alpha}\}$, where $0 < \alpha < 1/p$.
(ii) $\mathsf{P}(X > x) \overset{x}{\sim} \exp\{-x^{1/p}/(\log x)^{\gamma}\}$, where $\gamma > 0$.

In what follows, we provide five lemmas, which summarize the basic properties of $\mathcal{L}^p$.

Lemma A.1 *If $X \in \mathcal{L}^{1/\theta}$ (i.e., $X^{\theta} \in \mathcal{L}$) for some $0 < \theta \le 1$, the following are satisfied:*
(i) $\lim_{x\to\infty} e^{\varepsilon x^{\theta}} \mathsf{P}(X > x) = \infty$ for any $\varepsilon > 0$, i.e., $\mathsf{P}(X > x) = e^{-o(x^{\theta})}$.
(ii) $X \in \mathcal{L}^{1/\eta}$ for all $1 \le 1/\eta < 1/\theta$.

Proof: See Appendix B.1. □

Remark A.1 Lemma A.1 (ii) implies that $\mathcal{L}^{p_2} \subset \mathcal{L}^{p_1}$ for $1 \le p_1 < p_2$.

Lemma A.2 *For any $0 < \theta \le 1$, $X \in \mathcal{L}^{1/\theta}$ if and only if $\mathsf{P}(X > x - \xi x^{1-\theta}) \overset{x}{\sim} \mathsf{P}(X > x)$ for all $\xi \in \mathbb{R}$.*

Proof: See Appendix B.2. □

Lemma A.2 is an extension of Lemma 1 in [25]. The following lemma shows that the "if" part of Lemma A.2 holds under a weaker condition.

Lemma A.3 *For any $0 < \theta \le 1$, $X \in \mathcal{L}^{1/\theta}$ if $\mathsf{P}(X > x - \xi x^{1-\theta}) \overset{x}{\sim} \mathsf{P}(X > x)$ for some $\xi \in \mathbb{R}\backslash\{0\}$.*

Proof: See Appendix B.3. □

Remark A.2 The statements of Lemmas A.1–A.3 are presented in a slightly different way in a technical report [35] (see Lemmas 1–3 therein), where the statements are described in terms of h-insensitivity (see Chapter 2 in [16] for the definition of h-insensitivity).

Lemma A.4 below shows the inclusion relation between class $\mathcal{L}^p$ and the consistent variation class $\mathcal{C}$.

Lemma A.4 *$\mathcal{C} \subset \mathcal{L}^{\infty}$, i.e., $\mathcal{C} \subset \mathcal{L}^{1/\theta}$ for any $0 < \theta \le 1$.*

Proof: See Appendix B.4. □

The following lemma is used to prove Theorem 3.3.

Lemma A.5 *If $X \in \mathcal{L}^{1/\theta}$ for some $0 < \theta \le 1$, then for any $\varepsilon > 0$ there exists $\breve{x}_\varepsilon > 0$ such that for all $x > \breve{x}_\varepsilon$ and $0 \le u \le g(x)$,*

$$\mathsf{P}(X > x - u) \le \mathsf{P}(X > x) e^{\varepsilon(u^\theta + 1)},$$

where g is a nonnegative function on $[0, \infty)$ such that $\limsup_{x\to\infty} g(x)/x < 1$.

Proof: See Appendix B.5. □

A.2. Subexponential concave distributions

The subexponential concave class was first introduced by Nagaev [37]. According to Nagaev's definition of $\mathcal{SC}$, condition (iii) of Definition 2.3 is replaced by the following condition: (iii′) there exist $x_0 > 0$, $0 < \alpha < 1$ and $0 < \beta < 1$ such that for all $x \ge x_0$ and $\beta x \le u \le x$,

$$\frac{Q_X(x) - Q_X(u)}{Q_X(x)} \le \alpha \frac{x - u}{x}. \tag{A.1}$$

Actually, Nagaev's definition is equivalent to Definition 2.3. Lemma 3.1 (i) in [23] shows that Nagaev's definition implies Definition 2.3. The converse follows from Theorem 2 in [41], though the phrase "$Q(x)/f(x)$ is nondecreasing" should be replaced by "$Q(x)/f(x)$ is nonincreasing."

Remark A.3 Suppose that $Q \in \mathcal{SC}$ is differentiable. It then follows from (A.1) and (2.1) that

$$Q'(x) := \frac{\mathrm{d}}{\mathrm{d}x} Q(x) \le \frac{\alpha Q(x)}{x} \le C x^{\alpha - 1}, \qquad x > x_0. \tag{A.2}$$

Lemma A.6 below establishes the relationship between class $\mathcal{SC}$ and the higher-order long-tailed class.

Lemma A.6 *(i) $\mathcal{SC}_\alpha \subset \mathcal{L}^{1/\beta}$ for all $0 < \alpha < \beta \le 1$.*
(ii) $X^\alpha \in \mathcal{L}$ if $X \in \mathcal{SC}_\alpha$ for some $0 < \alpha < 1$ and

$$\lim_{x\to\infty} Q_X(x)/x^\alpha = 0. \tag{A.3}$$

Proof: See Appendix B.6. □

The following lemma plays an important role in the proof of Theorems 3.1 and 3.3.

Lemma A.7 *Assume $\mathsf{E}[(\Delta B_1)^2] < \infty$.*

(i) If $\mathsf{E}[(\Delta\tau_1)^2] < \infty$ and $\mathsf{E}[\exp\{Q(\Delta B_n^*)\}] < \infty$ $(n = 0, 1)$ for some $Q \in \mathcal{SC}$, then

$$\mathsf{P}\left(\sup_{0\le t\le x} \{B(t) - bt\} > u\right) \le C\left(e^{-cu^2/x} + e^{-cx} + x e^{-cQ(u)}\right), \quad \forall x \ge 0,\ \forall u \ge 0.$$

(ii) Let $\Delta B_0^{**} = \sup_{0 \le t \le \tau_0} \max(-B(t), 0)$ *and* $\Delta B_n^{**} = \sup_{\tau_{n-1} \le t \le \tau_n} (B(\tau_{n-1}) - B(t))$ *for* $n = 1, 2, \dots$. *If* $\mathsf{E}[\exp\{Q(\Delta B_n^{**} + b\Delta\tau_n)\}] < \infty$ $(n = 0, 1)$ *for some* $Q \in \mathcal{SC}$, *then*

$$\mathsf{P}\left(\inf_{0 \le t \le x} \{B(t) - bt\} < -u\right) \le C\left(e^{-cu^2/x} + e^{-cx} + xe^{-cQ(u)}\right), \quad \forall x \ge 0,\ \forall u \ge 0.$$

In the two above inequalities, C *and* c *are independent of* x *and* u.

Proof: See Appendix B.7. □

Remark A.4 Lemma A.7 (i) is a slight extension of Proposition 1 in [25], where the latter assumes that $\Delta B_1 \ge 0$ w.p.1.

Remark A.5 Suppose that $\{B(t)\}$ is nondecreasing with t. It then follows that $\Delta B_0^* = (\Delta B_0)^+$, $\Delta B_1^* = \Delta B_1$, $\Delta B_0^{**} = \max(-B(0), 0) = (-B(0))^+$ and $\Delta B_1^{**} = 0$. Therefore the condition $\mathsf{E}[\exp\{Q(\Delta B_n^*)\}] < \infty$ $(n = 0, 1)$ is reduced to

$$\mathsf{E}[\exp\{Q((\Delta B_0)^+)\}] < \infty \quad \text{and} \quad \mathsf{E}[\exp\{Q(\Delta B_1)\}] < \infty;$$

and the condition $\mathsf{E}[\exp\{Q(\Delta B_n^{**} + b\Delta\tau_n)\}] < \infty$ $(n = 0, 1)$ is reduced to

$$\mathsf{E}[\exp\{Q((-B(0))^+ + b\Delta\tau_0)\}] < \infty \quad \text{and} \quad \mathsf{E}[\exp\{Q(b\Delta\tau_1)\}] < \infty.$$

It should be noted that $\mathsf{E}[\exp\{Q(\Delta B_1)\}] < \infty$ and $\mathsf{E}[\exp\{Q(b\Delta\tau_1)\}] < \infty$ imply $\mathsf{E}[(\Delta B_1)^2] < \infty$ and $\mathsf{E}[(\Delta\tau_1)^2] < \infty$, respectively (see Remark 2.4).

A.3. Regular varying distributions

The regular variation class $\mathcal{R}$ is defined as follows:

Definition A.1 A nonnegative r.v. X and its d.f. F_X belong to class $\mathcal{R}(-\alpha)$ $(\alpha \ge 0)$ if $\overline{F}_X$ is regularly varying with index $-\alpha$, i.e.,

$$\lim_{x\to\infty} \frac{\overline{F}_X(vx)}{\overline{F}_X(x)} = v^{-\alpha}, \qquad \forall v > 0.$$

Further let $\mathcal{R} = \cup_{\alpha \ge 0} \mathcal{R}(-\alpha)$.

Remark A.6 If $F \in \mathcal{R}$, then $\overline{F}(x) = x^{-\alpha}\tilde{l}(x)$ for some $\alpha \ge 0$, where $\tilde{l}$ is a slowly varying function, i.e.,

$$\lim_{x\to\infty} \frac{\tilde{l}(vx)}{\tilde{l}(x)} = 1, \qquad \forall v > 0.$$

See [6] for the details of regularly varying functions.

Remark A.7 It is known that $\mathcal{R} \subset \mathcal{C}$ (see, e.g., [11, 13]). Thus $\mathcal{R} \subset \mathcal{C} \subset \mathcal{L}^\infty \subset \mathcal{L}^p \subset \mathcal{L}$ for any $p > 1$ (see Remark A.1 and Lemma A.4).

The following lemma is used to prove Theorem 3.2.

Lemma A.8 *Suppose that* U *is a r.v. with* $\mathsf{E}[|U|] < \infty$. *If* $\mathsf{P}(U > x) = o(\mathsf{P}(Y > x))$ *for some nonnegative r.v.* Y *with* $\mathsf{E}[Y] < \infty$, *then for any* $\mu > \mathsf{E}[U]$ *there exists some r.v.* Z *in* $\mathbb{R}$ *such that* $\mathsf{E}[U] < \mathsf{E}[Z] < \mu$, $\overline{F}_Z(x) \ge \overline{F}_U(x)$ *for all* $x \in \mathbb{R}$ *and*

$$\overline{F}_Z(x) = \tilde{l}(x)\overline{F}_Y(x) \qquad \text{for all sufficiently large } x > 0,$$

where $\tilde{l}$ *is some slowly varying function such that* $\lim_{x\to\infty} \tilde{l}(x) = 0$.

Proof: See Appendix B.8. □

A.4. Bounds on deviation probabilities

This subsection presents three lemmas on the deviation probabilities associated with i.i.d. r.v.s. The first one (Lemma A.9) is used to prove Theorems 3.2 and 3.4, and the other two are required by the proof of Theorem 3.2.

Lemma A.9 *If $X, X_1, X_2, \dots$ are i.i.d. nonnegative r.v.s with $\mathsf{E}[X] > 0$ and $\mathsf{E}[X^2] < \infty$, then for any $\delta > 0$ there exist finite constants $\widetilde{C} := \widetilde{C}(\delta) > 0$ and $\tilde{c} := \tilde{c}(\delta) > 0$ such that*

$$\mathsf{P}\left(N_X(x) - \frac{x}{\mathsf{E}[X]} > u\right) \le \widetilde{C}\exp\{-\tilde{c}u^2/x\}, \quad \forall x \ge 0, \quad 0 \le \forall u \le \delta x, \tag{A.4}$$

where $N_X(x) = \max\{k \ge 0; \sum_{n=1}^{k} X_n \le x\}$ for $x \in \mathbb{R}$.

Proof: See Appendix B.9. □

Remark A.8 Lemma A.9 is very similar to, but not exactly the same as Lemma 6 in [25]. The latter states that there exists *some* $\delta > 0$ such that (A.4) holds.

Lemmas A.10 and A.11 below are extensions of Lemma 2.3 in [44] and Lemma 2.2 in [29], respectively, to the maxima of partial sums of i.i.d. r.v.s.

Lemma A.10 *Suppose that $U, U_1, U_2, \dots$ are i.i.d. r.v.s in $\mathbb{R}$. If $\mathsf{E}[U] = 0$ and $\mathsf{E}[(U^+)^r] < \infty$ for some $r > 1$, then for any fixed $\gamma > 0$ and $p > 0$ there exist some $v := v(r,p) > 0$ and $\widetilde{C} := \widetilde{C}(v,\gamma)$ such that for all $n = 1, 2, \dots$,*

$$\mathsf{P}\left(\max_{1\le k\le n}\sum_{i=1}^{k} U_i \ge x\right) \le n\mathsf{P}(U > vx) + \widetilde{C}x^{-p}, \quad \forall x \ge \gamma n. \tag{A.5}$$

Proof: See Appendix B.10. □

Lemma A.11 *Suppose that $U, U_1, U_2, \dots$ are i.i.d. r.v.s in $\mathbb{R}$. If $0 \le \mathsf{E}[U] < \infty$ and $U^+ \in \mathcal{C}$, then for any $\gamma > \mathsf{E}[U]$ there exists some constant $\widetilde{C} := \widetilde{C}(\gamma) > 0$ such that for all $n = 1, 2, \dots$,*

$$\mathsf{P}\left(\max_{1\le k\le n}\sum_{i=1}^{k} U_i > x\right) \le \widetilde{C}n\mathsf{P}(U > x), \quad \forall x \ge \gamma n. \tag{A.6}$$

Proof: See Appendix B.11. □

A.5. Convolution tail of matrix-valued functions associated with subexponential distributions

Let $\boldsymbol{F} = (F_{i,j})_{1\le j\le m_1}^{1\le i\le m_0}$ and $\boldsymbol{G} = (G_{i,j})_{1\le j\le m_2}^{1\le i\le m_1}$ denote matrix-valued functions on $\mathbb{R}$. Assume that $F_{i,j}(x)$ and $G_{i,j}(x)$ are nonnegative and nondecreasing for all $x \in \mathbb{R}$ and that $F_{i,j}(\infty) := \lim_{x\to\infty} F_{i,j}(x) < \infty$ and $G_{i,j}(\infty) := \lim_{x\to\infty} G_{i,j}(x) < \infty$. We then define $\overline{\boldsymbol{F}}(x) = \boldsymbol{F}(\infty) - \boldsymbol{F}(x)$ and $\overline{\boldsymbol{G}}(x) = \boldsymbol{G}(\infty) - \boldsymbol{G}(x)$ for $x \in \mathbb{R}$.

Let $\boldsymbol{F} * \boldsymbol{G}$ denote the convolution of $\boldsymbol{F}$ and $\boldsymbol{G}$, i.e.,

$$\boldsymbol{F} * \boldsymbol{G}(x) = \int_{y\in\mathbb{R}} \boldsymbol{F}(x-y)\mathrm{d}\boldsymbol{G}(y) = \int_{y\in\mathbb{R}} \mathrm{d}\boldsymbol{F}(y)\boldsymbol{G}(x-y), \qquad x \in \mathbb{R}.$$

Let $\overline{\boldsymbol{F}*\boldsymbol{G}}(x)$ $(x \in \mathbb{R})$ denote

$$\overline{\boldsymbol{F}*\boldsymbol{G}}(x) = \boldsymbol{F}*\boldsymbol{G}(\infty) - \boldsymbol{F}*\boldsymbol{G}(x) = \boldsymbol{F}(\infty)\boldsymbol{G}(\infty) - \boldsymbol{F}*\boldsymbol{G}(x).$$

When $\boldsymbol{F}$ is a square matrix-valued function (i.e., $m_0 = m_1$), we define $\boldsymbol{F}^{*n}$ $(n = 1, 2, \dots)$ as the n-fold convolution of $\boldsymbol{F}$ itself, i.e.,

$$\boldsymbol{F}^{*n}(x) = \boldsymbol{F}^{*(n-1)} * \boldsymbol{F}(x), \quad x \in \mathbb{R},$$

and for convenience, define $\boldsymbol{F}^{*0}(x) = \boldsymbol{O}$ for $x < 0$ and $\boldsymbol{F}^{*0}(x) = \boldsymbol{I}$ for $x \ge 0$. Further for the n-fold convolution $\boldsymbol{F}^{*n}$, let $\overline{\boldsymbol{F}^{*n}}(x)$ $(x \in \mathbb{R})$ denote

$$\overline{\boldsymbol{F}^{*n}}(x) = \boldsymbol{F}^{*n}(\infty) - \boldsymbol{F}^{*n}(x) = (\boldsymbol{F}(\infty))^n - \boldsymbol{F}^{*n}(x).$$

The following lemma is the upper-limit version of Proposition A.3 in [31] and Lemma 6 in [22].

Lemma A.12 *Suppose that for some r.v. $Y \in \mathcal{S}$,*

$$\limsup_{x\to\infty} \frac{\overline{\boldsymbol{F}}(x)}{\mathsf{P}(Y > x)} \le \widetilde{\boldsymbol{F}}, \qquad \limsup_{x\to\infty} \frac{\overline{\boldsymbol{G}}(x)}{\mathsf{P}(Y > x)} \le \widetilde{\boldsymbol{G}}, \tag{A.7}$$

where $\widetilde{\boldsymbol{F}} = (\widetilde{F}_{i,j})$ and $\widetilde{\boldsymbol{G}} = (\widetilde{G}_{i,j})$ are finite, and where $\widetilde{\boldsymbol{F}} = \widetilde{\boldsymbol{G}} = \boldsymbol{O}$ is allowed. We then have

$$\limsup_{x\to\infty} \frac{\overline{\boldsymbol{F}*\boldsymbol{G}}(x)}{\mathsf{P}(Y > x)} \le \widetilde{\boldsymbol{F}}\boldsymbol{G}(\infty) + \boldsymbol{F}(\infty)\widetilde{\boldsymbol{G}}. \tag{A.8}$$

Further if $\boldsymbol{F}$ is a square matrix-valued function, then

$$\limsup_{x\to\infty} \frac{\overline{\boldsymbol{F}^{*n}}(x)}{\mathsf{P}(Y > x)} \le \sum_{\nu=0}^{n-1} (\boldsymbol{F}(\infty))^{\nu} \widetilde{\boldsymbol{F}} (\boldsymbol{F}(\infty))^{n-\nu-1}. \tag{A.9}$$

In addition to the above conditions, assume that $\sum_{n=0}^{\infty} (\boldsymbol{F}(\infty))^n = (\boldsymbol{I} - \boldsymbol{F}(\infty))^{-1} < \infty$. We then have

$$\limsup_{x\to\infty} \sum_{n=0}^{\infty} \frac{\overline{\boldsymbol{F}^{*n}}(x)}{\mathsf{P}(Y > x)} \le (\boldsymbol{I} - \boldsymbol{F}(\infty))^{-1} \widetilde{\boldsymbol{F}} (\boldsymbol{I} - \boldsymbol{F}(\infty))^{-1}. \tag{A.10}$$

Proof: See Appendix B.12. □

B. Proofs of Technical Lemmas

B.1. Proof of Lemma A.1

We first prove the statement (i). It follows from $X^{\theta} \in \mathcal{L}$ that $\lim_{y\to\infty} e^{\varepsilon y} \mathsf{P}(X^{\theta} > y) = \infty$ for any $\varepsilon > 0$. Thus letting $x = y^{1/\theta}$ for $y > 0$, we have

$$\lim_{x\to\infty} e^{\varepsilon x^{\theta}} \mathsf{P}(X > x) = \lim_{x\to\infty} e^{\varepsilon x^{\theta}} \mathsf{P}(X^{\theta} > x^{\theta}) = \lim_{y\to\infty} e^{\varepsilon y} \mathsf{P}(X^{\theta} > y) = \infty.$$

Next we prove the statement (ii). For all $x, y \ge 0$, we have

$$1 \ge \frac{\mathsf{P}(X^{\eta} > x + y)}{\mathsf{P}(X^{\eta} > x)} = \frac{\mathsf{P}(X^{\theta} > (x + y)^{\theta/\eta})}{\mathsf{P}(X^{\theta} > x^{\theta/\eta})}. \tag{B.1}$$

It follows from $0 < \theta/\eta < 1$ that for all $x, y \ge 0$,

$$(x+y)^{\theta/\eta} \le x^{\theta/\eta} + y^{\theta/\eta},$$

from which and $X^\theta \in \mathcal{L}$ we obtain

$$\lim_{x\to\infty} \frac{\mathsf{P}(X^\theta > (x+y)^{\theta/\eta})}{\mathsf{P}(X^\theta > x^{\theta/\eta})} \ge \lim_{x\to\infty} \frac{\mathsf{P}(X^\theta > x^{\theta/\eta} + y^{\theta/\eta})}{\mathsf{P}(X^\theta > x^{\theta/\eta})} = 1. \tag{B.2}$$

Combining (B.2) with (B.1) yields $\mathsf{P}(X^\eta > x+y) \overset{x}{\sim} \mathsf{P}(X^\eta > x)$ for any $y > 0$, i.e., $X^\eta \in \mathcal{L}$.

B.2. Proof of Lemma A.2

To prove Lemma A.2, we use the following proposition, whose proof is given in Appendix B.14.

Proposition B.1 *For any $\gamma > 0$ and $x > y \ge 0$,*

$$(x+y)^\gamma \le x^\gamma + C\left(1 - \frac{y}{x}\right)^{-1} yx^{\gamma-1}, \tag{B.3}$$

$$(x-y)^\gamma \ge x^\gamma - C\left(1 - \frac{y}{x}\right)^{-1} yx^{\gamma-1}, \tag{B.4}$$

where C is independent of x and y.

We first prove the "if" part of Lemma A.1. Proposition B.1 implies that $(x+y)^{1/\theta} \le x^{1/\theta} + Cyx^{1/\theta-1}$ for any $x > 0$ and $0 \le y < x/2$. Thus for any $y > 0$,

$$\begin{aligned} 1 \ge \lim_{x\to\infty} \frac{\mathsf{P}(X^\theta > x+y)}{\mathsf{P}(X^\theta > x)} &\ge \lim_{x\to\infty} \frac{\mathsf{P}(X > x^{1/\theta} + Cy\cdot(x^{1/\theta})^{1-\theta})}{\mathsf{P}(X > x^{1/\theta})} \\ &= \lim_{w\to\infty} \frac{\mathsf{P}(X > w + Cy\cdot w^{1-\theta})}{\mathsf{P}(X > w)} = 1, \end{aligned}$$

which shows that $X^\theta \in \mathcal{L}$.

Next we prove the "only if" part. We fix ξ such that $x^\theta > 2|\xi|$. It then follows from Proposition B.1 that

$$\begin{aligned} (x - \xi x^{1-\theta})^\theta &\ge x^\theta - C\left(1 - \frac{\xi}{x^\theta}\right)^{-1} \xi \ge x^\theta - 2C\xi, && \xi \ge 0, \\ (x - \xi x^{1-\theta})^\theta &\le x^\theta + C\left(1 - \frac{-\xi}{x^\theta}\right)^{-1} (-\xi) \le x^\theta + 2C(-\xi), && \xi < 0. \end{aligned}$$

Thus for $\xi \ge 0$,

$$1 \le \lim_{x\to\infty} \frac{\mathsf{P}(X > x - \xi x^{1-\theta})}{\mathsf{P}(X > x)} \le \lim_{x\to\infty} \frac{\mathsf{P}(X^\theta > x^\theta - C\xi)}{\mathsf{P}(X^\theta > x^\theta)} = 1,$$

where the last equality follows from $X^\theta \in \mathcal{L}$. Similarly, for $\xi < 0$,

$$1 \ge \lim_{x\to\infty} \frac{\mathsf{P}(X > x - \xi x^{1-\theta})}{\mathsf{P}(X > x)} \ge \lim_{x\to\infty} \frac{\mathsf{P}(X^\theta > x^\theta + C(-\xi))}{\mathsf{P}(X^\theta > x^\theta)} = 1.$$

As a result, $\mathsf{P}(X > x - \xi x^{1-\theta}) \overset{x}{\sim} \mathsf{P}(X > x)$ for any $\xi \in \mathbb{R}$.

B.3. Proof of Lemma A.3

It follows from Proposition B.1 that there exists some $\widetilde{C} > 0$ such that for all $x > 2\sigma > 0$,

$$(x-\sigma)^{1/\theta} \ge x^{1/\theta} - \sigma\widetilde{C} \cdot (x^{1/\theta})^{1-\theta}, \tag{B.5}$$

$$(x+\sigma)^{1/\theta} \le x^{1/\theta} + \sigma\widetilde{C} \cdot (x^{1/\theta})^{1-\theta}. \tag{B.6}$$

We fix $\sigma = |\xi|/\widetilde{C}$, where $\xi \in \mathbb{R} \setminus \{0\}$. Using (B.5), we then have

$$\begin{aligned} 1 \le \lim_{x\to\infty} \frac{\mathsf{P}(X^\theta > x - \sigma)}{\mathsf{P}(X^\theta > x)} &\le \lim_{x\to\infty} \frac{\mathsf{P}(X > x^{1/\theta} - \sigma\widetilde{C} \cdot (x^{1/\theta})^{1-\theta})}{\mathsf{P}(X > x^{1/\theta})} \\ &= \lim_{x\to\infty} \frac{\mathsf{P}(X > x - |\xi| x^{1-\theta})}{\mathsf{P}(X > x)}. \end{aligned} \tag{B.7}$$

Similarly from (B.6), we have

$$1 \ge \lim_{x\to\infty} \frac{\mathsf{P}(X^\theta > x + \sigma)}{\mathsf{P}(X^\theta > x)} \ge \lim_{x\to\infty} \frac{\mathsf{P}(X > x + |\xi| x^{1-\theta})}{\mathsf{P}(X > x)}. \tag{B.8}$$

We now suppose that $\mathsf{P}(X > x - \xi x^{1-\theta}) \overset{x}{\sim} \mathsf{P}(X > x)$ for some $\xi \in \mathbb{R} \setminus \{0\}$, which implies

$$\mathsf{P}(X > x - |\xi| x^{1-\theta}) \overset{x}{\sim} \mathsf{P}(X > x) \quad \text{or} \quad \mathsf{P}(X > x + |\xi| x^{1-\theta}) \overset{x}{\sim} \mathsf{P}(X > x).$$

It thus follows from (B.7) and (B.8) that $\mathsf{P}(X^\theta > x - \sigma) \overset{x}{\sim} \mathsf{P}(X^\theta > x)$ or $\mathsf{P}(X^\theta > x + \sigma) \overset{x}{\sim} \mathsf{P}(X^\theta > x)$, which shows $X^\theta \in \mathcal{L}$, i.e., $X \in \mathcal{L}^{1/\theta}$.

B.4. Proof of Lemma A.4

Suppose $X \in \mathcal{C}$. It then follows from Definition 1.2 that for any $v > 1$ there exists some $c(v) > 0$ such that $\lim_{v\downarrow 1} c(v) = 1$ and

$$\liminf_{x\to\infty} \frac{\mathsf{P}(X > vx)}{\mathsf{P}(X > x)} = c(v).$$

Since $x + 1 \le vx$ for any fixed $v > 1$ and all sufficiently large $x > 0$, we have for any $0 < \theta \le 1$,

$$\liminf_{x\to\infty} \frac{\mathsf{P}(X^\theta > x + 1)}{\mathsf{P}(X^\theta > x)} \ge \liminf_{x\to\infty} \frac{\mathsf{P}(X > (vx)^{1/\theta})}{\mathsf{P}(X > x^{1/\theta})} = c(v^{1/\theta}) \to 1 \quad \text{as } v \downarrow 1.$$

On the other hand, it is clear that $\mathsf{P}(X^\theta > x + 1) \lesssim_x \mathsf{P}(X^\theta > x)$. Therefore we obtain $\mathsf{P}(X^\theta > x + 1) \overset{x}{\sim} \mathsf{P}(X^\theta > x)$, i.e., $X^\theta \in \mathcal{L}$.

B.5. Proof of Lemma A.5

The case of $u = 0$ is obvious. Therefore we focus on the case of $u > 0$. For any $x \ge u$, we have

$$\frac{\mathsf{P}(X > x - u)}{\mathsf{P}(X > x)} = \frac{\mathsf{P}(X^\theta > (x-u)^\theta)}{\mathsf{P}(X^\theta > x^\theta)} \le \frac{\mathsf{P}(X^\theta > x^\theta - u^\theta)}{\mathsf{P}(X^\theta > x^\theta)}, \tag{B.9}$$

where we use $(x-u)^\theta \ge x^\theta - u^\theta$ for $0 \le u \le x$. Let y denote a nonnegative number such that $y = x^\theta - u^\theta$. We then have

$$\frac{\mathsf{P}(X^\theta > x^\theta - u^\theta)}{\mathsf{P}(X^\theta > x^\theta)} = \frac{\mathsf{P}(X^\theta > y)}{\mathsf{P}(X^\theta > y + u^\theta)} = \prod_{i=0}^{\lceil u^\theta \rceil - 1} \frac{\mathsf{P}\left(X^\theta > y + i\dfrac{u^\theta}{\lceil u^\theta \rceil}\right)}{\mathsf{P}\left(X^\theta > y + (i+1)\dfrac{u^\theta}{\lceil u^\theta \rceil}\right)}. \tag{B.10}$$

It follows from $X^\theta \in \mathcal{L}$ that for any $\varepsilon > 0$ there exists some $\breve{y}_\varepsilon > 0$ such that for all $y > \breve{y}_\varepsilon$,

$$\frac{\mathsf{P}(X^\theta > y)}{\mathsf{P}(X^\theta > y + \gamma)} \le \frac{\mathsf{P}(X^\theta > y)}{\mathsf{P}(X^\theta > y + 1)} \le e^\varepsilon, \qquad 0 \le \forall\gamma \le 1.$$

Thus since $0 < u^\theta / \lceil u^\theta \rceil \le 1$, we have

$$\prod_{i=0}^{\lceil u^\theta \rceil - 1} \frac{\mathsf{P}\left(X^\theta > y + i \dfrac{u^\theta}{\lceil u^\theta \rceil} \right)}{\mathsf{P}\left(X^\theta > y + (i+1) \dfrac{u^\theta}{\lceil u^\theta \rceil} \right)} \le e^{\varepsilon \lceil u^\theta \rceil} \le e^{\varepsilon (u^\theta + 1)}, \qquad y > \breve{y}_\varepsilon,$$

from which, (B.9) and (B.10) it follows that

$$\frac{\mathsf{P}(X > x - u)}{\mathsf{P}(X > x)} \le e^{\varepsilon(u^\theta + 1)} \quad \text{for all } x, u \ge 0 \text{ such that } x^\theta - u^\theta > \breve{y}_\varepsilon. \tag{B.11}$$

Note here that for all $0 < u \le g(x)$,

$$\liminf_{x\to\infty} (x^\theta - u^\theta) \ge \liminf_{x\to\infty} \left[x^\theta - \{g(x)\}^\theta \right] = \liminf_{x\to\infty} x^\theta \left[1 - \left(\frac{g(x)}{x} \right)^\theta \right] = \infty,$$

where the last equality is due to $\limsup_{x\to\infty} g(x)/x < 1$. As a result, there exists some $\breve{x}_\varepsilon > 0$ such that for all $x > \breve{x}_\varepsilon$ and $0 < u \le g(x)$

$$x^\theta - u^\theta > \breve{y}_\varepsilon,$$

and thus (B.11) holds.

B.6. Proof of Lemma A.6

For any $0 < \beta \le 1$, it follows from (A.1) that for all sufficiently large $x > 0$,

$$1 \le \frac{\mathsf{P}(X > x - x^{1-\beta})}{\mathsf{P}(X > x)} = \exp\{Q_X(x) - Q_X(x - x^{1-\beta})\} \le \exp\{\alpha Q_X(x)/x^\beta\}. \tag{B.12}$$

Further according to condition (iii) of Definition 2.3, there exists some $x_0 > 0$ such that

$$Q_X(x) \le C x^\alpha, \qquad \forall x \ge x_0.$$

Thus for any $\beta \in (\alpha, 1]$, we have

$$1 \le \frac{\mathsf{P}(X > x - x^{1-\beta})}{\mathsf{P}(X > x)} \le \exp\{C x^{\alpha - \beta}\} \to 1 \qquad \text{as } x \to \infty,$$

which implies $X^\beta \in \mathcal{L}$ due to Lemma A.3. In addition, if (A.3) holds, then substituting (A.3) into (B.12) with $\beta = \alpha$ yields $\mathsf{P}(X > x - x^{1-\alpha}) \overset{x}{\gtrsim} \mathsf{P}(X > x)$, i.e., $X^\alpha \in \mathcal{L}$.

B.7. Proof of Lemma A.7

To prove Lemma A.7, we consider another (possibly delayed) cumulative process $\{\breve{B}(t); t \ge 0\}$ on $\mathbb{R}$, which satisfies $|\breve{B}(0)| < \infty$ w.p.1 and has the same regenerative points as $\{B(t); t \ge 0\}$, i.e., $\{\breve{B}(t+\tau_n) - \breve{B}(\tau_n); t \ge 0\}$ $(n = 0, 1, \dots)$ is stochastically equivalent to $\{\breve{B}(t+\tau_0) - \breve{B}(\tau_0); t \ge 0\}$ and is independent of $\{\breve{B}(u); 0 \le u < \tau_n\}$. Let

$$\Delta\breve{B}_n = \begin{cases} \breve{B}(\tau_0), & n = 0, \\ \breve{B}(\tau_n) - \breve{B}(\tau_{n-1}), & n = 1, 2, \dots, \end{cases}$$

$$\Delta\breve{B}_n^* = \begin{cases} \sup\limits_{0 \le t \le \tau_0} \max(\breve{B}(t), 0), & n = 0, \\ \sup\limits_{\tau_{n-1} \le t \le \tau_n} \breve{B}(t) - \breve{B}(\tau_{n-1}), & n = 1, 2, \dots. \end{cases}$$

The $\Delta\breve{B}_n$'s (resp. $\Delta\breve{B}_n^*$'s) $(n = 1, 2, \dots)$ are i.i.d. and independent of $\Delta\breve{B}_0$ (resp. $\Delta\breve{B}_0^*$). We assume that

$$\mathsf{P}(0 \le \breve{B}_n^* < \infty) = 1 \quad (n = 0, 1), \quad \mathsf{E}[|\Delta\breve{B}_1|] < \infty, \quad \breve{b} := \mathsf{E}[\Delta\breve{B}_1]/\mathsf{E}[\Delta\tau_1] \neq 0.$$

Note that $\breve{b}$ can be negative.

The following lemma is an extension of Proposition 1 in [25]. Using the lemma, we can readily prove Lemma A.7.

Lemma B.1 *Let* $\Delta\Theta_n = \Delta\breve{B}_n^* - \min(\breve{b}, 0)\Delta\tau_n \ge 0$ *for* $n = 0, 1, \dots$. *If* $\mathsf{E}[(\Delta\breve{B}_1)^2] < \infty$, $\mathsf{E}[(\Delta\tau_1)^2] < \infty$ *and* $\mathsf{E}[\exp\{Q(\Delta\Theta_n)\}] < \infty$ $(n = 0, 1)$ *for some* $Q \in \mathcal{SC}$, *then for all* $x, u \ge 0$,

$$\mathsf{P}\left(\sup_{0 \le t \le x} \{\breve{B}(t) - \breve{b}t\} > u\right) \le C\left(e^{-cu^2/x} + e^{-cx} + xe^{-cQ(u)}\right), \tag{B.13}$$

where C *and* c *are independent of* x *and* u.

Note that if $\breve{B}(t) = B(t)$, then $\breve{b} = b > 0$, $\Delta\Theta_n = \Delta B_n^*$ and

$$\mathsf{P}\left(\sup_{0 \le t \le x} \{\breve{B}(t) - \breve{b}t\} > u\right) = \mathsf{P}\left(\sup_{0 \le t \le x} \{B(t) - bt\} > u\right).$$

On the other hand, suppose that $\breve{B}(t) = -B(t)$. We then have $\breve{b} = -b < 0$ and

$$\Delta\Theta_0 = \sup_{0 \le t \le \tau_0} \max(-B(t), 0) + b\Delta\tau_0 =: \Delta B_0^{**} + b\Delta\tau_0 \ge b\Delta\tau_0,$$
$$\Delta\Theta_n = \sup_{\tau_{n-1} \le t \le \tau_n} (B(\tau_{n-1}) - B(t)) + b\Delta\tau_n =: \Delta B_n^{**} + b\Delta\tau_n \ge b\Delta\tau_n, \quad n = 1, 2, \dots.$$

Thus $\mathsf{E}[\exp\{Q(\Delta\Theta_1)\}] < \infty$ implies $\mathsf{E}[(\Delta\tau_1)^2] < \infty$ (see Remark 2.4). We also have

$$\mathsf{P}\left(\sup_{0 \le t \le x} \{\breve{B}(t) - \breve{b}t\} > u\right) = \mathsf{P}\left(\inf_{0 \le t \le x} \{B(t) - bt\} < -u\right).$$

As a result, Lemma A.7 follows from Lemma B.1. As for the proof of Lemma B.1, see Appendix B.13.

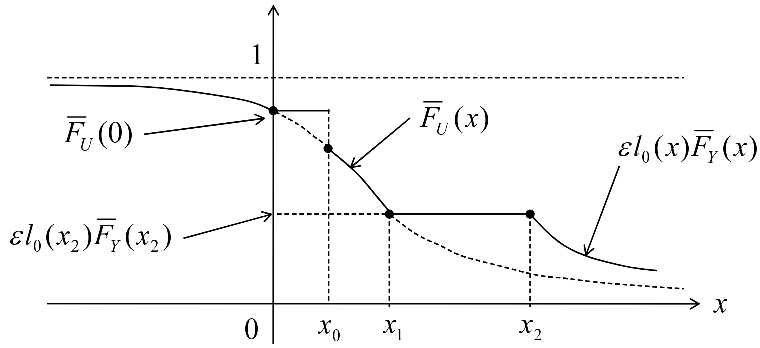

Figure 1: Tail distribution of $Z(\varepsilon, x_0)$

B.8. Proof of Lemma A.8

It follows from that Lemma 4.4 in [14] that there exists some nonincreasing slowly varying function l_0 such that $l_0(0) = 1$, $\lim_{x\to\infty} l_0(x) = 0$ and $\overline{F}_U(x) = o(l_0(x)\overline{F}_Y(x))$. Thus for any $\varepsilon > 0$ and $x_0 \ge 0$, there exists some $x_2 := x_2(\varepsilon, x_0) > x_0$ such that

$$\overline{F}_U(x) < \varepsilon l_0(x)\overline{F}_Y(x) \le \overline{F}_U(0), \qquad \forall x \ge x_2. \tag{B.14}$$

Let $x_1 := x_1(\varepsilon)$ denote

$$x_1 = \inf\left\{x \in [x_0, x_2]; \overline{F}_U(x) \le \varepsilon l_0(x_2)\overline{F}_Y(x_2)\right\},$$

from which and (B.14) we obtain $\overline{F}_U(x) \le \varepsilon l_0(x_2)\overline{F}_Y(x_2)$ for all $x_1 < x \le x_2$. Further since $\overline{F}_U$ is right-continuous,

$$\overline{F}_U(x) \le \varepsilon l_0(x_2)\overline{F}_Y(x_2), \qquad x_1 \le \forall x \le x_2.$$

We now define $Z(\varepsilon, x_0)$ as a r.v. in $\mathbb{R}$ such that (see Figure 1)

$$\overline{F}_{Z(\varepsilon,x_0)}(x) = \begin{cases} \overline{F}_U(x), & x < 0, \\ \overline{F}_U(0), & 0 \le x < x_0, \\ \overline{F}_U(x), & x_0 \le x < x_1, \\ \varepsilon l_0(x_2)\overline{F}_Y(x_2), & x_1 \le x < x_2, \\ \varepsilon l_0(x)\overline{F}_Y(x), & x \ge x_2. \end{cases} \tag{B.15}$$

Clearly, $\overline{F}_{Z(\varepsilon,x_0)}(x) \ge \overline{F}_U(x)$ for all $x \in \mathbb{R}$. Further it follows from (B.15) that

$$0 \le \mathsf{E}[Z(\varepsilon, x_0)] - \mathsf{E}[U] =: S_1(\varepsilon, x_0) + S_2(\varepsilon, x_0) + S_3(\varepsilon, x_0), \tag{B.16}$$

where

$$S_1(\varepsilon, x_0) = \int_0^{x_0} (\overline{F}_U(0) - \overline{F}_U(x))\mathrm{d}x, \tag{B.17}$$

$$S_2(\varepsilon, x_0) = \int_{x_1}^{x_2} \left(\varepsilon l_0(x_2)\overline{F}_Y(x_2) - \overline{F}_U(x)\right)\mathrm{d}x, \tag{B.18}$$

$$S_3(\varepsilon, x_0) = \int_{x_2}^{\infty} \left(\varepsilon l_0(x)\overline{F}_Y(x) - \overline{F}_U(x)\right)\mathrm{d}x. \tag{B.19}$$

From (B.16), (B.17) and $\int_0^\infty \overline{F}_U(x)\mathrm{d}x = \mathsf{E}[U^+] \le \mathsf{E}[|U|] < \infty$, we have

$$\lim_{x_0\to\infty} \sum_{j=1}^{3} S_j(\varepsilon, x_0) \ge \lim_{x_0\to\infty} S_1(\varepsilon, x_0) = \infty. \tag{B.20}$$

From (B.18) and (B.19), we also have

$$S_2(\varepsilon, x_0) + S_3(\varepsilon, x_0) \le \int_{x_1}^{\infty} \left(\varepsilon l_0(x)\overline{F}_Y(x) - \overline{F}_U(x)\right) \mathrm{d}x \le \int_{x_1}^{\infty} \varepsilon \overline{F}_Y(x)\mathrm{d}x,$$

where the second inequality follows from $l_0(x) \le 1$ for $x \ge 0$. Therefore since $\mathsf{E}[Y] < \infty$,

$$\lim_{\varepsilon\downarrow 0}(S_2(\varepsilon, x_0) + S_3(\varepsilon, x_0)) = 0,$$

which leads to

$$\lim_{x_0\downarrow 0}\lim_{\varepsilon\downarrow 0} \sum_{j=1}^{3} S_j(\varepsilon, x_0) = \lim_{x_0\downarrow 0}\lim_{\varepsilon\downarrow 0} S_1(\varepsilon, x_0) = 0. \tag{B.21}$$

According to (B.16), (B.20) and (B.21), we can fix $\mathsf{E}[Z(\varepsilon, x_0)] - \mathsf{E}[U] \in (0, y)$ for any $y > 0$. As a result, the statement of Lemma A.8 holds for $Z = Z(\varepsilon, x_0)$.

B.9. Proof of Lemma A.9

Let $\widetilde{X}_n$'s ($n = 1, 2, \dots$) are independent copies of $\widetilde{X} := X/\mathsf{E}[X]$. We then have

$$\begin{aligned}
&\{N_X(x) > u + x/\mathsf{E}[X]\} \\
&\subseteq \left\{ \sum_{n=1}^{\lfloor u+x/\mathsf{E}[X]\rfloor} X_n \le x \right\} = \left\{ \sum_{n=1}^{\lfloor u+x/\mathsf{E}[X]\rfloor} (1 - \widetilde{X}_n) \ge \lfloor u + x/\mathsf{E}[X]\rfloor - x/\mathsf{E}[X] \right\} \\
&\subseteq \left\{ \sum_{n=1}^{\lfloor u+x/\mathsf{E}[X]\rfloor} (1 - \widetilde{X}_n) \ge u - 1 \right\},
\end{aligned}$$

which leads to

$$\mathsf{P}\left(N_X(x) - \frac{x}{\mathsf{E}[X]} > u\right) \le \mathsf{P}\left(\sum_{n=1}^{\lfloor u+x/\mathsf{E}[X]\rfloor} (1 - \widetilde{X}_n) \ge u - 1 \right). \tag{B.22}$$

Using Markov's inequality (see, e.g., [45]), we have for any $s > 0$,

$$\begin{aligned}
\mathsf{P}\left(\sum_{n=1}^{\lfloor u+x/\mathsf{E}[X]\rfloor} (1 - \widetilde{X}_n) \ge u - 1 \right) &\le e^{-s(u-1)} \left(\mathsf{E}[e^{s(1-\widetilde{X})}]\right)^{\lfloor u+x/\mathsf{E}[X]\rfloor} \\
&\le e^{-s(u-1)} \left(\mathsf{E}[e^{s(1-\widetilde{X})}]\right)^{u+x/\mathsf{E}[X]} \\
&= e^{s(1+x/\mathsf{E}[X])} \left(\mathsf{E}[e^{-s\widetilde{X}}]\right)^{u+x/\mathsf{E}[X]},
\end{aligned} \tag{B.23}$$

where the second inequality follows from $\mathsf{E}[e^{s(1-\widetilde{X})}] \ge \exp\{s(1-\mathsf{E}[\widetilde{X}])\} = 1$ due to $\mathsf{E}[\widetilde{X}] = 1$ and Jensen's inequality (see, e.g., [45]). Further for any $s > 0$,

$$\mathsf{E}[e^{-s\widetilde{X}}] \le 1 - s\mathsf{E}[\widetilde{X}] + s^2\mathsf{E}[\widetilde{X}^2] = 1 - s + s^2\mathsf{E}[\widetilde{X}^2], \tag{B.24}$$

because $e^{-x} \le 1 - x + x^2$ for all $x \ge 0$ and $\widetilde{X} \ge 0$ w.p.1. Substituting (B.24) into (B.23), we obtain

$$\begin{aligned}
\mathsf{P}\left(\sum_{n=1}^{\lfloor u+x/\mathsf{E}[X] \rfloor} (1-\widetilde{X}_n) \ge u-1 \right) &\le e^{s(1+x/\mathsf{E}[X])} \left(1 - s + s^2 \mathsf{E}[\widetilde{X}^2]\right)^{u+x/\mathsf{E}[X]} \\
&\le e^{s(1+x/\mathsf{E}[X])} e^{(-s+s^2\mathsf{E}[\widetilde{X}^2])(u+x/\mathsf{E}[X])} \\
&\le e^s \exp\left\{-su + s^2 \cdot \mathsf{E}[\widetilde{X}^2](\delta + 1/\mathsf{E}[X]) \cdot x\right\} \\
&=: e^s \exp\left\{-su + s^2 \widetilde{K}(\delta) x\right\}, \qquad \text{(B.25)}
\end{aligned}$$

where we use $1 + x \le e^x$ $(x \in \mathbb{R})$ and $u \le \delta x$ in the second and third inequalities.

Finally, letting $s = (u/x)\{2\widetilde{K}(\delta)\}^{-1}$ in (B.25) and using $u/x \le \delta$, we obtain

$$\begin{aligned}
\mathsf{P}\left(\sum_{n=1}^{\lfloor u+x/\mathsf{E}[X] \rfloor} (1-\widetilde{X}_n) \ge u-1 \right) &\le \exp\left\{ \frac{u}{x} \frac{1}{2\widetilde{K}(\delta)} \right\} \cdot \exp\left\{ -\frac{1}{4\widetilde{K}(\delta)} \frac{u^2}{x} \right\} \\
&\le \exp\left\{ \frac{\delta}{2\widetilde{K}(\delta)} \right\} \cdot \exp\left\{ -\frac{1}{4\widetilde{K}(\delta)} \frac{u^2}{x} \right\}. \qquad \text{(B.26)}
\end{aligned}$$

Note here that $\widetilde{K}(\delta) = \mathsf{E}[\widetilde{X}^2](\delta + 1/\mathsf{E}[X])$ is finite and positive for any fixed $\delta > 0$. As a result, substituting (B.26) into (B.22) yields (A.4).

B.10. Proof of Lemma A.10

For all $n = 1, 2, \ldots$ and $k = 1, 2, \ldots, n$,

$$\begin{aligned}
\left\{ \max_{1\le k\le n} \sum_{i=1}^{k} U_i \ge x \right\} &= \bigcup_{1\le k\le n} \left\{ \sum_{i=1}^{k} U_i \ge x \right\}, \quad x > 0, \\
\left\{ \sum_{i=1}^{k} U_i \ge x \right\} &\subseteq \bigcup_{1\le i\le k} \{U_i \ge x/k\} \subseteq \bigcup_{1\le i\le k} \{U_i \ge x/n\}, \quad x > 0.
\end{aligned}$$

Thus for any fixed positive integer n_0 and all $n = 1, 2, \ldots, n_0$, we have

$$\left\{ \max_{1\le k\le n} \sum_{i=1}^{k} U_i \ge x \right\} \subseteq \bigcup_{1\le i\le n} \{U_i \ge x/n\} \subseteq \bigcup_{1\le i\le n} \{U_i \ge x/n_0\}, \quad x > 0,$$

which leads to

$$\mathsf{P}\left(\max_{1\le k\le n} \sum_{i=1}^{k} U_i \ge x \right) \le n\mathsf{P}(U \ge x/n_0), \quad n = 1, 2, \ldots, n_0,\ x > 0.$$

Therefore it suffices to prove that (A.5) holds for all sufficiently large n.

Let $\widetilde{U}_i = \min(U_i, vx)$ for $i = 1, 2, \ldots$, where $0 < v < 1/n_0$ is a constant. Since $\mathsf{E}[U] = 0$, we have $\mathsf{E}[\widetilde{U}_1] \le 0$. Thus for all $x > 0$,

$$\begin{aligned}\mathsf{P}\left(\max_{1\le k\le n}\sum_{i=1}^k U_i \ge x\right) &\le \mathsf{P}\left(\max_{1\le i\le n} U_i > vx\right) + \mathsf{P}\left(\max_{1\le k\le n}\sum_{i=1}^k U_i \ge x, \max_{1\le i\le n} U_i \le vx\right)\\ &\le n\mathsf{P}(U > vx) + \mathsf{P}\left(\max_{1\le k\le n}\sum_{i=1}^k \widetilde{U}_i \ge x\right)\\ &\le n\mathsf{P}(U > vx) + \mathsf{P}\left(\max_{1\le k\le n}\sum_{i=1}^k W_i \ge x\right), \end{aligned} \tag{B.27}$$

where $W_i = \widetilde{U}_i - \mathsf{E}[\widetilde{U}_1]$ for $i = 1, 2, \dots$. In what follows, we estimate the second term on the right hand side of (B.27).

Since $\{\sum_{i=1}^k W_i\}; k = 1, 2, \dots\}$ is martingale, $\{\exp\{s\sum_{i=1}^k W_i\}; k = 1, 2, \dots\}$ is submartingale for any $s > 0$ (see, e.g., [45, Section 14.6, Lemma (b)]). It thus follows from Doob's submartingale inequality (see, e.g., [45, Section 14.6, Theorem (a)]) that for any $s > 0$,

$$\begin{aligned}\mathsf{P}\left(\max_{1\le k\le n}\sum_{i=1}^k W_i \ge x\right) &= \mathsf{P}\left(\max_{1\le k\le n}\exp\left\{s\sum_{i=1}^k W_i\right\} \ge e^{sx}\right)\\ &\le e^{-sx}\mathsf{E}\left[\exp\left\{s\sum_{i=1}^n W_i\right\}\right] = e^{-sx}\left(\mathsf{E}[e^{s\widetilde{U}_1}]\right)^n e^{-sn\mathsf{E}[\widetilde{U}_1]}. \end{aligned} \tag{B.28}$$

We first estimate $e^{-sx}(\mathsf{E}[e^{s\widetilde{U}_1}])^n$ on the right hand side of (B.28). Let $1 < q < \min(r, 2)$ and

$$s = \frac{1}{vx}\log\left(\frac{v^{q-1}x^q}{n\mathsf{E}[(U^+)^q]} + 1\right). \tag{B.29}$$

Following the estimation of the right hand side of (2.4) in [44], we can prove that there exist some positive constant $\widetilde{C}_1 := \widetilde{C}_1(v, \gamma)$ and some positive integer n_1 such that

$$e^{-sx}\left(\mathsf{E}[e^{s\widetilde{U}_1}]\right)^n \le \widetilde{C}_1 x^{-(q-1)/(2v)}, \qquad \forall x \ge \gamma n,\ \forall n \ge n_1.$$

Fix $n_0 = n_1$ and $v := v(r, p)$ such that $0 < v < 1/n_0$ and $(q-1)/(2v) > p$. We then have

$$e^{-sx}\left(\mathsf{E}[e^{s\widetilde{U}_1}]\right)^n \le \widetilde{C}_1 x^{-p}, \qquad \forall x \ge \gamma n,\ \forall n \ge n_0. \tag{B.30}$$

Next we estimate $e^{-sn\mathsf{E}[\widetilde{U}_1]}$ on the right hand side of (B.28). From (B.29), $\mathsf{E}[\widetilde{U}_1] \le 0$, $x \ge \gamma n$ and $n \ge 1$, we have

$$-sn\mathsf{E}[\widetilde{U}_1] \le \frac{1}{v\gamma}\log\left(\frac{v^{q-1}x^q}{\mathsf{E}[(U^+)^q]} + 1\right)(-\mathsf{E}[\widetilde{U}_1]). \tag{B.31}$$

Note here that

$$\begin{aligned}\mathsf{E}[\widetilde{U}_1] &= \mathsf{E}[U \cdot \mathbb{1}(U \le vx)] + vx\mathsf{P}(U > vx),\\ \mathsf{E}[U \cdot \mathbb{1}(U \le vx)] &+ \mathsf{E}[U \cdot \mathbb{1}(U > vx)] = \mathsf{E}[U] = 0\end{aligned}$$

It thus follows from $\mathsf{P}(U^+ > x) = o(x^{-r})$ (due to $\mathsf{E}[(U^+)^r] < \infty$) that for all $x > 0$,

$$\begin{aligned} -\mathsf{E}[\widetilde{U}_1] &\le -\mathsf{E}[U \cdot 1\!\!1(U \le vx)] = \mathsf{E}[U \cdot 1\!\!1(U > vx)] = \mathsf{E}[U^+ \cdot 1\!\!1(U^+ > vx)] \\ &= vx\mathsf{P}(U^+ > vx) + \int_{vx}^{\infty} \mathsf{P}(U^+ > y)\mathrm{d}y = o(x^{-r+1}). \end{aligned}$$

This equation and (B.31) imply that for all $x \ge \gamma n$ and $n = 1, 2, \dots$,

$$e^{-sn\mathsf{E}[\widetilde{U}_1]} \le \widetilde{C}_2 < \infty, \tag{B.32}$$

where

$$\widetilde{C}_2 := \widetilde{C}_2(v,\gamma) = \sup_{x\ge\gamma} \exp\left\{ \frac{1}{v\gamma} \log\left(\frac{v^{q-1}x^q}{\mathsf{E}[(U^+)^q]} + 1 \right) Cx^{-r+1} \right\}.$$

Substituting (B.30) and (B.32) into (B.28) and letting $\widetilde{C} := \widetilde{C}(v,\gamma) = \widetilde{C}_1(v,\gamma)\widetilde{C}_2(v,\gamma)$ yield

$$\mathsf{P}\left(\max_{1\le k\le n} \sum_{i=1}^{k} W_i \ge x \right) \le \widetilde{C}x^{-p}, \qquad \forall x \ge \gamma n,\ \forall n \ge n_0.$$

This inequality and (B.27) show that (A.5) holds for all $n \ge n_0$.

B.11. Proof of Lemma A.11

Let V_i's $(i = 1, 2, \dots)$ denote independent copies of $V := U - \mathsf{E}[U] - \varepsilon$, where $\varepsilon > 0$. Clearly, $V \le U$ and $\mathsf{E}[V] = -\varepsilon < 0$. Further since $U^+ \in \mathcal{C}$, we have $V^+ \in \mathcal{C} \subset \mathcal{S}^*$ (see Remark 2.5). It thus follows from the theorem in [27] that for all $x \ge (\mathsf{E}[U] + \varepsilon)n$ and $n = 1, 2, \dots$,

$$\begin{aligned} \mathsf{P}\left(\max_{1\le k\le n} \sum_{i=1}^{k} U_i \ge x \right) &\le \mathsf{P}\left(\max_{1\le k\le n} \sum_{i=1}^{k} V_i \ge x - (\mathsf{E}[U] + \varepsilon)n \right) \\ &\le \frac{C}{\varepsilon} \int_{x-(\mathsf{E}[U]+\varepsilon)n}^{x-\mathsf{E}[U]n} \mathsf{P}(V > y)\mathrm{d}y \\ &\le Cn\mathsf{P}(V > x - (\mathsf{E}[U] + \varepsilon)n) \\ &\le Cn\mathsf{P}(U > x - (\mathsf{E}[U] + \varepsilon)n). \end{aligned} \tag{B.33}$$

It also follows from $U^+ \in \mathcal{C} \subset \mathcal{D}$ that for all $x \ge (1+\varepsilon)(\mathsf{E}[U]+\varepsilon)n$, $n = 1, 2, \dots$ and $\varepsilon > 0$,

$$\mathsf{P}(U > x - (\mathsf{E}[U] + \varepsilon)n) \le \mathsf{P}(U > \varepsilon x/(1+\varepsilon)) \le \widehat{C}\mathsf{P}(U > x), \tag{B.34}$$

where $\widehat{C} := \widehat{C}(\gamma) \in (0, \infty)$ is given by

$$\widehat{C} = \sup_{x\ge\gamma} \frac{\mathsf{P}(U > \varepsilon x/(1+\varepsilon))}{\mathsf{P}(U > x)} \quad \text{with } \gamma := \gamma(\varepsilon) = (1+\varepsilon)(\mathsf{E}[U] + \varepsilon).$$

According to (B.33) and (B.34), there exists some $\widetilde{C} := \widetilde{C}(\gamma) \in (0, \infty)$ such that

$$\mathsf{P}\left(\max_{1\le k\le n} \sum_{i=1}^{k} U_i \ge x \right) \le \widetilde{C}n\mathsf{P}(U > x), \quad x \ge \gamma n,\ n = 1, 2, \dots.$$

B.12. Proof of Lemma A.12

It follows from (A.7) that for any $\varepsilon > 0$ there exists some $x_0 := x_0(\varepsilon) > 0$ such that for all $x \ge x_0$,

$$\begin{aligned} \overline{F}_{i,j}(x) &\le (\widetilde{F}_{i,j} + \varepsilon)\mathsf{P}(Y > x), \qquad & 1 \le i \le m_0,\ 1 \le j \le m_1, \\ \overline{G}_{i,j}(x) &\le (\widetilde{G}_{i,j} + \varepsilon)\mathsf{P}(Y > x), \qquad & 1 \le i \le m_1,\ 1 \le j \le m_2. \end{aligned}$$

Without loss of generality, we assume that $\lim_{\varepsilon\downarrow 0} x_0(\varepsilon) = \infty$.

We now define $\boldsymbol{P} = (P_{i,j})$ and $\boldsymbol{Q} = (Q_{i,j})$ as $m_0 \times m_1$ and $m_1 \times m_2$ matrix-valued functions on $\mathbb{R}$ such that $\overline{P}_{i,j}(x) := P_{i,j}(\infty) - P_{i,j}(x)$ and $\overline{Q}_{i,j}(x) := Q_{i,j}(\infty) - Q_{i,j}(x)$ are given by

$$\begin{aligned} \overline{P}_{i,j}(x) &= \begin{cases} \max\left(\overline{F}_{i,j}(x), (\widetilde{F}_{i,j} + \varepsilon)\mathsf{P}(Y > x_0)\right), & x < x_0, \\ (\widetilde{F}_{i,j} + \varepsilon)\mathsf{P}(Y > x), & x \ge x_0, \end{cases} \\ \overline{Q}_{i,j}(x) &= \begin{cases} \max\left(\overline{G}_{i,j}(x), (\widetilde{G}_{i,j} + \varepsilon)\mathsf{P}(Y > x_0)\right), & x < x_0, \\ (\widetilde{G}_{i,j} + \varepsilon)\mathsf{P}(Y > x), & x \ge x_0, \end{cases} \end{aligned}.$$

Clearly, $\overline{F}_{i,j}(x) \le \overline{P}_{i,j}(x)$ and $\overline{G}_{i,j}(x) \le \overline{Q}_{i,j}(x)$ for all $x \in \mathbb{R}$. Further,

$$\lim_{x\to\infty} \frac{\overline{P}_{i,j}(x)}{\mathsf{P}(Y > x)} = \widetilde{F}_{i,j} + \varepsilon =: \widetilde{P}_{i,j}, \quad \lim_{x\to\infty} \frac{\overline{Q}_{i,j}(x)}{\mathsf{P}(Y > x)} = \widetilde{G}_{i,j} + \varepsilon =: \widetilde{Q}_{i,j}.$$

Therefore using Proposition A.3 in [31] yields

$$\limsup_{x\to\infty} \frac{\overline{\boldsymbol{F} * \boldsymbol{G}}(x)}{\mathsf{P}(Y > x)} \le \lim_{x\to\infty} \frac{\overline{\boldsymbol{P} * \boldsymbol{Q}}(x)}{\mathsf{P}(Y > x)} = \widetilde{\boldsymbol{P}}\boldsymbol{Q}(\infty) + \boldsymbol{P}(\infty)\widetilde{\boldsymbol{Q}}, \tag{B.35}$$

where $\widetilde{\boldsymbol{P}} = (\widetilde{P}_{i,j})$ and $\widetilde{\boldsymbol{Q}} = (\widetilde{Q}_{i,j})$. Note here that

$$\lim_{\varepsilon\downarrow 0} \left(\widetilde{\boldsymbol{P}}\boldsymbol{Q}(\infty) + \boldsymbol{P}(\infty)\widetilde{\boldsymbol{Q}}\right) = \widetilde{\boldsymbol{F}}\boldsymbol{G}(\infty) + \boldsymbol{F}(\infty)\widetilde{\boldsymbol{G}}.$$

Combining this with (B.35), we have the first statement (A.8). The second statement (A.9) can be proved by induction using the first statement.

Finally we prove the third statement (A.10). Since $\sum_{n=0}^{\infty}(\boldsymbol{F}(\infty))^n = (\boldsymbol{I} - \boldsymbol{F}(\infty))^{-1}$,

$$\sum_{n=0}^{\infty}(\boldsymbol{P}(\infty))^n = (\boldsymbol{I} - \boldsymbol{P}(\infty))^{-1} \quad \text{for any sufficiently small } \varepsilon > 0.$$

It thus follows from $\overline{\boldsymbol{F}^{*n}}(x) \le \overline{\boldsymbol{P}^{*n}}(x)$ ($x \in \mathbb{R}$, $n = 0, 1, \dots$) and Lemma 6 in [22] that

$$\limsup_{x\to\infty} \sum_{n=0}^{\infty} \frac{\overline{\boldsymbol{F}^{*n}}(x)}{\mathsf{P}(Y > x)} \le \lim_{x\to\infty} \sum_{n=0}^{\infty} \frac{\overline{\boldsymbol{P}^{*n}}(x)}{\mathsf{P}(Y > x)} = (\boldsymbol{I} - \boldsymbol{P}(\infty))^{-1}\widetilde{\boldsymbol{P}}(\boldsymbol{I} - \boldsymbol{P}(\infty))^{-1}. \tag{B.36}$$

Letting $\varepsilon \downarrow 0$ in (B.36) and using the dominated convergence theorem, we obtain the third statement (A.10).

B.13. Proof of Lemma B.1

It follows from condition (iii) of Definition 2.3 that there exists some $x_* > 0$ such that

$$Q(x/3) \ge Q(x)/3^\alpha \ge Q(x)/3, \qquad \forall x \ge x_*. \tag{B.37}$$

Let η denote any fixed positive number such that $\eta x_*^2 \ge 1$. We then discuss three cases: (a) $0 \le x < \eta x_*^2$, (b) $x > \eta u^2$ and (c) $\eta x_*^2 \le x \le \eta u^2$ separately. In case (a), (B.13) holds for $C \ge e^{(\eta x_*)^2}$ because $Ce^{-\eta x} > Ce^{-(\eta x_*)^2} \ge 1$. In case (b), (B.13) also holds for $C \ge e$ because $Ce^{-\eta u^2/x} > Ce^{-1} \ge 1$. Therefore in what follows, we consider case (c).

For all $t \ge 0$, we have

$$\begin{aligned}\breve{B}(t) - \breve{b}t &\le \Delta\breve{B}_0^* + \Delta\breve{B}_{N(t-\tau_0)+1}^* - \min(\breve{b},0)(\Delta\tau_0 + \Delta\tau_{N(t-\tau_0)+1}) + \sum_{i=1}^{N(t-\tau_0)} (\Delta\breve{B}_i - \breve{b}\Delta\tau_i) \\ &= \Delta\Theta_0 + \Delta\Theta_{N(t-\tau_0)+1} + \sum_{i=1}^{N(t-\tau_0)} (\Delta\breve{B}_i - \breve{b}\Delta\tau_i),\end{aligned}$$

where $N(t) = \max\{n \ge 0; \sum_{i=1}^n \Delta\tau_i \le t\}$ for $t \in \mathbb{R}$. Thus we obtain

$$\begin{aligned}&\mathsf{P}\left(\sup_{0\le t\le x}\{\breve{B}(t) - \breve{b}t\} > u\right) \\ &\quad\le \mathsf{P}\left(\Delta\Theta_0 > \frac{u}{3}\right) + \mathsf{P}\left(\Delta\Theta_1 > \frac{u}{3}\right) + \mathsf{P}\left(\max_{1\le n\le N(x-\tau_0)} \sum_{i=1}^n (\Delta\breve{B}_i - \breve{b}\Delta\tau_i) > \frac{u}{3}\right) \\ &\quad\le \mathsf{P}\left(\Delta\Theta_0 > \frac{u}{3}\right) + \mathsf{P}\left(\Delta\Theta_1 > \frac{u}{3}\right) + \mathsf{P}\left(\max_{1\le n\le N(x)} \sum_{i=1}^n (\Delta\breve{B}_i - \breve{b}\Delta\tau_i) > \frac{u}{3}\right), \end{aligned} \tag{B.38}$$

where we use the inequality $\mathsf{P}(X^{(1)} + X^{(2)} + X^{(3)} > u) \le \sum_{m=1}^3 \mathsf{P}(X^{(m)} > u/3)$ for any triple of r.v.s $X^{(m)}$'s $(m = 1,2,3)$. Note here that $\mathsf{P}(\Delta\Theta_n > x) \le Ce^{-Q(x)}$ for all $x \ge 0$ due to $\mathsf{E}[\exp\{Q(\Delta\Theta_n)\}] < \infty$ $(n = 0,1)$. It then follows from $\eta x_*^2 \ge 1$ that for all x and u such that $\eta x_*^2 \le x \le \eta u^2$,

$$\mathsf{P}\left(\Delta\Theta_n > \frac{u}{3}\right) \le Ce^{-Q(u/3)} \le C\eta x_*^2 e^{-Q(u/3)} \le Cxe^{-Q(u/3)}, \qquad n = 0,1,$$

from which and (B.38) we have

$$\mathsf{P}\left(\sup_{0\le t\le x}\{\breve{B}(t) - \breve{b}t\} > u\right) \le Cxe^{-Q(u/3)} + \mathsf{P}\left(\max_{1\le n\le N(x)} \sum_{i=1}^n (\Delta\breve{B}_i - \breve{b}\Delta\tau_i) > \frac{u}{3}\right). \tag{B.39}$$

We now fix $\delta > 0$ arbitrarily and then have

$$\begin{aligned}&\mathsf{P}\left(\max_{1\le n\le N(x)} \sum_{i=1}^n (\Delta\breve{B}_i - \breve{b}\Delta\tau_i) > \frac{u}{3}\right) \\ &\quad\le \mathsf{P}\left(N(x) - \frac{x}{\mathsf{E}[\Delta\tau_1]} > \delta x\right) + \mathsf{P}\left(\max_{1\le n\le (\delta+1/\mathsf{E}[\Delta\tau_1])x} \sum_{i=1}^n (\Delta\breve{B}_i - \breve{b}\Delta\tau_i) > \frac{u}{3}\right). \end{aligned} \tag{B.40}$$

Applying Lemma A.9 (which requires $\mathsf{E}[(\Delta\tau_1)^2] < \infty$) to the first term on the right hand side of (B.40), we have

$$\mathsf{P}\left(N(x) - \frac{x}{\mathsf{E}[\Delta\tau_1]} > \delta x\right) \le Ce^{-cx}, \qquad x \ge 0. \tag{B.41}$$

Note that $\Delta\Theta_1 \ge 0$ and $\Delta\Theta_1 \ge \Delta\breve{B}_1 - \breve{b}\Delta\tau_1$, which lead to $\Delta\Theta_1 \ge (\Delta\breve{B}_1 - \breve{b}\Delta\tau_1)^+$. Thus $\mathsf{E}[\exp\{Q(\Delta\Theta_1)\}] < \infty$ yields

$$\mathsf{E}[\exp\{Q((\Delta\breve{B}_1 - \breve{b}\Delta\tau_1)^+)\}] < \infty.$$

Further it follows from $\mathsf{E}[(\Delta\tau_1)^2] < \infty$, $\mathsf{E}[(\Delta\breve{B}_1)^2] < \infty$ and Hölder's inequality (see, e.g., [45]) that

$$\left|\mathsf{E}[\Delta\breve{B}_1\Delta\tau_1]\right| \le \sqrt{\mathsf{E}[(\Delta\breve{B}_1)^2]}\sqrt{\mathsf{E}[(\Delta\tau_1)^2]} < \infty,$$

which implies $\mathsf{E}[(\Delta\breve{B}_1 - \breve{b}\Delta\tau_1)^2] < \infty$.

We now need the following result:

Proposition B.2 (Lemma 5 in [25]) *Suppose that $U, U_1, U_2, \dots$ are i.i.d. r.v.s in $\mathbb{R}$. If $\mathsf{E}[U^2] < \infty$ and $\mathsf{E}[e^{Q(U^+)}] < \infty$ for some $Q \in \mathcal{SC}$, then for all $x, u \ge 0$,*

$$\mathsf{P}\left(\max_{1\le n\le x}\left\{\sum_{i=1}^{n} U_i - n\mathsf{E}[U]\right\} > u\right) \le C\left(e^{-cu^2/x} + xe^{-(1/2)Q(u)}\right),$$

where C and c are independent of x and u.

Remark B.1 Although $\mathsf{E}[U^2] < \infty$ is not explicitly assumed in Lemma 5 in [25], this condition is required to prove the lemma (see p. 110 therein).

Applying Proposition B.2 to the second term on the right hand side of (B.40) and using $\mathsf{E}[\Delta\breve{B}_1 - \breve{b}\Delta\tau_1] = 0$, we obtain

$$\mathsf{P}\left(\max_{1\le n\le(\delta+1/\mathsf{E}[\Delta\tau_1])x}\sum_{i=1}^{n}(\Delta\breve{B}_i - \breve{b}\Delta\tau_i) > \frac{u}{3}\right) \le C\left(e^{-cu^2/x} + xe^{-(1/2)Q(u/3)}\right). \tag{B.42}$$

Substituting (B.41) and (B.42) into (B.40) yields

$$\mathsf{P}\left(\max_{1\le n\le N(x)}\sum_{i=1}^{n}(\Delta\breve{B}_i - \breve{b}\Delta\tau_i) > \frac{u}{3}\right) \le C\left(e^{-cx} + e^{-cu^2/x} + xe^{-(1/2)Q(u/3)}\right),$$

from which and (B.39), we have

$$\mathsf{P}\left(\sup_{0\le t\le x}\{\breve{B}(t) - \breve{b}t\} > u\right) \le C\left(e^{-cx} + e^{-cu^2/x} + xe^{-(1/2)Q(u/3)}\right). \tag{B.43}$$

Recall that in case (c), we have $u \ge x_*$ and thus $Q(u/3) \ge (1/3)Q(u)$ due to (B.37). Finally, (B.43) yields (B.13).

B.14. Proof of Proposition B.1

We first prove (B.3). The Taylor expansion of $(x+y)^\gamma$ is given by

$$(x+y)^\gamma = \sum_{n=0}^{\infty}\frac{\gamma(\gamma-1)\cdots(\gamma-n+1)}{n!}y^n x^{\gamma-n}, \quad x > y \ge 0,$$

from which we have

$$(x+y)^\gamma \le x^\gamma + \sum_{n=1}^{\infty}\frac{|\gamma(\gamma-1)\cdots(\gamma-n+1)|}{n!}\left(\frac{y}{x}\right)^{n-1} yx^{\gamma-1}, \quad x > y \ge 0. \tag{B.44}$$

Note here that

$$\begin{aligned}|\gamma(\gamma-1)\cdots(\gamma-n+1)| &= \gamma(\gamma-1)\cdots(\gamma-\lfloor\gamma\rfloor)\cdot\prod_{i=1}^{n-1-\lfloor\gamma\rfloor}(i+\lfloor\gamma\rfloor-\gamma)\\ &\le \gamma(\gamma-1)\cdots(\gamma-\lfloor\gamma\rfloor)\cdot n!.\end{aligned}$$

Thus since $y/x<1$,

$$\begin{aligned}\sum_{n=1}^{\infty}\frac{|\gamma(\gamma-1)\cdots(\gamma-n+1)|}{n!}\left(\frac{y}{x}\right)^{n-1} &\le \gamma(\gamma-1)\cdots(\gamma-\lfloor\gamma\rfloor)\cdot\sum_{n=1}^{\infty}\left(\frac{y}{x}\right)^{n-1}\\ &= \gamma(\gamma-1)\cdots(\gamma-\lfloor\gamma\rfloor)\cdot\left(1-\frac{y}{x}\right)^{-1}. \qquad \text{(B.45)}\end{aligned}$$

Substituting (B.45) into (B.44) yields (B.3).

In the same way as the proof of (B.3), we have

$$(x-y)^{\gamma} \ge x^{\gamma} - \sum_{n=1}^{\infty}\frac{|\gamma(\gamma-1)\cdots(\gamma-n+1)|}{n!}\left(\frac{y}{x}\right)^{n-1} yx^{\gamma-1},$$

from which and (B.45) we obtain (B.4).

C. Proofs of Main Results

C.1. Proof of Theorem 3.1

Since $\{B(t)\}$ is nondecreasing with t, we have for $x \ge 1$,

$$\begin{aligned}\mathsf{P}(B(T)>x) &\le \mathsf{P}(T>x-x^{2/3}) + \mathsf{P}(B(T)>x, T\le x-x^{2/3})\\ &\le \mathsf{P}(T>x-x^{2/3}) + \mathsf{P}(B(x-x^{2/3})>x),\\ \mathsf{P}(B(T)>x) &\ge \mathsf{P}(B(T)>x, T>x+x^{2/3})\\ &= \mathsf{P}(T>x+x^{2/3}) - \mathsf{P}(B(T)\le x, T>x+x^{2/3})\\ &\ge \mathsf{P}(T>x+x^{2/3}) - \mathsf{P}(B(x+x^{2/3})\le x).\end{aligned}$$

Note here that condition (i) is equivalent to $T\in\mathcal{L}^3$ (see Lemma A.1 (ii)). It thus follows from Lemma A.2 that

$$\mathsf{P}(T>x+x^{2/3}) \stackrel{x}{\sim} \mathsf{P}(T>x-x^{2/3}) \stackrel{x}{\sim} \mathsf{P}(T>x).$$

Therefore it suffices to show that

$$\begin{aligned}\mathsf{P}(B(x-x^{2/3})>x) &= o(\mathsf{P}(T>x-x^{2/3})),\\ \mathsf{P}(B(x+x^{2/3})\le x) &= o(\mathsf{P}(T>x+x^{2/3})).\end{aligned}$$

For $x\ge 1$, we have

$$\begin{aligned}\mathsf{P}(B(x-x^{2/3})>x) &= \mathsf{P}\left(B(x-x^{2/3})-(x-x^{2/3})>x^{2/3}\right)\\ &\le \mathsf{P}\left(\sup_{0\le t\le x-x^{2/3}}(B(t)-t)>x^{2/3}\right)\\ &\le \mathsf{P}\left(\sup_{0\le t\le x}(B(t)-t)>x^{2/3}\right), \qquad \text{(C.1)}\end{aligned}$$

and

$$\begin{aligned}\mathsf{P}(B(x+x^{2/3}) \le x) &\le \mathsf{P}(B(x+x^{2/3}) < x + (1/2)x^{2/3}) \\ &= \mathsf{P}\left(B(x+x^{2/3}) - (x+x^{2/3}) < -(1/2)x^{2/3}\right) \\ &\le \mathsf{P}\left(\inf_{0\le t\le x+x^{2/3}} (B(t)-t) < -(1/2)x^{2/3}\right) \\ &\le \mathsf{P}\left(\inf_{0\le t\le 2x} (B(t)-t) < -(1/2)x^{2/3}\right). \qquad \text{(C.2)}\end{aligned}$$

Applying Lemma A.7 (i) to the right hand side of (C.1), we obtain

$$\mathsf{P}(B(x-x^{2/3}) > x) \le C\left(e^{-cx^{1/3}} + e^{-cx} + xe^{-cQ(x^{2/3})}\right), \qquad x \ge 1. \qquad \text{(C.3)}$$

Since $T^\theta \in \mathcal{L}$, we have $\mathsf{P}(T > x) = e^{-o(x^\theta)}$ (see Lemma A.1 (i)) and thus for any $0 < \theta \le 1/3$,

$$\begin{aligned}\limsup_{x\to\infty} \frac{e^{-cx^{1/3}}}{\mathsf{P}(T > x - x^{2/3})} &= \limsup_{x\to\infty} e^{-cx^{1/3}+o(x^\theta)} = 0, \\ \limsup_{x\to\infty} \frac{e^{-cx}}{\mathsf{P}(T > x - x^{2/3})} &= \limsup_{x\to\infty} e^{-cx+o(x^\theta)} = 0.\end{aligned}$$

Further it follows from (3.1) and $\mathsf{P}(T > x) = e^{-o(x^\theta)}$ that

$$\limsup_{x\to\infty} \frac{xe^{-cQ(x^{2/3})}}{\mathsf{P}(T > x - x^{2/3})} \le \limsup_{x\to\infty} e^{-cx^\theta + \log x + o(x^\theta)} = 0.$$

As a result, we have $\mathsf{P}(B(x-x^{2/3}) > x) = o(\mathsf{P}(T > x - x^{2/3}))$.

Next we estimate $\mathsf{P}(B(x+x^{2/3}) \le x)$. Applying Lemma A.7 (ii) to the right hand side of (C.2) yields

$$\mathsf{P}(B(x+x^{2/3}) \le x) \le C\left(e^{-cx^{1/3}} + e^{-cx} + xe^{-cQ((1/2)x^{2/3})}\right).$$

Therefore similarly to the estimation of (C.3), we can readily show $\mathsf{P}(B(x+x^{2/3}) \le x) = o(\mathsf{P}(T > x + x^{2/3}))$. □

C.2. Proof of Theorem 3.2

We fix $\varepsilon \in (0,1)$ arbitrarily. Since $\{B(t)\}$ is nondecreasing with t, we have for $x > 0$,

$$\begin{aligned}\mathsf{P}(B(T) > x) &\le \mathsf{P}(T > (1-\varepsilon)x) + \mathsf{P}(B(T) > x, T \le (1-\varepsilon)x) \\ &\le \mathsf{P}(T > (1-\varepsilon)x) + \mathsf{P}(B((1-\varepsilon)x) > x), \\ \mathsf{P}(B(T) > x) &\ge \mathsf{P}(B(T) > x, T > (1+\varepsilon)x) \\ &= \mathsf{P}(T > (1+\varepsilon)x) - \mathsf{P}(B(T) \le x, T > (1+\varepsilon)x) \\ &\ge \mathsf{P}(T > (1+\varepsilon)x) - \mathsf{P}(B((1+\varepsilon)x) \le x).\end{aligned}$$

Since $T \in \mathcal{C}$ (see Definition 1.2),

$$\lim_{\varepsilon\downarrow 0} \liminf_{x\to\infty} \frac{\mathsf{P}(T > (1+\varepsilon)x)}{\mathsf{P}(T > x)} = 1, \qquad \text{(C.4)}$$

$$\lim_{\varepsilon\downarrow 0} \limsup_{x\to\infty} \frac{\mathsf{P}(T > (1-\varepsilon)x)}{\mathsf{P}(T > x)} = 1. \qquad \text{(C.5)}$$

Therefore it suffices to show that

$$\mathsf{P}(B((1-\varepsilon)x) > x) = o(\mathsf{P}(T > x)), \tag{C.6}$$
$$\mathsf{P}(B((1+\varepsilon)x) \le x) = o(\mathsf{P}(T > x)). \tag{C.7}$$

For $x > 0$, we have

$$\begin{aligned}
\mathsf{P}(B((1-\varepsilon)x) > x) &= \mathsf{P}(B((1-\varepsilon)x) - (1-\varepsilon)x > \varepsilon x) \\
&\le \mathsf{P}\left(\sup_{0 \le t \le (1-\varepsilon)x} (B(t) - t) > \varepsilon x\right) \\
&\le \mathsf{P}\left(\sup_{0 \le t \le (1+\varepsilon)x} (B(t) - t) > \varepsilon x\right), \qquad \text{(C.8)}\\
\mathsf{P}(B((1+\varepsilon)x) \le x) &= \mathsf{P}(B((1+\varepsilon)x) - (1+\varepsilon)x \le -\varepsilon x) \\
&\le \mathsf{P}(B((1+\varepsilon)x) - (1+\varepsilon)x < -\varepsilon x/2) \\
&\le \mathsf{P}\left(\sup_{0 \le t \le (1+\varepsilon)x} (t - B(t)) > \varepsilon x/2\right). \qquad \text{(C.9)}
\end{aligned}$$

Note here that

$$\begin{aligned}
\sum_{i=1}^{N(t-\tau_0)} \Delta\tau_i \le t \le \sum_{i=0}^{N(t-\tau_0)+1} \Delta\tau_i, \\
\sum_{i=1}^{N(t-\tau_0)} \Delta B_i \le B(t) - B(0) \le \sum_{i=0}^{N(t-\tau_0)+1} \Delta B_i - B(0),
\end{aligned}$$

where $N(t) = \max\{n \ge 0; \sum_{i=1}^{n} \Delta\tau_i \le t\}$ for $t \in \mathbb{R}$. We thus have

$$B(t) - t \le \Delta B_0 + \Delta B_{N(t-\tau_0)+1} + \sum_{i=1}^{N(t-\tau_0)} (\Delta B_i - \Delta\tau_i), \tag{C.10}$$

$$t - B(t) \le \Delta\tau_0 + \Delta\tau_{N(t-\tau_0)+1} + \sum_{i=1}^{N(t-\tau_0)} (\Delta\tau_i - \Delta B_i) - B(0). \tag{C.11}$$

Therefore similarly to (B.38), it follows from (C.8) and (C.10) that

$$\begin{aligned}
\mathsf{P}(B((1-\varepsilon)x) > x) &\le \mathsf{P}(\Delta B_0 > \varepsilon x/3) + \mathsf{P}(\Delta B_1 > \varepsilon x/3) \\
&\quad + \mathsf{P}\left(\max_{1 \le k \le N((1+\varepsilon)x)} \sum_{i=1}^{k} (\Delta B_i - \Delta\tau_i) > \frac{\varepsilon}{3} x\right), \quad x > 0, \qquad \text{(C.12)}
\end{aligned}$$

and it follows from (C.9) and (C.11) that

$$\begin{aligned}
\mathsf{P}(B((1+\varepsilon)x) \le x) &\le \mathsf{P}(-B(0) > \varepsilon x/8) + \mathsf{P}(\Delta\tau_0 > \varepsilon x/8) + \mathsf{P}(\Delta\tau_1 > \varepsilon x/8) \\
&\quad + \mathsf{P}\left(\max_{1 \le k \le N((1+\varepsilon)x)} \sum_{i=1}^{k} (\Delta\tau_i - \Delta B_i) > \frac{\varepsilon}{8} x\right), \quad x > 0. \qquad \text{(C.13)}
\end{aligned}$$

Since $T \in \mathcal{C} \subset \mathcal{D}$, condition (iii) yields

$$\lim_{x\to\infty} \frac{\mathsf{P}(-B(0) > \varepsilon x/8)}{\mathsf{P}(T > x)} \le \limsup_{x\to\infty} \frac{\mathsf{P}(-B(0) > \varepsilon x/8)}{\mathsf{P}(T > \varepsilon x/8)} \limsup_{x\to\infty} \frac{\mathsf{P}(T > \varepsilon x/8)}{\mathsf{P}(T > x)} = 0,$$

which shows that $\mathsf{P}(-B(0) > \varepsilon x/8) = o(\mathsf{P}(T > x))$. In addition, $\mathsf{P}(\Delta\tau_n > \varepsilon x/8) = o(\mathsf{P}(T > x))$ and $\mathsf{P}(\Delta B_n > \varepsilon x/3) = o(\mathsf{P}(T > x))$ $(n = 0, 1)$.

As for the last terms in (C.12) and (C.13), we have for any $\delta > 0$,

$$\begin{aligned}
&\mathsf{P}\left(\max_{1\le k\le N((1+\varepsilon)x)} \sum_{i=1}^{k} (\Delta B_i - \Delta\tau_i) > \frac{\varepsilon}{3}x\right) \\
&\quad\le \mathsf{P}\left(N((1+\varepsilon)x) - \frac{(1+\varepsilon)x}{\mathsf{E}[\Delta\tau_1]} > \delta x\right) \\
&\qquad + \mathsf{P}\left(N((1+\varepsilon)x) - \frac{(1+\varepsilon)x}{\mathsf{E}[\Delta\tau_1]} \le \delta x, \max_{1\le k\le N((1+\varepsilon)x)} \sum_{i=1}^{k} (\Delta B_i - \Delta\tau_i) > \frac{\varepsilon}{3}x\right) \\
&\quad\le \mathsf{P}\left(N((1+\varepsilon)x) - \frac{(1+\varepsilon)x}{\mathsf{E}[\Delta\tau_1]} > \delta x\right) \\
&\qquad + \mathsf{P}\left(\max_{1\le k\le \{\delta+(1+\varepsilon)/\mathsf{E}[\Delta\tau_1]\}x} \sum_{i=1}^{k} (\Delta B_i - \Delta\tau_i) > \frac{\varepsilon}{3}x\right), \qquad x > 0, \qquad \text{(C.14)}
\end{aligned}$$

and

$$\begin{aligned}
&\mathsf{P}\left(\max_{1\le k\le N((1+\varepsilon)x)} \sum_{i=1}^{k} (\Delta\tau_i - \Delta B_i) > \frac{\varepsilon}{8}x\right) \\
&\quad\le \mathsf{P}\left(N((1+\varepsilon)x) - \frac{(1+\varepsilon)x}{\mathsf{E}[\Delta\tau_1]} > \delta x\right) \\
&\qquad + \mathsf{P}\left(\max_{1\le k\le \{\delta+(1+\varepsilon)/\mathsf{E}[\Delta\tau_1]\}x} \sum_{i=1}^{k} (\Delta\tau_i - \Delta B_i) > \frac{\varepsilon}{8}x\right), \qquad x > 0. \qquad \text{(C.15)}
\end{aligned}$$

According to Lemma A.9, the first terms in (C.14) and (C.15) are bounded from above by $Ce^{-cx} = o(\mathsf{P}(T > x))$. Let

$$\gamma = \frac{\varepsilon}{8} \cdot \frac{1}{\delta + (1+\varepsilon)/\mathsf{E}[\Delta\tau_1]}. \qquad \text{(C.16)}$$

We then have $\varepsilon x/(8\gamma) = \{\delta + (1+\varepsilon)/\mathsf{E}[\Delta\tau_1]\}x$ and thus for $x > 0$,

$$\mathsf{P}\left(\max_{1\le k\le \{\delta+(1+\varepsilon)/\mathsf{E}[\Delta\tau_1]\}x} \sum_{i=1}^{k} (\Delta B_i - \Delta\tau_i) > \frac{\varepsilon}{3}x\right) \le \mathsf{P}\left(\max_{1\le k\le \varepsilon x/(8\gamma)} \sum_{i=1}^{k} (\Delta B_i - \Delta\tau_i) > \frac{\varepsilon}{8}x\right).$$

As a result, to prove (C.6) and (C.7), it suffices to show that

$$\mathsf{P}\left(\max_{1\le k\le \varepsilon x/(8\gamma)} \sum_{i=1}^{k} (\Delta B_i - \Delta\tau_i) > \frac{\varepsilon}{8}x\right) = o(\mathsf{P}(T > x)), \qquad \text{(C.17)}$$

$$\mathsf{P}\left(\max_{1\le k\le \varepsilon x/(8\gamma)} \sum_{i=1}^{k} (\Delta\tau_i - \Delta B_i) > \frac{\varepsilon}{8}x\right) = o(\mathsf{P}(T > x)). \qquad \text{(C.18)}$$

In what follows, we prove (C.17) under condition (v.a) and condition (v.b), separately. We omit the proof of (C.18), which is almost the same as that of (C.17).

C.2.1. Condition (v.a)

Suppose condition (v.a) holds. It then follows from Lemma A.10 that for any fixed $p > 0$,

$$\mathsf{P}\left(\max_{1 \le k \le \varepsilon x/(8\gamma)} \sum_{i=1}^{k} (\Delta B_i - \Delta\tau_i) > \frac{\varepsilon}{8} x \right) \le Cx\mathsf{P}(\Delta B_1 - \Delta\tau_1 > vx) + Cx^{-p} \quad \text{for all sufficiently large } x > 0,$$

where $v := v(r,p)$ is some finite positive constant. Further from $T \in \mathcal{C} \subset \mathcal{D}$ and condition (iv), we have

$$\lim_{x\to\infty} \frac{x\mathsf{P}(\Delta B_1 - \Delta\tau_1 > vx)}{\mathsf{P}(T > x)} \le \limsup_{x\to\infty} \frac{x\mathsf{P}(\Delta B_1 - \Delta\tau_1 > vx)}{\mathsf{P}(T > vx)} \limsup_{x\to\infty} \frac{\mathsf{P}(T > vx)}{\mathsf{P}(T > x)} = 0.$$

We now fix $p > r_+$, where r_+ denotes the upper Matuszewska index of the d.f. of $\Delta B_1 - \Delta\tau_1$ (see subsection 2.2). It then follows from Proposition 2.1 and condition (iv) that $x^{-p} = o(\mathsf{P}(\Delta B_1 - \Delta\tau_1 > x))$ and thus $x^{-p} = o(\mathsf{P}(T > x))$. Therefore (C.17) holds.

C.2.2. Condition (v.b)

Suppose that condition (v.b) holds and define Y as a nonnegative r.v. such that

$$\mathsf{P}(Y > x) = \min(1, c\mathsf{P}(T > x)/x), \qquad x > 0.$$

It then follows from $T \in \mathcal{C}$ and conditions (iv) and (v.b) that

$$Y \in \mathcal{C}, \quad \mathsf{E}[Y] < \infty, \quad \mathsf{P}(\Delta B_1 - \Delta\tau_1 > x) = o(\mathsf{P}(T > x)/x) = o(\mathsf{P}(Y > x)).$$

Therefore Lemma A.8 implies that there exists a r.v. Z in $\mathbb{R}$ such that $0 < \mathsf{E}[Z] < \gamma$,

$$\mathsf{P}(Z > x) \ge \mathsf{P}(\Delta B_1 - \Delta\tau_1 > x) \quad \text{for all } x \in \mathbb{R} \text{ and,} \tag{C.19}$$

$$\mathsf{P}(Z > x) = \tilde{l}(x)\mathsf{P}(Y > x) \quad \text{for all sufficiently large } x > 0, \tag{C.20}$$

where γ is given in (C.16) and $\tilde{l}$ is some slowly varying function such that $\lim_{x\to\infty} \tilde{l}(x) = 0$.

The inequality (C.19) enables us to assume that Z and $\Delta B_1 - \Delta\tau_1$ are on the same probability space and $Z \ge \Delta B_1 - \Delta\tau_1$, without loss of generality (see, e.g., Theorem 1.2.4 in [36]). We thus have

$$\mathsf{P}\left(\max_{1 \le k \le \varepsilon x/(8\gamma)} \sum_{i=1}^{k} (\Delta B_i - \Delta\tau_i) > \frac{\varepsilon}{8} x \right) \le \mathsf{P}\left(\max_{1 \le k \le \varepsilon x/(8\gamma)} \sum_{i=1}^{k} Z_i > \frac{\varepsilon}{8} x \right), \tag{C.21}$$

where Z_i's ($i = 1, 2, \dots$) are independent copies of Z. Note here that $Z \in \mathcal{C}$ due to $Y \in \mathcal{C}$ and (C.20). Therefore applying Lemma A.11 to the right hand side of (C.21) yields

$$\mathsf{P}\left(\max_{1 \le k \le \varepsilon x/(8\gamma)} \sum_{i=1}^{k} \Delta B_i - \Delta\tau_i > \frac{\varepsilon}{8} x \right) \le Cx\mathsf{P}(Z > \varepsilon x/8) \quad \text{for all sufficiently large } x > 0.$$

In addition, it follows from condition (iv) and the definitions of Y and Z that $\mathsf{P}(Z > x) = o(\mathsf{P}(T > x)/x)$. Using this and $T \in \mathcal{C} \subset \mathcal{D}$, we have

$$\lim_{x\to\infty} \frac{x\mathsf{P}(Z > \varepsilon x/8)}{\mathsf{P}(T > x)} \le \limsup_{x\to\infty} \frac{x\mathsf{P}(Z > \varepsilon x/8)}{\mathsf{P}(T > \varepsilon x/8)} \limsup_{x\to\infty} \frac{\mathsf{P}(T > \varepsilon x/8)}{\mathsf{P}(T > x)} = 0.$$

As a result, we obtain (C.17).

C.3. Proof of Theorem 3.3

The asymptotic lower bound $\mathsf{P}(B(T) > x) \gtrsim_x \mathsf{P}(T > x)$ can be proved in the same way as the proof of Theorem 3 in [25]. Thus we here prove only the asymptotic upper bound $\mathsf{P}(M(T) > x) \lesssim_x \mathsf{P}(T > x)$.

We fix δ $(0 < \delta < 1)$ arbitrarily and also fix x such that $0 < \delta x \le x - \xi\sqrt{x}$ and $\xi \ge 1$, which leads to $\sqrt{x} \ge \xi/(1-\delta) > 1$. We then have

$$\begin{aligned}\mathsf{P}(M(T) > x) &= \mathsf{P}(M(T) > x, T > x - \xi\sqrt{x}) \\ &\quad + \mathsf{P}(M(T) > x, \delta x < T \le x - \xi\sqrt{x}) + \mathsf{P}(M(T) > x, T \le \delta x) \\ &\le \mathsf{P}(T > x - \xi\sqrt{x}) \\ &\quad + \mathsf{P}(M(T) > x, \delta x < T \le x - \xi\sqrt{x}) + \mathsf{P}(M(\delta x) > x). \end{aligned} \tag{C.22}$$

Since $\mathsf{P}(T > x - \xi\sqrt{x}) \overset{x}{\sim} \mathsf{P}(T > x)$ (due to condition (i); see Lemmas A.1 and A.2), it suffices to show that the second and third terms in (C.22) are $o(\mathsf{P}(T > x))$.

Note first that $\sup_{0\le t\le \delta x} B(t) - \delta x \le \sup_{0 \le t \le \delta x}(B(t) - t)$ and thus

$$\mathsf{P}(M(\delta x) > x) \le \mathsf{P}\left(\sup_{0\le t\le \delta x}(B(t) - t) > (1-\delta)x\right). \tag{C.23}$$

Applying Lemma A.7 (i) to (C.23) yields

$$\mathsf{P}(M(\delta x) > x) \le C\left(e^{-cx} + xe^{-cQ((1-\delta)x)}\right) = o(\mathsf{P}(T > x)) + Cxe^{-cQ((1-\delta)x)}.$$

Further since $x^\theta = O(Q(x))$ and $\mathsf{P}(T > x) = e^{-o(x^\theta)}$ (due to $T \in \mathcal{L}^{1/\theta}$; see Lemma A.1 (i)),

$$\limsup_{x\to\infty} \frac{xe^{-cQ((1-\delta)x)}}{\mathsf{P}(T > x)} \le \limsup_{x\to\infty} \exp\left\{-cx^\theta/C + \log x + o(x^\theta)\right\} = 0. \tag{C.24}$$

Consequently, we have $\mathsf{P}(M(\delta x) > x) = o(\mathsf{P}(T > x))$.

Next we consider the second term on the right hand side of (C.22). Note that

$$\begin{aligned}&\mathsf{P}(M(T) > x, \delta x < T \le x - \xi\sqrt{x}) \\ &\quad = \int_{\delta x}^{x-\xi\sqrt{x}} \mathsf{P}(M(u) > x)\mathrm{d}\mathsf{P}(T \le u) \\ &\quad \le \int_{\delta x}^{x-\xi\sqrt{x}} \mathsf{P}\left(\sup_{0\le t\le u}(B(t) - t) > x - u\right)\mathrm{d}\mathsf{P}(T \le u). \end{aligned} \tag{C.25}$$

Applying Lemma A.7 (i) to the right hand side of (C.25) and using $\delta x \le u \le x$, we obtain

$$\begin{aligned}&\mathsf{P}(M(T) > x, \delta x < T \le x - \xi\sqrt{x}) \\ &\quad \le \int_{\delta x}^{x-\xi\sqrt{x}} C\left(e^{-c(x-u)^2/u} + e^{-cu} + ue^{-cQ(x-u)}\right)\mathrm{d}\mathsf{P}(T \le u) \\ &\quad \le Ce^{-c\delta x} + C\int_{\delta x}^{x-\xi\sqrt{x}} \left(e^{-c(x-u)^2/x} + xe^{-cQ(x-u)}\right)\mathrm{d}\mathsf{P}(T \le u) \\ &\quad = o(\mathsf{P}(T > x)) + Cf_1(x) + Cf_2(x), \end{aligned}$$

where

$$f_1(x) = \int_{\delta x}^{x-\xi\sqrt{x}} e^{-c(x-u)^2/x}\mathrm{d}\mathsf{P}(T \le u), \tag{C.26}$$

$$f_2(x) = \int_{\delta x}^{x-\xi\sqrt{x}} xe^{-cQ(x-u)}\mathrm{d}\mathsf{P}(T \le u). \tag{C.27}$$

In what follows, we prove $f_1(x) = o(\mathsf{P}(T > x))$ and $f_2(x) = o(\mathsf{P}(T > x))$.

Note that $e^{-c(x-u)^2/x}$ is differentiable with respect to u. Thus integrating the right hand side of (C.26) by parts (see, e.g., Theorems 6.1.7 and 6.2.2 in [9]) and letting $y = (x-u)/\sqrt{x}$ yield

$$\begin{aligned} f_1(x) &\le e^{-c(1-\delta)^2 x} + \int_{\delta x}^{x-\xi\sqrt{x}} \mathsf{P}(T > u) \mathrm{d}_u (e^{-c(x-u)^2/x}) \\ &= e^{-c(1-\delta)^2 x} + \int_{\delta x}^{x-\xi\sqrt{x}} \mathsf{P}(T > u) \frac{2c(x-u)}{x} e^{-c(x-u)^2/x} \mathrm{d}u \\ &= o(\mathsf{P}(T > x)) + \int_{\xi}^{(1-\delta)\sqrt{x}} \mathsf{P}(T > x - y\sqrt{x}) 2cye^{-cy^2} \mathrm{d}y \\ &\le o(\mathsf{P}(T > x)) + \int_{\xi}^{(1-\delta)\sqrt{x}} \mathsf{P}(\sqrt{T} > \sqrt{x} - y) 2cye^{-cy^2} \mathrm{d}y, \end{aligned} \tag{C.28}$$

where the last inequality holds because $(x - y\sqrt{x})^{1/2} \ge \sqrt{x} - y$ for $0 \le y \le \sqrt{x}$. It thus follows from $\sqrt{T} \in \mathcal{L}$ and Lemma A.5 that for any $\varepsilon > 0$,

$$\begin{aligned} &\lim_{\xi\to\infty} \limsup_{x\to\infty} \int_{\xi}^{(1-\delta)\sqrt{x}} \frac{\mathsf{P}(\sqrt{T} > \sqrt{x} - y)}{\mathsf{P}(T > x)} 2cye^{-cy^2} \mathrm{d}y \\ &\quad = \lim_{\xi\to\infty} \limsup_{x\to\infty} \int_{\xi}^{(1-\delta)\sqrt{x}} \frac{\mathsf{P}(\sqrt{T} > \sqrt{x} - y)}{\mathsf{P}(\sqrt{T} > \sqrt{x})} 2cye^{-cy^2} \mathrm{d}y \\ &\quad \le e^{\varepsilon} \lim_{\xi\to\infty} \limsup_{x\to\infty} \int_{\xi}^{(1-\delta)\sqrt{x}} 2cy \exp\{-cy^2 + \varepsilon y\} \mathrm{d}y \\ &\quad \le e^{\varepsilon} \lim_{\xi\to\infty} \int_{\xi}^{\infty} 2cy \exp\{-cy^2 + \varepsilon y\} \mathrm{d}y = 0. \end{aligned} \tag{C.29}$$

Combining (C.28) with (C.29) yields $f_1(x) = o(\mathsf{P}(T > x))$.

We proceed to the proof of $f_2(x) = o(\mathsf{P}(T > x))$. Since Q is eventually concave (see Definition 2.3), Q is continuous for all sufficiently large $x > 0$. Therefore without loss of generality, we fix x to be sufficiently large such that $Q(x-u)$ is continuous for all $\delta x \le u \le x - \xi\sqrt{x}$.

For $\delta x \le u \le x - \xi\sqrt{x}$, we have

$$e^{-cQ(x-u)} = e^{-(c/2)Q(x-u)} e^{-(c/2)Q(x-u)} \le e^{-(c/2)Q(\xi\sqrt{x})} e^{-(c/2)Q(x-u)}.$$

Substituting this into the right hand side of (C.27) and integrating it by parts yield

$$\begin{aligned} f_2(x) &\le xe^{-cQ(\xi\sqrt{x})} \int_{\delta x}^{x-\xi\sqrt{x}} \{-e^{-cQ(x-u)}\} \mathrm{d}\mathsf{P}(T > u) \\ &\le xe^{-cQ(\xi\sqrt{x})} \left[e^{-cQ((1-\delta)x)} + \int_{\delta x}^{x-\xi\sqrt{x}} \mathsf{P}(T > u) \mathrm{d}_u (e^{-cQ(x-u)}) \right] \\ &= xe^{-cQ(\xi\sqrt{x})} \left[o(\mathsf{P}(T > x)) + \int_{\delta x}^{x-\xi\sqrt{x}} \mathsf{P}(T > u) \mathrm{d}_u (e^{-cQ(x-u)}) \right], \end{aligned}$$

where the last equality follows from $e^{-cQ((1-\delta)x)} = o(\mathsf{P}(T > x))$ due to (C.24). Further using $\log x = o(Q(x))$ and $x^\theta = O(Q(x))$, we have

$$\lim_{x\to\infty} xe^{-cQ(\xi\sqrt{x})} = \lim_{x\to\infty} e^{-cQ(\xi\sqrt{x}) + 2\log\sqrt{x}} = \lim_{x\to\infty} e^{-cQ(\xi\sqrt{x}) + o(Q(\xi\sqrt{x}))} = 0.$$

Finally, it follows from $T^\theta \in \mathcal{L}$ and Lemma A.5 that for sufficiently small $\varepsilon > 0$,

$$\begin{aligned}
&\limsup_{x\to\infty} \int_{\delta x}^{x-\xi\sqrt{x}} \frac{\mathsf{P}(T>u)}{\mathsf{P}(T>x)} \mathrm{d}_u(e^{-cQ(x-u)}) \\
&\quad \le e^\varepsilon \limsup_{x\to\infty} \int_{\delta x}^{x-\xi\sqrt{x}} e^{\varepsilon(x-u)^\theta} \mathrm{d}_u(e^{-cQ(x-u)}) \\
&\quad \le e^\varepsilon \limsup_{x\to\infty} \left[e^{\varepsilon(\xi\sqrt{x})^\theta - cQ(\xi\sqrt{x})} + \int_{\delta x}^{x-\xi\sqrt{x}} \varepsilon\theta(x-u)^{\theta-1} e^{\varepsilon(x-u)^\theta - cQ(x-u)} \mathrm{d}u \right] \\
&\quad \le e^\varepsilon \limsup_{x\to\infty} \left[e^{\varepsilon(\xi\sqrt{x})^\theta - cQ(\xi\sqrt{x})} + \varepsilon\theta \int_{\xi\sqrt{x}}^{(1-\delta)x} e^{\varepsilon y^\theta - cQ(y)} \mathrm{d}y \right] = 0,
\end{aligned}$$

where the last equality is due to $x^\theta = O(Q(x))$. As a result, we have $f_2(x) = o(\mathsf{P}(T > x))$.

C.4. Proof of Theorem 3.4

For any $\varepsilon > 0$, we have

$$\begin{aligned}
\mathsf{P}(B(T) > x) &\ge \int_{(1+\varepsilon)x}^{\infty} \mathsf{P}(B(u) > x) \mathrm{d}\mathsf{P}(T \le u) \\
&\ge \inf_{u>(1+\varepsilon)x} \mathsf{P}(B(u) > x) \mathsf{P}(T > (1+\varepsilon)x) \\
&= \inf_{u>(1+\varepsilon)x} \mathsf{P}\left(\frac{B(u)-u}{u} > \frac{x-u}{u} \right) \mathsf{P}(T > (1+\varepsilon)x) \\
&\ge \inf_{u>(1+\varepsilon)x} \mathsf{P}\left(\frac{B(u)-u}{u} > \frac{-\varepsilon}{1+\varepsilon} \right) \mathsf{P}(T > (1+\varepsilon)x). \qquad \text{(C.30)}
\end{aligned}$$

It follows from the SLLN for $\{B(t)\}$ (see [3, Chapter VI, Theorem 3.1]) that for any $\varepsilon > 0$,

$$\lim_{x\to\infty} \inf_{u>(1+\varepsilon)x} \mathsf{P}\left(\frac{B(u)-u}{u} > \frac{-\varepsilon}{1+\varepsilon} \right) \ge \lim_{x\to\infty} \inf_{u>(1+\varepsilon)x} \mathsf{P}\left(\left| \frac{B(u)-u}{u} \right| < \frac{\varepsilon}{1+\varepsilon} \right) = 1.$$

Note here that (C.4) holds due to $T \in \mathcal{C}$. Thus from (C.30), we have $\mathsf{P}(B(T) > x) \gtrsim_x \mathsf{P}(T > x)$.

In what follows, we prove $\mathsf{P}(M(T) > x) \lesssim_x \mathsf{P}(T > x)$. For any $\varepsilon \in (0, 1)$,

$$\mathsf{P}(M(T) > x) \le \mathsf{P}(T > (1-\varepsilon)x) + \mathsf{P}(M(T) > x, T \le (1-\varepsilon)x).$$

Since (C.5) holds, it suffices to show $\mathsf{P}(M(T) > x, T \le (1-\varepsilon)x) = o(\mathsf{P}(T > x))$.

It follows from $M(u) - u \le \sup_{0\le t\le u}(B(t) - t)$ $(u \ge 0)$ that for $x > 0$,

$$\begin{aligned}
\mathsf{P}(M(T) > x, T \le (1-\varepsilon)x) &\le \int_0^{(1-\varepsilon)x} \mathsf{P}\left(\sup_{0\le t\le u} \{B(t) - t\} > x - u \right) \mathrm{d}\mathsf{P}(T \le u) \\
&\le \int_0^{(1-\varepsilon)x} \mathsf{P}\left(\sup_{0\le t\le u} \{B(t) - t\} > \varepsilon x \right) \mathrm{d}\mathsf{P}(T \le u).
\end{aligned}$$

Similarly to (B.38), we estimate the integrand on the right hand side of the above inequality as follows:

$$\begin{aligned}
&\mathsf{P}\left(\sup_{0\le t\le u} \{B(t) - t\} > \varepsilon x \right) \\
&\quad \le \mathsf{P}\left(\Delta B_0^* > \varepsilon x/3\right) + \mathsf{P}\left(\Delta B_1^* > \varepsilon x/3\right) + \mathsf{P}\left(\max_{1\le k\le N(u)} \sum_{i=1}^{k} (\Delta B_i - \Delta\tau_i) > \frac{\varepsilon x}{3} \right).
\end{aligned}$$

From conditions (i) and (iii), we have $\mathsf{P}\left(\Delta B_n^* > \varepsilon x/3\right) = o(\mathsf{P}(T > x))$ $(n = 0, 1)$. Therefore it remains to show that

$$\int_0^{(1-\varepsilon)x} \mathsf{P}\left(\max_{1\le k\le N(u)} \sum_{i=1}^{k} (\Delta B_i - \Delta\tau_i) > \frac{\varepsilon x}{3}\right) \mathrm{d}\mathsf{P}(T \le u) = o(\mathsf{P}(T > x)). \tag{C.31}$$

Fix a positive number γ such that

$$\frac{\varepsilon}{3\gamma} > \frac{1-\varepsilon}{\mathsf{E}[\Delta\tau_1]}. \tag{C.32}$$

We then decompose the left hand side of (C.31) into $R_1(x) + R_2(x)$ in the following way:

$$\begin{aligned} R_1(x) &= \int_0^{(1-\varepsilon)x} \mathrm{d}\mathsf{P}(T \le u)\mathsf{P}\left(\max_{1\le k\le N(u)} \sum_{i=1}^{k} (\Delta B_i - \Delta\tau_i) > \frac{\varepsilon x}{3},\ N(u) > \frac{\varepsilon x}{3\gamma}\right), \\ R_2(x) &= \int_0^{(1-\varepsilon)x} \mathrm{d}\mathsf{P}(T \le u)\mathsf{P}\left(\max_{1\le k\le N(u)} \sum_{i=1}^{k} (\Delta B_i - \Delta\tau_i) > \frac{\varepsilon x}{3},\ N(u) \le \frac{\varepsilon x}{3\gamma}\right). \end{aligned} \tag{C.33}$$

For $x > 0$, we have

$$R_1(x) \le \int_0^{(1-\varepsilon)x} \mathsf{P}\left(N(u) > \frac{\varepsilon x}{3\gamma}\right) \mathrm{d}\mathsf{P}(T \le u) \le \mathsf{P}\left(N((1-\varepsilon)x) > \frac{\varepsilon x}{3\gamma}\right). \tag{C.34}$$

Note here that $\varepsilon/(3\gamma) - (1-\varepsilon)/\mathsf{E}[\Delta\tau_1] > 0$ due to (C.32). Thus Lemma A.9 yields

$$\begin{aligned} \mathsf{P}\left(N((1-\varepsilon)x) > \frac{\varepsilon x}{3\gamma}\right) &= \mathsf{P}\left(N((1-\varepsilon)x) - \frac{(1-\varepsilon)x}{\mathsf{E}[\Delta\tau_1]} > \left(\frac{\varepsilon}{3\gamma} - \frac{1-\varepsilon}{\mathsf{E}[\Delta\tau_1]}\right)x\right) \\ &\le Ce^{-cx} = o(\mathsf{P}(T > x)). \end{aligned}$$

Combining this with (C.34), we have $R_1(x) = o(\mathsf{P}(T > x))$.

Next we consider $R_2(x)$. From (C.33), we have

$$R_2(x) \le \int_0^{(1-\varepsilon)x} \mathrm{d}\mathsf{P}(T \le u)\mathsf{P}\left(\max_{1\le k\le \varepsilon x/(3\gamma)} \sum_{i=1}^{k} (\Delta B_i - \Delta\tau_i) > \frac{\varepsilon x}{3}\right).$$

Following the proof of (C.17), we can show that

$$\mathsf{P}\left(\max_{1\le k\le \varepsilon x/(3\gamma)} \sum_{i=1}^{k} (\Delta B_i - \Delta\tau_i) > \frac{\varepsilon x}{3}\right) = o(\mathsf{P}(T > x)),$$

which leads to $R_2(x) = o(\mathsf{P}(T > x))$ (see subsection C.2.1 and C.2.2 in the proof of Theorem 3.2).

Remark C.1 Except for the estimation of $R_2(x)$, conditions (i)–(iii) and the independence between $\{B(t)\}$ and T are sufficient for the proof of Theorem 3.4. Conditions (iv) and (v) are required by the estimation of $R_2(x)$.

C.5. Proof of Lemma 3.1

We first partition $\widetilde{\boldsymbol{\beta}}$ and $\widetilde{\boldsymbol{H}}$ as

$$\widetilde{\boldsymbol{\beta}} = \begin{array}{c} \begin{array}{cc} \{0\} & \mathbb{D}\setminus\{0\} \end{array} \\ \begin{pmatrix} \widetilde{\beta}_0 & \widetilde{\boldsymbol{\beta}}_+ \end{pmatrix} \end{array}, \quad \widetilde{\boldsymbol{H}} = \begin{array}{c} \{0\} \\ \mathbb{D}\setminus\{0\} \end{array} \overset{\begin{array}{cc} \{0\} & \mathbb{D}\setminus\{0\} \end{array}}{\begin{pmatrix} \widetilde{H}_{0,0} & \widetilde{\boldsymbol{\eta}}_+ \\ \widetilde{\boldsymbol{h}}_+ & \widetilde{\boldsymbol{H}}_+ \end{pmatrix}}.$$

We then fix $z = 1$ in (3.3) and (3.4) and take the inverse of them with respect to ξ. Thus

$$\mathsf{P}(\Delta B_0 \le x) = \beta_0(x) + \boldsymbol{\beta}_+ * \sum_{n=0}^{\infty} \boldsymbol{H}_+^{*n} * \boldsymbol{h}_+(x), \tag{C.35}$$

$$\mathsf{P}(\Delta B_1 \le x) = H_{0,0}(x) + \boldsymbol{\eta}_+ * \sum_{n=0}^{\infty} \boldsymbol{H}_+^{*n} * \boldsymbol{h}_+(x), \tag{C.36}$$

where the symbol $*$ denotes the operator of convolution and the superscript $*n$ represents the nth-fold convolution (see Appendix A.5), and where

$$\boldsymbol{\beta}(x) = \begin{array}{c} \begin{array}{cc} \{0\} & \mathbb{D}\setminus\{0\} \end{array} \\ \begin{pmatrix} \beta_0(x) & \boldsymbol{\beta}_+(x) \end{pmatrix} \end{array}, \quad \boldsymbol{H}(x) = \begin{array}{c} \{0\} \\ \mathbb{D}\setminus\{0\} \end{array} \overset{\begin{array}{cc} \{0\} & \mathbb{D}\setminus\{0\} \end{array}}{\begin{pmatrix} H_{0,0}(x) & \boldsymbol{\eta}_+(x) \\ \boldsymbol{h}_+(x) & \boldsymbol{H}_+(x) \end{pmatrix}}.$$

Applying Lemma A.12 to (C.35) and (C.36), we obtain

$$\begin{aligned} \limsup_{x\to\infty} \frac{\mathsf{P}(\Delta B_0 > x)}{\mathsf{P}(Y > x)} &\le \tilde{c}\widetilde{\beta}_0 + \tilde{c}\widetilde{\boldsymbol{\beta}}_+(\boldsymbol{I} - \boldsymbol{H}_+(\infty))^{-1}\boldsymbol{h}_+(\infty) \\ &\quad + \boldsymbol{\beta}_+(\infty)(\boldsymbol{I} - \boldsymbol{H}_+(\infty))^{-1}(\tilde{c}\widetilde{\boldsymbol{H}}_+)(\boldsymbol{I} - \boldsymbol{H}_+(\infty))^{-1}\boldsymbol{h}_+(\infty) \\ &\quad + \boldsymbol{\beta}_+(\infty)(\boldsymbol{I} - \boldsymbol{H}_+(\infty))^{-1}(\tilde{c}\widetilde{\boldsymbol{h}}_+) \\ &= \tilde{c}\left[\widetilde{\boldsymbol{\beta}}\boldsymbol{e} + \boldsymbol{\beta}_+(\infty)(\boldsymbol{I} - \boldsymbol{H}_+(\infty))^{-1}(\widetilde{\boldsymbol{H}}_+\boldsymbol{e} + \widetilde{\boldsymbol{h}}_+)\right] \le \tilde{c}C, \\ \limsup_{x\to\infty} \frac{\mathsf{P}(\Delta B_1 > x)}{\mathsf{P}(Y > x)} &\le \tilde{c}\left[(\widetilde{H}_{0,0} + \widetilde{\boldsymbol{\eta}}_+\boldsymbol{e}) + \boldsymbol{\eta}_+(\infty)(\boldsymbol{I} - \boldsymbol{H}_+(\infty))^{-1}(\widetilde{\boldsymbol{H}}_+\boldsymbol{e} + \widetilde{\boldsymbol{h}}_+)\right] \\ &= \tilde{c}(1/\varpi_0)\left[\varpi_0(\widetilde{H}_{0,0} + \widetilde{\boldsymbol{\eta}}_+\boldsymbol{e}) + \boldsymbol{\varpi}_+(\widetilde{\boldsymbol{H}}_+\boldsymbol{e} + \widetilde{\boldsymbol{h}}_+)\right] \\ &= \tilde{c}(1/\varpi_0)\boldsymbol{\varpi}\widetilde{\boldsymbol{H}}\boldsymbol{e} \le \tilde{c}C, \end{aligned}$$

where we use $(\boldsymbol{I} - \boldsymbol{H}_+(\infty))^{-1}\boldsymbol{h}_+(\infty) = \boldsymbol{e}$ (which is due to $\boldsymbol{h}_+(\infty) + \boldsymbol{H}_+(\infty)\boldsymbol{e} = \boldsymbol{e}$); and also use $\boldsymbol{\varpi}_+ := (\varpi_i)_{i\in\mathbb{D}\setminus\{0\}} = \varpi_0\boldsymbol{\eta}_+(\infty)(\boldsymbol{I} - \boldsymbol{H}_+(\infty))^{-1}$ and $\varpi_0 = 1/\mathsf{E}[\Delta\tau_1]$ (see, e.g., [7, Chapter 3, Theorems 2.1 and 3.2]).

C.6. Proof of Lemma 3.2

Let $\psi_1(k, \xi)$ $(k = 1, 2, \dots)$ denote

$$\psi_1(k, \xi) = \mathsf{E}[\mathbb{1}(\Delta\tau_1 = k)e^{\mathrm{i}\xi\Delta B_1}] = \frac{1}{k!}\lim_{z\to 0}\frac{\partial^k}{\partial z^k}\widehat{\psi}_1(z, \xi).$$

It then follows from (3.4) that for $k = 1, 2, \dots,$

$$\psi_1(k, \xi) = \mathbb{1}(k = 1)\widehat{H}_{0,0}(\xi) + \mathbb{1}(k \ge 2)\widehat{\boldsymbol{\eta}}_+(\xi) \cdot (\widehat{\boldsymbol{H}}_+(\xi))^{k-2} \cdot \widehat{\boldsymbol{h}}_+(\xi). \tag{C.37}$$

Taking the inverse of (C.37) with respect to ξ and applying Lemma A.12 to the resulting equation, we obtain

$$\begin{aligned}
&\limsup_{x\to\infty} \frac{\mathsf{P}(\Delta\tau_1 = k, \Delta B_1 > x)}{\mathsf{P}(Y > x)} \\
&\quad \le \tilde{c}1\!\!1(k=1)\widetilde{H}_{0,0} + \tilde{c}1\!\!1(k\ge 2)\left\{\widetilde{\boldsymbol{\eta}}_+(\boldsymbol{H}_+(\infty))^{k-2}\boldsymbol{h}_+(\infty)\right\} \\
&\qquad + \tilde{c}1\!\!1(k\ge 2)\left\{\boldsymbol{\eta}_+(\infty)\cdot\sum_{\nu=0}^{k-3}(\boldsymbol{H}_+(\infty))^{\nu}\widetilde{\boldsymbol{H}}_+(\boldsymbol{H}_+(\infty))^{k-\nu-3}\cdot\boldsymbol{h}_+(\infty)\right\} \\
&\qquad + \tilde{c}1\!\!1(k\ge 2)\left\{\boldsymbol{\eta}_+(\infty)(\boldsymbol{H}_+(\infty))^{k-2}\widetilde{\boldsymbol{h}}_+\right\}, \qquad \forall k = 1,2,\dots. \qquad \text{(C.38)}
\end{aligned}$$

Note here that $\beta_i(\infty) = 0$ (resp. $H_{i,j}(\infty) = 0$) implies $\widetilde{\beta}_i = 0$ (resp. $\widetilde{H}_{i,j} = 0$) and thus $\widetilde{\boldsymbol{\beta}} \le C\boldsymbol{\beta}(\infty) = C\widehat{\boldsymbol{\beta}}(0)$ and $\widetilde{\boldsymbol{H}} \le C\boldsymbol{H}(\infty) = C\widehat{\boldsymbol{H}}(0)$. Therefore from (C.38) and (3.5), we have for all $k = 1,2,\dots$,

$$\begin{aligned}
&\limsup_{x\to\infty} \frac{\mathsf{P}(\Delta\tau_1 = k, \Delta B_1 > x)}{\mathsf{P}(Y > x)} \\
&\le \tilde{c}C\left[1\!\!1(k=1)k\widehat{H}_{0,0}(0) + 1\!\!1(k\ge 2)k\left\{\widehat{\boldsymbol{\eta}}_+(0)\left(\widehat{\boldsymbol{H}}_+(0)\right)^{k-2}\widehat{\boldsymbol{h}}_+(0)\right\}\right] = \tilde{c}Ck\mathsf{P}(\Delta\tau_1 = k),
\end{aligned}$$

where C is independent of k.

C.7. Proof of Lemma 3.3

Since $\mathsf{P}(\sum_{i=1}^{0}\Delta B_i > 0 \mid N(t) = 0) = \mathsf{P}(\{\emptyset\}) = 0$, (3.6) holds for all $t \ge 0$ if $m = 0$.

In what follows, we consider the case of $m \ge 1$. Under Assumption 3.1, $\Delta\tau_1 \ge 1$ and $N(t) = N(\lfloor t \rfloor) \le \lfloor t \rfloor$ for all $t \ge 0$. Therefore we fix $t = n \in \{1,2,\dots\}$ without loss of generality.

Note that $\{N(n) = m\}$ is equivalent to $\{\sum_{i=1}^{m}\Delta\tau_i \le n, \sum_{i=1}^{m+1}\Delta\tau_i > n\}$ and that $\Delta\tau_{m+1}$ is independent of $\Delta\tau_i$ and ΔB_i $(i = 1,2,\dots,m)$. We then have

$$\begin{aligned}
\mathsf{P}\left(N(n) = m, \sum_{i=1}^{m}\Delta B_i > x\right) &= \mathsf{P}\left(\sum_{i=1}^{m}\Delta\tau_i \le n, \sum_{i=1}^{m+1}\Delta\tau_i > n, \sum_{i=1}^{m}\Delta B_i > x\right) \\
&= \sum_{k=1}^{n}\mathsf{P}\left(\sum_{i=1}^{m}\Delta\tau_i = k, \sum_{i=1}^{m}\Delta B_i > x\right) \\
&\qquad \times \mathsf{P}(\Delta\tau_{m+1} > n-k). \qquad \text{(C.39)}
\end{aligned}$$

Note also that ΔB_i is independent of $\Delta\tau_j$'s $(j \ne i)$. We thus have

$$\begin{aligned}
&\mathsf{P}\left(\sum_{i=1}^{m}\Delta\tau_i = k, \sum_{i=1}^{m}\Delta B_i > x\right) \\
&\quad = \sum_{k_1+\cdots+k_m=k}\prod_{i=1}^{m}\mathsf{P}(\Delta\tau_i = k_i)\cdot\mathsf{P}\left(\sum_{i=1}^{m}\Delta B_i > x \,\middle|\, \Delta\tau_i = k_i,\ i = 1,2,\dots,m\right) \\
&\quad = \sum_{k_1+\cdots+k_m=k}\prod_{i=1}^{m}\mathsf{P}(\Delta\tau_i = k_i)\cdot\mathsf{P}\left(\sum_{i=1}^{m}(\Delta B_i \,|\, \{\Delta\tau_i = k_i\}) > x\right), \qquad \text{(C.40)}
\end{aligned}$$

where $\Delta B_i \,|\, \{\Delta\tau_i = k_i\}$ denotes the conditional random variable ΔB_i given $\Delta\tau_i = k_i$. Further it follows from Lemmas 3.2 and A.12 that for $(k_1, \dots, k_m)$ such that $\sum_{i=1}^m k_i = k$,

$$\limsup_{x\to\infty} \frac{\mathsf{P}\left(\sum_{i=1}^m (\Delta B_i \,|\, \{\Delta\tau_i = k_i\}) > x\right)}{\mathsf{P}(Y > x)} \le \tilde{c}C \cdot (k_1 + \cdots + k_m) = \tilde{c}Ck.$$

Combining this with (C.40) yields

$$\begin{aligned}
&\limsup_{x\to\infty} \frac{\mathsf{P}\left(\sum_{i=1}^m \Delta\tau_i = k, \sum_{i=1}^m \Delta B_i > x\right)}{\mathsf{P}(Y > x)} \\
&\qquad \le \tilde{c}Ck \sum_{k_1+\cdots+k_m=k} \prod_{i=1}^m \mathsf{P}(\Delta\tau_i = k_i) = \tilde{c}Ck \cdot \mathsf{P}\left(\sum_{i=1}^m \Delta\tau_i = k\right). \qquad \text{(C.41)}
\end{aligned}$$

From (C.39) and (C.41), we obtain for all $n = 0, 1, \dots$ and $m = 0, 1, \dots, n$,

$$\begin{aligned}
\limsup_{x\to\infty} \frac{\mathsf{P}\left(N(n) = m, \sum_{i=1}^m \Delta B_i > x\right)}{\mathsf{P}(Y > x)} &\le \tilde{c}C \sum_{k=1}^n k\mathsf{P}\left(\sum_{i=1}^m \Delta\tau_i = k\right) \mathsf{P}(\Delta\tau_{m+1} > n - k) \\
&\le \tilde{c}Cn \sum_{k=1}^n \mathsf{P}\left(\sum_{i=1}^m \Delta\tau_i = k\right) \mathsf{P}(\Delta\tau_{m+1} > n - k) \\
&= \tilde{c}Cn\mathsf{P}\left(\sum_{i=1}^m \Delta\tau_i \le n, \sum_{i=1}^{m+1} \Delta\tau_i > n\right) \\
&= \tilde{c}Cn\mathsf{P}(N(n) = m).
\end{aligned}$$

C.8. Proof of Theorem 3.5

As shown later, the conditions of Theorem 3.5 imply conditions (i), (ii) and (iii) of Theorem 3.4. Thus according to Remark C.1, we can follow the proof of Theorem 3.4, except for the estimation of $R_2(x)$ in (C.33). In addition, we can prove that $R_2(x) = o(\mathsf{P}(T > x))$ as follows.

From (C.33), we have

$$R_2(x) \le \int_0^{(1-\varepsilon)x} \sum_{n \le \varepsilon x/(3\gamma)} \mathsf{P}(N(u) = n)\mathsf{P}\left(\sum_{i=1}^n \Delta B_i > \frac{\varepsilon x}{3} \,\middle|\, N(u) = n\right) \mathrm{d}\mathsf{P}(T \le u). \quad \text{(C.42)}$$

Note here that condition (iii) of Theorem 3.4 implies $\mathsf{P}(\Delta B_1 > x) = o(\mathsf{P}(T > x))$. Thus by using Lemma 3.3 with $Y = T \in \mathcal{C}$ and $\tilde{c} = 0$, we obtain

$$\mathsf{P}\left(\sum_{i=1}^n \Delta B_i > \frac{\varepsilon x}{3} \,\middle|\, N(u) = n\right) = u \cdot o(\mathsf{P}(T > x)).$$

Substituting this into (C.42) yields

$$\begin{aligned}
R_2(x) &\le \int_0^{(1-\varepsilon)x} \sum_{n \le \varepsilon x/(3\gamma)} \mathsf{P}(N(u) = n) u \mathrm{d}\mathsf{P}(T \le u) \cdot o(\mathsf{P}(T > x)) \\
&\le \mathsf{E}[T] \cdot o(\mathsf{P}(T > x)),
\end{aligned}$$

which implies that $R_2(x) = o(\mathsf{P}(T > x))$ due to $\mathsf{E}[T] < \infty$.

In what follows, we confirm that conditions (ii) and (iii) of Theorem 3.4 are satisfied (condition (i) is obvious). For simplicity, we assume $h = b = 1$, which does not lose generality.

We first introduce a cumulative process $\{B^{\#}(t); t \ge 0\}$ such that $B^{\#}(t) = \sum_{n=0}^{\lfloor t \rfloor} |X_n|$ for $t \ge 0$. Clearly, $\{B^{\#}(t)\}$ and $\{B(t)\}$ have the common regenerative points τ_n's. Further $\{(B^{\#}(n), J_n); n = 0, 1, \dots\}$ is a Markov additive process with initial distribution $\boldsymbol{\beta}^{\#}(x)$ and kernel $\boldsymbol{H}^{\#}(x)$ $(x \in \mathbb{R})$, where $\boldsymbol{\beta}^{\#}(x) = \int_{|y| \le x} \mathrm{d}\boldsymbol{\beta}(y)$ and $\boldsymbol{H}^{\#}(x) = \int_{|y| \le x} \mathrm{d}\boldsymbol{H}(y)$.

Let $\Delta B_n^{\#}$ $(n = 0, 1, \dots)$ denote

$$\Delta B_n^{\#} = \begin{cases} B^{\#}(\tau_0), & n = 0, \\ B^{\#}(\tau_n) - B^{\#}(\tau_{n-1}), & n = 1, 2, \dots. \end{cases}$$

We then have

$$\begin{aligned} \Delta B_0^{\#} &\ge \sup_{0 \le t \le \tau_0} |B(t)| \ge \Delta B_0^* \ge \Delta B_0, \\ \Delta B_n^{\#} &\ge \sup_{\tau_{n-1} \le t \le \tau_n} |B(t) - B(\tau_{n-1})| \ge \Delta B_n^* \ge \Delta B_n, \quad n = 1, 2, \dots. \end{aligned}$$

Thus, similarly to the proof of Proposition 3.1, we readily obtain

$$\mathsf{E}\left[\sup_{\tau_0 \le t \le \tau_1} |B(t) - B(\tau_0)|\right] \le \mathsf{E}[\Delta B_1^{\#}] = \boldsymbol{\varpi} \int_{x \in \mathbb{R}} |x| \mathrm{d}\boldsymbol{H}(x)\boldsymbol{e} \cdot \mathsf{E}[\Delta\tau_1] < \infty,$$

where the last inequality is due to Assumption 3.1 (iii). Recall here that $\Delta\tau_n$ follows a phase-type distribution and thus $\mathsf{E}[(\Delta\tau_n)^2] < \infty$ $(n = 0, 1)$. Therefore condition (ii) of Theorem 3.4 is satisfied. Further following the proof of Lemma 3.1 with $Y = T$ and $\tilde{c} = 0$, we can prove that

$$\mathsf{P}(\Delta B_n^* > x) \le \mathsf{P}(\Delta B_n^{\#} > x) = o(\mathsf{P}(T > x)), \qquad n = 0, 1,$$

which shows that condition (iii) of Theorem 3.4 is satisfied. As a result, the conditions of Theorem 3.5 imply conditions (i), (ii) and (iii) of Theorem 3.4.

C.9. Proof of Theorem 3.6

Note that (3.8) and (3.9) yield (3.7) and thus the conditions of Theorem 3.6 imply those of Theorem 3.5, except for $\mathsf{E}[T] < \infty$. Note also that $\mathsf{E}[T] < \infty$ is not covered by conditions (i), (ii) and (iii) of Theorem 3.4. Therefore the conditions of Theorem 3.6 imply conditions (i), (ii) and (iii) of Theorem 3.4 (see the proof of Theorem 3.5 in subsection C.8). As a result, it suffices to prove $R_2(x) = o(\mathsf{P}(T > x))$ (see Remark C.1).

It follows from (C.42) and Lemma 3.3 that

$$R_2(x) \le C \int_0^{(1-\varepsilon)x} u \mathrm{d}\mathsf{P}(T \le u) \sum_{n \le \varepsilon x/(3\gamma)} \mathsf{P}(N(u) = n) \cdot \mathsf{P}(Y > x), \tag{C.43}$$

where γ is a positive number satisfying (C.32). We now fix γ to be

$$\frac{1 - \varepsilon}{\mathsf{E}[\Delta\tau_1]} < \frac{\varepsilon}{3\gamma} \le \frac{1}{\mathsf{E}[\Delta\tau_1]}.$$

As a result, from (C.43), we have

$$\begin{aligned} R_2(x) &\le C \int_0^{(1-\varepsilon)x} u \mathrm{d}\mathsf{P}(T \le u) \sum_{n \le x/\mathsf{E}[\Delta\tau_1]} \mathsf{P}(N(u) = n) \cdot \mathsf{P}(Y > x) \\ &= C\mathsf{E}[T \cdot \mathbb{1}(T \le x, N(T) \le x/\mathsf{E}[\Delta\tau_1])] \cdot \mathsf{P}(Y > x) = o(\mathsf{P}(T > x)), \end{aligned}$$

where the last equality is due to (3.9).

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