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AN EXTENSION OF THE MATRIX-ANALYTIC METHOD FOR M/G/1-TYPE MARKOV PROCESSES

Yoshiaki Inoue
Osaka University
Research Fellow of JSPS

Tetsuya Takine
Osaka University

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Abstract We consider a bivariate Markov process $\{(U(t), S(t));\ t \geq 0\}$, where $U(t)$ $(t \geq 0)$ takes values in $[0, \infty)$ and $S(t)$ $(t \geq 0)$ takes values in a finite set. We assume that $U(t)$ $(t \geq 0)$ is skip-free to the left, and therefore we call it the M/G/1-type Markov process. The M/G/1-type Markov process was first introduced as a generalization of the workload process in the MAP/G/1 queue and its stationary distribution was analyzed under a strong assumption that the conditional infinitesimal generator of the underlying Markov chain $S(t)$ given $U(t) > 0$ is irreducible. In this paper, we extend known results for the stationary distribution to the case that the conditional infinitesimal generator of the underlying Markov chain given $U(t) > 0$ is reducible. With this extension, those results become applicable to the analysis of a certain class of queueing models.

Keywords: Queue, bivariate Markov process, skip-free to the left, matrix-analytic method, reducible infinitesimal generator, MAP/G/1 queue

1. Introduction

We consider a bivariate Markov process $\{(U(t), S(t));\ t \geq 0\}$, where $U(t)$ and $S(t)$ are referred to as the level and the phase, respectively, at time t. $U(t)$ $(t \geq 0)$ takes values in $[0, \infty)$ and $S(t)$ $(t \geq 0)$ takes values in a finite set $\mathcal{M} = \{1, 2, \dots, M\}$. $\{U(t);\ t \geq 0\}$ either decreases at rate one or has upward jump discontinuities, so that $\{U(t);\ t \geq 0\}$ is skip-free to the left. We assume that when $(U(t-), S(t-)) = (x, i)$ $(x > 0,\ i \in \mathcal{M})$, an upward jump (possibly with size zero) occurs at a rate $\sigma^{[i]}$ $(\sigma^{[i]} > 0)$ and the phase $S(t)$ becomes j $(j \in \mathcal{M})$ with probability $p^{[i,j]}$. On the other hand, when $(U(t-), S(t-)) = (0, i)$ $(i \in \mathcal{M})$, an upward jump occurs with probability one and the phase $S(t)$ becomes j $(j \in \mathcal{M})$ with probability $\overline{p}^{[i,j]}$. Note here that for $i \in \mathcal{M}$,

$$\sum_{j \in \mathcal{M}} p^{[i,j]} = 1, \quad \sum_{j \in \mathcal{M}} \overline{p}^{[i,j]} = 1.$$

When $U(t) > 0$ (resp. $U(t) = 0$), the sizes of upward jumps with phase transitions from $S(t-) = i$ to $S(t) = j$ are independent and identically distributed (i.i.d.) according to a general distribution function $B^{[i,j]}(x)$ $(x \geq 0)$ (resp. $\overline{B}^{[i,j]}(x)$ $(x \geq 0)$). To avoid trivialities, we assume $B^{[i,i]}(0) = 0$ $(i \in \mathcal{M})$ and $\overline{B}^{[i,j]}(0) = 0$ $(i, j \in \mathcal{M})$.

We introduce $M \times M$ matrices $\boldsymbol{C}$, $\boldsymbol{D}(x)$ $(x \geq 0)$, and $\overline{\boldsymbol{B}}(x)$ $(x \geq 0)$ to deal with this Markov process.

$$[\boldsymbol{C}]_{i,j} = \begin{cases} -\sigma^{[i]}, & i = j, \\ \sigma^{[i]} p^{[i,j]} B^{[i,j]}(0), & i \neq j, \end{cases}$$

$$[\boldsymbol{D}(0)]_{i,j} = 0, \qquad [\boldsymbol{D}(x)]_{i,j} = \sigma^{[i]} p^{[i,j]} B^{[i,j]}(x), \quad x > 0,$$
$$[\overline{\boldsymbol{B}}(x)]_{i,j} = \overline{p}^{[i,j]} \overline{B}^{[i,j]}(x).$$

We define $\boldsymbol{D}^*(s)$ ($\mathrm{Re}(s) > 0$) and $\overline{\boldsymbol{B}}^*(s)$ ($\mathrm{Re}(s) > 0$) as the Laplace-Stieltjes transforms (LSTs) of $\boldsymbol{D}(x)$ and $\overline{\boldsymbol{B}}(x)$, respectively.

$$\boldsymbol{D}^*(s) = \int_0^\infty \exp(-sx) d\boldsymbol{D}(x), \quad \overline{\boldsymbol{B}}^*(s) = \int_0^\infty \exp(-sx) d\overline{\boldsymbol{B}}(x).$$

Further we define $M \times M$ matrices $\boldsymbol{D}$ and $\overline{\boldsymbol{B}}$ as

$$\boldsymbol{D} = \lim_{x\to\infty} \boldsymbol{D}(x) = \lim_{s\to 0+} \boldsymbol{D}^*(s), \quad \overline{\boldsymbol{B}} = \lim_{x\to\infty} \overline{\boldsymbol{B}}(x) = \lim_{s\to 0+} \overline{\boldsymbol{B}}^*(s).$$

By definition, $\boldsymbol{C} + \boldsymbol{D}$ represents the infinitesimal generator of a continuous-time Markov chain defined on finite state space $\mathcal{M}$. Also, $\overline{\boldsymbol{B}}$ represents the transition probability matrix of a discrete-time Markov chain defined on finite state space $\mathcal{M}$. Therefore, $\boldsymbol{C} + \boldsymbol{D}$ and $\overline{\boldsymbol{B}}$ satisfy

$$(\boldsymbol{C} + \boldsymbol{D})\boldsymbol{e} = \boldsymbol{0}, \qquad \overline{\boldsymbol{B}}\boldsymbol{e} = \boldsymbol{e},$$

respectively, where $\boldsymbol{e}$ denotes an $M \times 1$ vector whose elements are all equal to one.

The Markov process described above was first introduced in [7] as a continuous analog of Markov chains of the M/G/1 type [5]. We thus refer to this Markov process as the M/G/1-type Markov *process*. In [7], the M/G/1-type Markov process is regarded as a generalization of the workload process in the queueing model with customer arrivals of Markovian arrival process (MAP), and the LST of the stationary distribution is derived under the assumption that $\boldsymbol{C} + \boldsymbol{D}$ is *irreducible*. This assumption is appropriate when we consider the stationary behavior of the ordinary MAP/G/1 queues, because it is equivalent to assume that the underlying Markov chain governing the arrival process is irreducible. However, the irreducibility of $\boldsymbol{C} + \boldsymbol{D}$ is *not* necessary for the irreducibility of $\{(U(t), S(t)); t \geq 0\}$. This assumption is thus too strong and restricts its applicability to queueing models.

In this paper, we assume that an $M \times M$ infinitesimal generator

$$\boldsymbol{C} + \boldsymbol{D} + \overline{\boldsymbol{B}} - \boldsymbol{I}$$

is irreducible, where $\boldsymbol{I}$ denotes an $M \times M$ unit matrix. It is easy to see that $\{(U(t), S(t)); t \geq 0\}$ is irreducible if and only if $\boldsymbol{C} + \boldsymbol{D} + \overline{\boldsymbol{B}} - \boldsymbol{I}$ is irreducible. Therefore, even when $\boldsymbol{C} + \boldsymbol{D}$ is reducible, $\{(U(t), S(t)); t \geq 0\}$ is irreducible if all states in $\mathcal{M}$ can be reached from each other with transitions governed by $\boldsymbol{C} + \boldsymbol{D}$ and $\overline{\boldsymbol{B}}$. Note that for discrete-time M/G/1 type Markov chains, analytical results for the case corresponding to reducible $\boldsymbol{C} + \boldsymbol{D}$ is found in [5, section 3.5]. To the best of our knowledge, however, a continuous analog of such results have not been reported in the literature.

The rest of this paper is organized as follows. In Section 2, we explain the application of the M/G/1-type Markov process to the analysis of queueing models. We show through some examples that its applicability is extended allowing $\boldsymbol{C} + \boldsymbol{D}$ to be reducible. In Section 3, we briefly review known results for the M/G/1-type Markov process with irreducible $\boldsymbol{C} + \boldsymbol{D}$ [7]. In Section 4, we first show that results in [7] are not applicable directly to reducible $\boldsymbol{C} + \boldsymbol{D}$, and then derive a formula applicable to the reducible case. In addition, we provide a recursion to compute the moments of the stationary distribution, and consider an efficient computational procedure of a fundamental matrix for reducible $\boldsymbol{C} + \boldsymbol{D}$. Finally, we conclude this paper in Section 5.

2. Applications of the M/G/1-Type Markov Process to Queueing Models

In this section, we shortly explain applications of the M/G/1-type Markov process to the analysis of queueing models. We first make an explanation about queueing models formulated to be the M/G/1-type Markov process with irreducible $\boldsymbol{C}+\boldsymbol{D}$. Next, we present some examples of queueing models that can be analyzed using the M/G/1-type Markov process with reducible $\boldsymbol{C}+\boldsymbol{D}$, which emphasize the motivation of this paper.

As mentioned in Section 1, the M/G/1-type Markov process was first introduced as a continuous analog of the M/G/1 type Markov chain. The M/G/1-type Markov chain is utilized typically for the analysis of MAP/G/1 queues. For the MAP/G/1 queue, the embedded queue length process at the departure instants can be described by the M/G/1-type Markov chain. On the other hand, the censored workload process in the MAP/G/1 queue obtained by observing only busy periods can be described by the M/G/1-type Markov process. Note here that the censored workload process is a stochastic process whose sample paths are identical to those of the original workload process, except that time periods of the system being empty are removed from the time axis and a busy period starts immediately after the system becomes empty. Specifically, the censored workload process in the MAP/G/1 queue characterized by a MAP $(\boldsymbol{C}_{\mathrm{MAP}}, \boldsymbol{D}_{\mathrm{MAP}})$ and a service time distribution $B(x)$ $(x \geq 0)$ corresponds to the M/G/1-type Markov process with

$$\boldsymbol{C} = \boldsymbol{C}_{\mathrm{MAP}}, \quad \boldsymbol{D}(x) = B(x)\boldsymbol{D}_{\mathrm{MAP}}, \quad \overline{\boldsymbol{B}}(x) = B(x)(-\boldsymbol{C}_{\mathrm{MAP}})^{-1}\boldsymbol{D}_{\mathrm{MAP}}.$$

Analysis of the workload process is important when we consider the multi-class FIFO MAP/G/1 queue, i.e., the FIFO queue with marked MAP (MMAP) arrivals [2] with different service time distributions among classes. The queue length process in such a model is difficult to analyze directly because the embedded queue length process at the departure instants is no longer of the M/G/1 type, and we need to keep track of the class of every waiting customer [2]. On the other hand, analysis of the workload process does not have such difficulty. Consider the MMAP/G/1 queue characterized by a MMAP $(\boldsymbol{C}_{\mathrm{MAP}}, \boldsymbol{D}_{\mathrm{MAP},k})$ $(k = 1, 2, \ldots, K)$ and service time distributions $B_k(x)$ $(k = 1, 2, \ldots, K,\ x \geq 0)$. For this model, the censored workload process obtained by observing only busy periods is described by the M/G/1-type Markov process with

$$\boldsymbol{C} = \boldsymbol{C}_{\mathrm{MAP}}, \quad \boldsymbol{D}(x) = \sum_{k=1}^{K} B_k(x)\boldsymbol{D}_{\mathrm{MAP},k}, \quad \overline{\boldsymbol{B}}(x) = (-\boldsymbol{C}_{\mathrm{MAP}})^{-1}\sum_{k=1}^{K} B_k(x)\boldsymbol{D}_{\mathrm{MAP},k}. \tag{1}$$

As shown in [8], the joint distribution of the numbers of customers in the stationary system is given in terms of the stationary workload distribution.

In the analysis of the ordinary MAP/G/1 and MMAP/G/1 queues, it is usually assumed that the underlying Markov chain is irreducible because the existence of transient states have no effect on performance measures of the queues in steady state. In accordance with this convention, the analytical results for the M/G/1-type Markov process reported in [7] are derived under an assumption that $\boldsymbol{C}+\boldsymbol{D}$ is irreducible.

As shown in examples below, however, there exist queueing models whose (censored) workload processes are formulated as M/G/1-type Markov processes with reducible $\boldsymbol{C}+\boldsymbol{D}$.

Example 1-A. Consider a MAP/G/1 queue with two types of busy periods $\{1, 2\}$, where the customer arrival process is governed by $(\boldsymbol{C}^{(i)}_{\mathrm{MAP}}, \boldsymbol{D}^{(i)}_{\mathrm{MAP}}(x))$ during busy periods of type i $(i = 1, 2)$. Transitions of busy-period type occur only when the system is empty. The censored workload process obtained by observing only busy periods is formulated as an M/G/1-type Markov process with

$$\boldsymbol{C} = \begin{pmatrix} \boldsymbol{C}_{\mathrm{MAP}}^{(1)} & \boldsymbol{O} \\ \boldsymbol{O} & \boldsymbol{C}_{\mathrm{MAP}}^{(2)} \end{pmatrix}, \quad \boldsymbol{D}(x) = \begin{pmatrix} \boldsymbol{D}_{\mathrm{MAP}}^{(1)}(x) & \boldsymbol{O} \\ \boldsymbol{O} & \boldsymbol{D}_{\mathrm{MAP}}^{(2)}(x) \end{pmatrix},$$
$$\overline{\boldsymbol{B}}(x) = \begin{pmatrix} \overline{\boldsymbol{B}}^{(11)}(x) & \overline{\boldsymbol{B}}^{(12)}(x) \\ \overline{\boldsymbol{B}}^{(21)}(x) & \overline{\boldsymbol{B}}^{(22)}(x) \end{pmatrix}.$$

An example of a queueing system with two (or more) types of busy periods is a host machine in a distributed server system with dedicated task assignment policy [1]. Each host is dedicated to either "short" or "long" jobs during a busy period so that variability of job sizes to be processed at each host becomes low. Furthermore, when a host becomes idle, its role may be changed to the other one, which improves the utilization of the system.

Example 1-B.

Consider a MAP/G/1 queue with multiple vacations and exhaustive service discipline [4]. For queueing models with vacations, lengths of vacations are usually assumed to be i.i.d. random variables. Using the M/G/1-type Markov process with reducible $\boldsymbol{C} + \boldsymbol{D}$, we can describe a MAP/G/1 queue with semi-Markovian vacation times, where a sequence of vacation lengths forms a semi-Markov process. For example, consider a 2-state semi-Markov process $\{S_{\mathrm{V}}(t);\ t \geq 0\}$, where $S_{\mathrm{V}}(t)$ takes value in $\{1, 2\}$. Let $V^{[i,j]}(x)$ $(x \geq 0,\ i, j = 1, 2)$ denote the joint probability that a state-transition from state i to state j occurs when the sojourn time in state i is elapsed, and the sojourn time in state i is not greater than x. The workload process in a MAP/G/1 queue with vacations whose lengths are governed by this semi-Markov process is then represented by the M/G/1-type Markov process with

$$\boldsymbol{C} = \begin{pmatrix} \boldsymbol{C}_{\mathrm{MAP}} & \boldsymbol{O} \\ \boldsymbol{O} & \boldsymbol{C}_{\mathrm{MAP}} \end{pmatrix}, \quad \boldsymbol{D}(x) = \begin{pmatrix} \boldsymbol{D}_{\mathrm{MAP}}(x) & \boldsymbol{O} \\ \boldsymbol{O} & \boldsymbol{D}_{\mathrm{MAP}}(x) \end{pmatrix},$$
$$\overline{\boldsymbol{B}}(x) = \begin{pmatrix} V^{[1,1]}(x)\boldsymbol{I}_{\mathrm{MAP}} & V^{[1,2]}(x)\boldsymbol{I}_{\mathrm{MAP}} \\ V^{[2,1]}(x)\boldsymbol{I}_{\mathrm{MAP}} & V^{[2,2]}(x)\boldsymbol{I}_{\mathrm{MAP}} \end{pmatrix},$$

where $\boldsymbol{I}_{\mathrm{MAP}}$ denotes a unit matrix with the same size as $\boldsymbol{C}_{\mathrm{MAP}}$. Note that in this case, vacations can be regarded as service times of virtual customers who arrive immediately after the system becomes empty, so that this M/G/1-type Markov process represents the original workload process in the exhaustive-service MAP/G/1 vacation queue with semi-Markovian vacation times.

Example 2-A. Consider a MAP/G/1 queue, where the underlying Markov chain governing the arrival process is transient, and its state gets reset when the system becomes empty. The censored workload process obtained by observing only busy periods is formulated as an M/G/1-type Markov process with

$$\boldsymbol{C} = \begin{pmatrix} \boldsymbol{C}_{\mathrm{T}} & \boldsymbol{C}_{\mathrm{T,N}} \\ \boldsymbol{O} & \boldsymbol{C}_{\mathrm{N}} \end{pmatrix}, \quad \boldsymbol{D}(x) = \begin{pmatrix} \boldsymbol{D}_{\mathrm{T}}(x) & \boldsymbol{D}_{\mathrm{T,N}}(x) \\ \boldsymbol{O} & \boldsymbol{D}_{\mathrm{N}}(x) \end{pmatrix},$$
$$\overline{\boldsymbol{B}}(x) = \begin{pmatrix} \overline{\boldsymbol{B}}_{\mathrm{T,T}}(x) & \boldsymbol{O} \\ \overline{\boldsymbol{B}}_{\mathrm{N,T}}(x) & \boldsymbol{O} \end{pmatrix},$$

where "T" and "N" represent "transient" and "normal", respectively. Data streams generated by the slow-start mechanism of the transmission control protocol (TCP), whose behavior is different during start-up periods, is an example of such a transient arrival process. The analysis of this queueing model enables us to examine the trade-off between the data throughput and the queueing delay caused by congestion in a communication channel.

Example 2-B. As a modified version of Example 2-A, consider a queueing model in which the processing rate is given by $\gamma > 0$ during the transient periods. By means of the change of time scale, the censored workload process can be converted to an M/G/1-type Markov process with [10]

$$\boldsymbol{C} = \begin{pmatrix} \boldsymbol{C}_{\mathrm{T}}/\gamma & \boldsymbol{C}_{\mathrm{T,N}}/\gamma \\ \boldsymbol{O} & \boldsymbol{C}_{\mathrm{N}} \end{pmatrix}, \quad \boldsymbol{D}(x) = \begin{pmatrix} \boldsymbol{D}_{\mathrm{T}}(x)/\gamma & \boldsymbol{D}_{\mathrm{T,N}}(x)/\gamma \\ \boldsymbol{O} & \boldsymbol{D}_{\mathrm{N}}(x) \end{pmatrix}, \tag{2}$$

$$\overline{\boldsymbol{B}}(x) = \begin{pmatrix} \overline{\boldsymbol{B}}_{\mathrm{T,T}}(x) & \boldsymbol{O} \\ \overline{\boldsymbol{B}}_{\mathrm{N,T}}(x) & \boldsymbol{O} \end{pmatrix}. \tag{3}$$

This queueing model is referred to as a queue with working vacations, which was introduced in [6] as a model of an access router in a reconfigurable wavelength division multiplexing (WDM) optical access network. In queueing models with working vacations, when the system becomes empty, the server starts a period called a working vacation, during which the server serves arriving customers with a service rate different from normal periods. By considering the queueing models with working vacations, we can discuss the effectiveness of the adaptive resource allocation mechanisms in reconfigurable WDM optical access networks.

Remark 1. *For simplicity of notations, we considered a single-class model in each example above. These models can be easily extended to the case of MMAP arrivals in the same way as in (1).*

In Section 4, we develop analytical methods for the M/G/1-type Markov processes with reducible $\boldsymbol{C} + \boldsymbol{D}$. The results in Section 4 enable us to obtain performance measures in varieties of queueing models including those described in the examples above.

3. Known Results for Irreducible $\boldsymbol{C} + \boldsymbol{D}$ [7]

In this section, we review known results in [7], assuming that $\boldsymbol{C}+\boldsymbol{D}$ is irreducible. Owing to this assumption, $\boldsymbol{C}+\boldsymbol{D}$ has its invariant probability vector $\boldsymbol{\pi}$, which is uniquely determined by

$$\boldsymbol{\pi}(\boldsymbol{C} + \boldsymbol{D}) = \boldsymbol{0}, \quad \boldsymbol{\pi}\boldsymbol{e} = 1.$$

Let $\boldsymbol{\beta}$ and $\overline{\boldsymbol{\beta}}$ denote $M \times 1$ vectors given by

$$\boldsymbol{\beta} = \int_0^\infty x d\boldsymbol{D}(x)\boldsymbol{e}, \qquad \overline{\boldsymbol{\beta}} = \int_0^\infty x d\overline{\boldsymbol{B}}(x)\boldsymbol{e}. \tag{4}$$

Throughout this section, we assume that

$$\overline{\boldsymbol{\beta}} < \boldsymbol{\infty}, \qquad \boldsymbol{\pi}\boldsymbol{\beta} < 1,$$

which ensures the irreducible Markov process $\{(U(t), S(t));\ t \geq 0\}$ being positive recurrent [7, Theorem 1]. Let $\boldsymbol{u}(x)$ $(x > 0)$ denote a $1 \times M$ vector whose jth $(j \in \mathcal{M})$ element represents the joint probability that the level is not greater than x and the phase is equal to j in steady state and we define $\boldsymbol{u}^*(s)$ $(\mathrm{Re}(s) > 0)$ as the LST of $\boldsymbol{u}(x)$.

$$[\boldsymbol{u}(x)]_j = \lim_{t\to\infty} \Pr(U(t) \leq x, S(t) = j), \quad j \in \mathcal{M},$$

$$\boldsymbol{u}^*(s) = \int_0^\infty \exp(-sx) d\boldsymbol{u}(x).$$

We can derive the following lemma from the balance equation for steady state.

Lemma 1. (Theorem 2 in [7]) $\boldsymbol{u}^*(s)$ *(*$\mathrm{Re}(s) > 0$*) satisfies*

$$\boldsymbol{u}^*(s)[s\boldsymbol{I} + \boldsymbol{C} + \boldsymbol{D}^*(s)] = \acute{\boldsymbol{u}}(0)[\boldsymbol{I} - \overline{\boldsymbol{B}}^*(s)], \qquad \mathrm{Re}(s) > 0, \tag{5}$$

where $\acute{\boldsymbol{u}}(0)$ *denotes the right derivative of* $\boldsymbol{u}(x)$ *at* $x = 0$*.*

$$\acute{\boldsymbol{u}}(0) = \lim_{x \to 0+} \frac{\boldsymbol{u}(x) - \boldsymbol{u}(0)}{x}.$$

Let c denote the reciprocal of the mean recurrence time of the set of states $\{(0, i); i \in \mathcal{M}\}$. Further let $\boldsymbol{\eta}^{\mathrm{E}}$ denote the stationary probability vector of the phase just before the level becomes 0. $\acute{\boldsymbol{u}}(0)$ is then given by

$$\acute{\boldsymbol{u}}(0) = c\boldsymbol{\eta}^{\mathrm{E}}. \tag{6}$$

In order to determine c and $\boldsymbol{\eta}^{\mathrm{E}}$, we consider the first passage time to level 0. Let T^{E} denote the first passage time to level 0 after time 0.

$$T^{\mathrm{E}} = \begin{cases} 0, & U(0) = 0, \\ \inf\{t;\, U(t) = 0, t > 0\}, & \text{otherwise.} \end{cases}$$

We define $\boldsymbol{P}(t \mid x)$ ($t \geq 0$, $x \geq 0$) as an $M \times M$ matrix whose (i, j)th element ($i, j \in \mathcal{M}$) represents the joint probability that the first passage time is not greater than t and the phase is equal to j at the end of the first passage time, given that the level is equal to x and the phase is equal to i at time 0.

$$[\boldsymbol{P}(t \mid x)]_{i,j} = \Pr(T^{\mathrm{E}} \leq t, S(T^{\mathrm{E}}-) = j \mid U(0) = x, S(0) = i).$$

Let $\boldsymbol{P}^*(s \mid x)$ ($\mathrm{Re}(s) > 0$, $x \geq 0$) denote the LST of $\boldsymbol{P}(t \mid x)$ with respect to t.

$$\boldsymbol{P}^*(s \mid x) = \int_{t=0}^{\infty} \exp(-st) d\boldsymbol{P}(t \mid x).$$

Using

$$\boldsymbol{P}^*(s \mid x + y) = \boldsymbol{P}^*(s \mid x)\boldsymbol{P}^*(s \mid y), \qquad x \geq 0, y \geq 0,$$

[11] shows that $\boldsymbol{P}^*(s \mid x)$ ($x \geq 0$) is given by

$$\boldsymbol{P}^*(s \mid x) = \exp(\boldsymbol{Q}^*(s)x), \tag{7}$$

where $\boldsymbol{Q}^*(s)$ ($\mathrm{Re}(s) > 0$) denotes an $M \times M$ matrix that satisfies

$$\boldsymbol{Q}^*(s) = -s\boldsymbol{I} + \boldsymbol{C} + \int_0^{\infty} d\boldsymbol{D}(y) \exp(\boldsymbol{Q}^*(s)y). \tag{8}$$

Let $\boldsymbol{P}(x)$ ($x \geq 0$) denote an $M \times M$ transition probability matrix whose (i, j)th element ($i, j \in \mathcal{M}$) is given by

$$[\boldsymbol{P}(x)]_{i,j} = \Pr(S(T^{\mathrm{E}}-) = j \mid U(0) = x, S(0) = i).$$

By definition, we have

$$\boldsymbol{P}(x) = \lim_{s \to 0+} \boldsymbol{P}^*(s \mid x) = \exp(\boldsymbol{Q}x), \quad x \geq 0, \tag{9}$$

where

$$\boldsymbol{Q} = \lim_{s \to 0+} \boldsymbol{Q}^*(s).$$

Because of (8), $\boldsymbol{Q}$ satisfies

$$\boldsymbol{Q} = \boldsymbol{C} + \int_0^\infty d\boldsymbol{D}(y) \exp(\boldsymbol{Q}y). \tag{10}$$

Remark 2. *As shown in [11], $\boldsymbol{Q}$ is given by the limit $\lim_{n\to\infty} \boldsymbol{Q}^{(n)}$ of an elementwise increasing sequence of matrices $\{\boldsymbol{Q}^{(n)}\}_{n=0,1,\dots}$ given by the following recursion.*

$$\boldsymbol{Q}^{(0)} = \boldsymbol{C}, \qquad \boldsymbol{Q}^{(n)} = \boldsymbol{C} + \int_0^\infty d\boldsymbol{D}(y) \exp\bigl(\boldsymbol{Q}^{(n-1)}y\bigr), \quad n = 1, 2, \dots. \tag{11}$$

Because the integral on the right-side of this equation can be computed with uniformization [12], we can numerically obtain $\boldsymbol{Q} = \lim_{n\to\infty} \boldsymbol{Q}^{(n)}$ with an adequate stopping criterion. More specifically, for a given allowable error $\epsilon > 0$, we may stop the iteration at n^* satisfying $\max_{i\in\mathcal{M}} \bigl|[\boldsymbol{Q}^{(n^*)}\boldsymbol{e}]_i\bigr| < \epsilon$.

$\boldsymbol{Q}$ is known to be an infinitesimal generator of a Markov chain on $\mathcal{M}$, and it is irreducible if $\boldsymbol{C}+\boldsymbol{D}$ is irreducible [9, 11]. Therefore, because of the assumption of the irreducible $\boldsymbol{C}+\boldsymbol{D}$, $\boldsymbol{Q}$ has its invariant probability vector $\boldsymbol{\kappa}$, which is uniquely determined by

$$\boldsymbol{\kappa}\boldsymbol{Q} = \boldsymbol{0}, \quad \boldsymbol{\kappa}\boldsymbol{e} = 1. \tag{12}$$

We define $\boldsymbol{f}(x)$ $(x \geq 0)$ as an $M \times 1$ vector whose ith $(i \in \mathcal{M})$ element represents the mean first passage time to level 0, given that the level is equal to x and the phase is equal to i at time 0.

$$[\boldsymbol{f}(x)]_i = \mathrm{E}[T^{\mathrm{E}} \mid U(0) = x, S(0) = i].$$

Noting (7) and (8), we obtain $\boldsymbol{f}(x)$ through a straightforward calculation.

$$\begin{aligned}
\boldsymbol{f}(x) &= (-1) \cdot \lim_{s\to 0+} \frac{\partial}{\partial s} \boldsymbol{P}^*(s \mid x)\boldsymbol{e} \\
&= \left(\sum_{n=1}^{\infty} \frac{x^n \boldsymbol{Q}^{n-1}}{n!}\right) \left((-1) \cdot \lim_{s\to 0+} \frac{\partial}{\partial s} \boldsymbol{Q}^*(s)\boldsymbol{e}\right) &\quad (13) \\
&= [x\boldsymbol{e}\boldsymbol{\kappa} - \exp(\boldsymbol{Q}x) + \boldsymbol{I}][(\boldsymbol{e} - \boldsymbol{\beta})\boldsymbol{\kappa} - \boldsymbol{C} - \boldsymbol{D}]^{-1}\boldsymbol{e}, &\quad (14)
\end{aligned}$$

because

$$\left(\sum_{n=1}^{\infty} \frac{x^n \boldsymbol{Q}^{n-1}}{n!}\right) = [x\boldsymbol{e}\boldsymbol{\kappa} - \exp(\boldsymbol{Q}x) + \boldsymbol{I}](\boldsymbol{e}\boldsymbol{\kappa} - \boldsymbol{Q})^{-1}, \tag{15}$$

$$(-1) \cdot \lim_{s\to 0+} \frac{\partial}{\partial s} \boldsymbol{Q}^*(s)\boldsymbol{e} = (\boldsymbol{e}\boldsymbol{\kappa} - \boldsymbol{Q})[(\boldsymbol{e} - \boldsymbol{\beta})\boldsymbol{\kappa} - \boldsymbol{C} - \boldsymbol{D}]^{-1}\boldsymbol{e}. \tag{16}$$

It is known that both of $\boldsymbol{e}\boldsymbol{\kappa} - \boldsymbol{Q}$ and $(\boldsymbol{e} - \boldsymbol{\beta})\boldsymbol{\kappa} - \boldsymbol{C} - \boldsymbol{D}$ are non-singular when $\boldsymbol{C} + \boldsymbol{D}$ is irreducible.

c and $\boldsymbol{\eta}^{\mathrm{E}}$ on the right-hand side of (6) is then given by the following lemma.

Lemma 2. *(Theorem 3 in [7]) $\boldsymbol{\eta}^{\mathrm{E}}$ is uniquely determined by*

$$\boldsymbol{\eta}^{\mathrm{E}} \int_0^\infty d\overline{\boldsymbol{B}}(x) \exp(\boldsymbol{Q}x) = \boldsymbol{\eta}^{\mathrm{E}}, \quad \boldsymbol{\eta}^{\mathrm{E}}\boldsymbol{e} = 1, \tag{17}$$

and c *is given by*

$$c = \frac{1}{\boldsymbol{\eta}^{\mathrm{E}} \displaystyle\int_0^\infty d\overline{\boldsymbol{B}}(x)\boldsymbol{f}(x)} = \frac{1}{\boldsymbol{\eta}^{\mathrm{E}}(\overline{\boldsymbol{\beta}}\boldsymbol{\kappa} + \overline{\boldsymbol{B}} - \boldsymbol{I})[(\boldsymbol{e} - \boldsymbol{\beta})\boldsymbol{\kappa} - \boldsymbol{C} - \boldsymbol{D}]^{-1}\boldsymbol{e}}. \tag{18}$$

Before closing this section, we derive an alternative formula for $\boldsymbol{u}^*(s)$, which is similar to that given in [9]. We define $M \times M$ matrices $\boldsymbol{R}^*(s)$ $(\mathrm{Re}(s) > 0)$ and $\overline{\boldsymbol{R}}^*(s)$ $(\mathrm{Re}(s) > 0)$ as

$$\begin{aligned}
\boldsymbol{R}^*(s) &= \int_0^\infty \exp(-sx)dx \int_x^\infty d\boldsymbol{D}(y)\exp(\boldsymbol{Q}(y-x)),\\
\overline{\boldsymbol{R}}^*(s) &= \int_0^\infty \exp(-sx)dx \int_x^\infty d\overline{\boldsymbol{B}}(y)\exp\big(\boldsymbol{Q}(y-x)\big).
\end{aligned}$$

By definition, $\boldsymbol{R}^*(s)$ and $\overline{\boldsymbol{R}}^*(s)$ satisfy

$$[\boldsymbol{I} - \boldsymbol{R}^*(s)](s\boldsymbol{I} + \boldsymbol{Q}) = s\boldsymbol{I} + \boldsymbol{C} + \boldsymbol{D}^*(s), \quad \mathrm{Re}(s) > 0, \tag{19}$$

$$\overline{\boldsymbol{R}}^*(s)(s\boldsymbol{I} + \boldsymbol{Q}) = \int_0^\infty d\overline{\boldsymbol{B}}(y)\exp(\boldsymbol{Q}y) - \overline{\boldsymbol{B}}^*(s), \quad \mathrm{Re}(s) > 0. \tag{20}$$

It follows from (6), (17), and (20) that

$$\acute{\boldsymbol{u}}(0)\overline{\boldsymbol{R}}^*(s)(s\boldsymbol{I} + \boldsymbol{Q}) = \acute{\boldsymbol{u}}(0)[\boldsymbol{I} - \overline{\boldsymbol{B}}^*(s)], \qquad \mathrm{Re}(s) > 0.$$

With (19), (5) is then rewritten to be

$$\boldsymbol{u}^*(s)[\boldsymbol{I} - \boldsymbol{R}^*(s)](s\boldsymbol{I} + \boldsymbol{Q}) = \acute{\boldsymbol{u}}(0)\overline{\boldsymbol{R}}^*(s)(s\boldsymbol{I} + \boldsymbol{Q}), \qquad \mathrm{Re}(s) > 0. \tag{21}$$

In the same way as in [3, P. 66], it can be shown that (21) implies

$$\boldsymbol{u}^*(s)[\boldsymbol{I} - \boldsymbol{R}^*(s)] = \acute{\boldsymbol{u}}(0)\overline{\boldsymbol{R}}^*(s), \qquad \mathrm{Re}(s) > 0.$$

We thus obtain the following theorem.

Theorem 1. $\boldsymbol{u}^*(s)$ *is given by*

$$\boldsymbol{u}^*(s) = \acute{\boldsymbol{u}}(0)\overline{\boldsymbol{R}}^*(s)[\boldsymbol{I} - \boldsymbol{R}^*(s)]^{-1}, \qquad \mathrm{Re}(s) > 0. \tag{22}$$

Remark 3. *[9] shows that* $\boldsymbol{I} - \boldsymbol{R}^*(s)$ *(*$\mathrm{Re}(s) > 0$*) is non-singular when* $\boldsymbol{C} + \boldsymbol{D}$ *is irreducible.*

4. Results for Reducible $\boldsymbol{C} + \boldsymbol{D}$

In this section, we generalize the results in Section 3 to the case of reducible $\boldsymbol{C} + \boldsymbol{D}$. More specifically, we assume that the infinitesimal generator $\boldsymbol{C} + \boldsymbol{D}$ is reducible and it has H closed irreducible classes of states. We define $\mathcal{H} = \{1, 2, \ldots, H\}$ as the set of such irreducible classes. $\boldsymbol{C}$ and $\boldsymbol{D}$ are then written in the following form.

$$\boldsymbol{C} = \begin{pmatrix} \boldsymbol{C}_{\mathrm{T}} & \boldsymbol{C}_{\mathrm{T},1} & \boldsymbol{C}_{\mathrm{T},2} & \cdots & \boldsymbol{C}_{\mathrm{T},H} \\ \boldsymbol{O} & \boldsymbol{C}_1 & \boldsymbol{O} & \cdots & \boldsymbol{O} \\ \boldsymbol{O} & \boldsymbol{O} & \boldsymbol{C}_2 & \cdots & \boldsymbol{O} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \boldsymbol{O} & \boldsymbol{O} & \boldsymbol{O} & \cdots & \boldsymbol{C}_H \end{pmatrix}, \quad \boldsymbol{D} = \begin{pmatrix} \boldsymbol{D}_{\mathrm{T}} & \boldsymbol{D}_{\mathrm{T},1} & \boldsymbol{D}_{\mathrm{T},2} & \cdots & \boldsymbol{D}_{\mathrm{T},H} \\ \boldsymbol{O} & \boldsymbol{D}_1 & \boldsymbol{O} & \cdots & \boldsymbol{O} \\ \boldsymbol{O} & \boldsymbol{O} & \boldsymbol{D}_2 & \cdots & \boldsymbol{O} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \boldsymbol{O} & \boldsymbol{O} & \boldsymbol{O} & \cdots & \boldsymbol{D}_H \end{pmatrix}, \tag{23}$$

where $\boldsymbol{C}_{\mathrm{T}}$ denotes an $M_{\mathrm{T}} \times M_{\mathrm{T}}$ defective infinitesimal generator, $\boldsymbol{C}_h$ $(h \in H)$ denotes an $M_h \times M_h$ defective infinitesimal generator, and $\boldsymbol{C}_{\mathrm{T},h}$ $(h \in H)$ denotes an $M_{\mathrm{T}} \times M_h$ transition rate matrix. Also, $\boldsymbol{D}_{\mathrm{T}}$ denotes an $M_{\mathrm{T}} \times M_{\mathrm{T}}$ transition rate matrix, $\boldsymbol{D}_h$ $(h \in H)$ denotes an $M_h \times M_h$ transition rate matrix, and $\boldsymbol{D}_{\mathrm{T},h}$ $(h \in H)$ denotes an $M_{\mathrm{T}} \times M_h$ transition rate matrix. Because $\boldsymbol{C}_h + \boldsymbol{D}_h$ $(h \in H)$ represents an irreducible infinitesimal generator, it has its invariant probability vector $\boldsymbol{\pi}_h$, which is uniquely determined by

$$\boldsymbol{\pi}_h(\boldsymbol{C}_h + \boldsymbol{D}_h) = \boldsymbol{0}, \quad \boldsymbol{\pi}_h \boldsymbol{e}_h = 1,$$

where $\boldsymbol{e}_h$ $(h \in \mathcal{H})$ denotes an $M_h \times 1$ vector whose elements are all equal to one.

Throughout this paper, for any $M \times M$ block upper-triangular matrix similar to $\boldsymbol{C}$ and $\boldsymbol{D}$ in (23), we denote the $(0,0)$th block by the subscript "T", the $(0,h)$th block $(h \in \mathcal{H})$ by the subscript "T, h", and the (h,h)th block $(h \in \mathcal{H})$ by the subscript "h". We define $M_{\mathrm{T}} \times 1$ vector $\boldsymbol{\beta}_{\mathrm{T}}$ and $M_h \times 1$ vector $\boldsymbol{\beta}_h$ $(h \in \mathcal{H})$ as

$$\boldsymbol{\beta}_h = \int_0^\infty x d\boldsymbol{D}_h(x)\boldsymbol{e}_h, \quad \boldsymbol{\beta}_{\mathrm{T}} = \int_0^\infty x d\boldsymbol{D}_{\mathrm{T}}(x)\boldsymbol{e}_{\mathrm{T}} + \sum_{h \in \mathcal{H}} \int_0^\infty x d\boldsymbol{D}_{\mathrm{T},h}(x)\boldsymbol{e}_h,$$

respectively, where $\boldsymbol{e}_{\mathrm{T}}$ denotes an $M_{\mathrm{T}} \times 1$ vector whose elements are all equal to one (cf. (4)). We assume that an $M \times M$ infinitesimal generator $\boldsymbol{C} + \boldsymbol{D} + \overline{\boldsymbol{B}} - \boldsymbol{I}$ is irreducible, which is a necessary and sufficient condition for $\{(U(t), S(t));\, t \geq 0\}$ to be irreducible as noted in Section 1. We also assume that

$$\overline{\boldsymbol{\beta}} < \infty, \quad \boldsymbol{\beta}_{\mathrm{T}} < \infty, \quad \boldsymbol{\pi}_h \boldsymbol{\beta}_h < 1, \ \ h \in \mathcal{H}. \tag{24}$$

With Theorem 1 in [7], it is easy to see that $\{(U(t), S(t));\, t \geq 0\}$ is positive recurrent if and only if (24) holds.

As mentioned in Section 3, the assumption of the irreducible $\boldsymbol{C} + \boldsymbol{D}$ is a sufficient condition for the followings to hold:

(i) The matrix $\boldsymbol{Q}$ is irreducible, so that its invariant probability vector $\boldsymbol{\kappa}$ is uniquely determined by (12).

(ii) Both $\boldsymbol{e}\boldsymbol{\kappa} - \boldsymbol{Q}$ and $(\boldsymbol{e} - \boldsymbol{\beta})\boldsymbol{\kappa} - \boldsymbol{C} - \boldsymbol{D}$ are non-singular, and therefore $\boldsymbol{f}(x)$ $(x \geq 0)$ is given by (14).

(iii) $\boldsymbol{I} - \boldsymbol{R}^*(s)$ on the right-hand side of (22) is non-singular for $\mathrm{Re}(s) > 0$.

Note that these are the only things in the discussion of Section 3, which are related to the irreducibility of $\boldsymbol{C} + \boldsymbol{D}$.

We can prove that (iii) still holds for reducible $\boldsymbol{C} + \boldsymbol{D}$. We provide an outline of its proof in Appendix A. As shown below, on the other hand, neither of (i) and (ii) above is valid when $\boldsymbol{C} + \boldsymbol{D}$ is reducible with more than one irreducible classes of states (i.e., $H \geq 2$).

Noting that $\boldsymbol{Q}$ is given by the limit of the sequence of matrices $\{\boldsymbol{Q}^{(n)}\}_{n=0,1,\dots}$ defined as (11), it is easy to see that $\boldsymbol{Q}$ takes the form

$$\boldsymbol{Q} = \begin{pmatrix} \boldsymbol{Q}_{\mathrm{T}} & \boldsymbol{Q}_{\mathrm{T},1} & \boldsymbol{Q}_{\mathrm{T},2} & \cdots & \boldsymbol{Q}_{\mathrm{T},H} \\ \boldsymbol{O} & \boldsymbol{Q}_1 & \boldsymbol{O} & \cdots & \boldsymbol{O} \\ \boldsymbol{O} & \boldsymbol{O} & \boldsymbol{Q}_2 & \cdots & \boldsymbol{O} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \boldsymbol{O} & \boldsymbol{O} & \boldsymbol{O} & \cdots & \boldsymbol{Q}_H \end{pmatrix}, \tag{25}$$

where $\boldsymbol{Q}_{\mathrm{T}}$ denotes a defective infinitesimal generator, $\boldsymbol{Q}_{\mathrm{T},h}$ $(h \in \mathcal{H})$ denotes a transition rate matrix, and $\boldsymbol{Q}_h$ $(h \in \mathcal{H})$ denotes an irreducible infinitesimal generator. $\boldsymbol{Q}$ is thus no longer irreducible. Furthermore, when $H \geq 2$, there are infinitely many invariant probability vectors of $\boldsymbol{Q}$, which are given by linear combinations of the invariant probability vectors of $\boldsymbol{Q}_1$, $\boldsymbol{Q}_2$, ..., and $\boldsymbol{Q}_H$. The following lemma shows that $\boldsymbol{e}\boldsymbol{\kappa} - \boldsymbol{Q}$ and $(\boldsymbol{e} - \boldsymbol{\beta})\boldsymbol{\kappa} - \boldsymbol{C} - \boldsymbol{D}$ are no longer non-singular for any invariant probability vector $\boldsymbol{\kappa}$ of $\boldsymbol{Q}$ if $H \geq 2$.

Lemma 3. *Consider an $M \times M$ reducible infinitesimal generator $\boldsymbol{Y}$ with H closed irreducible classes of states.*

$$\boldsymbol{Y} = \begin{pmatrix} \boldsymbol{Y}_{\mathrm{T}} & \boldsymbol{Y}_{\mathrm{T},1} & \boldsymbol{Y}_{\mathrm{T},2} & \cdots & \boldsymbol{Y}_{\mathrm{T},H} \\ \boldsymbol{O} & \boldsymbol{Y}_1 & \boldsymbol{O} & \cdots & \boldsymbol{O} \\ \boldsymbol{O} & \boldsymbol{O} & \boldsymbol{Y}_2 & \cdots & \boldsymbol{O} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \boldsymbol{O} & \boldsymbol{O} & \boldsymbol{O} & \cdots & \boldsymbol{Y}_H \end{pmatrix}.$$

Let $\boldsymbol{\gamma}_h$ $(h \in \mathcal{H})$ denote the invariant probability vector of $\boldsymbol{Y}_h$.

$$\boldsymbol{\gamma}_h \boldsymbol{Y}_h = \boldsymbol{0}, \qquad \boldsymbol{\gamma}_h \boldsymbol{e}_h = 1, \quad h \in \mathcal{H}.$$

If $H \geq 2$, $\boldsymbol{v}\boldsymbol{\alpha} - \boldsymbol{Y}$ is singular for any $1 \times M$ real vector $\boldsymbol{\alpha}$ and any $M \times 1$ real vector $\boldsymbol{v}$ satisfying

$$\boldsymbol{v} = \begin{pmatrix} \boldsymbol{v}_{\mathrm{T}} \\ \boldsymbol{v}_1 \\ \boldsymbol{v}_2 \\ \vdots \\ \boldsymbol{v}_H \end{pmatrix}, \qquad \boldsymbol{\gamma}_h \boldsymbol{v}_h \neq 0 \quad \text{for some } h \in \mathcal{H}, \tag{26}$$

where $\boldsymbol{v}_{\mathrm{T}}$ and $\boldsymbol{v}_h$ $(h \in \mathcal{H})$ denote $M_{\mathrm{T}} \times 1$ and $M_h \times 1$ vectors, respectively.

We prove Lemma 3 in Appendix B.

When $H \geq 2$, we can verify that $\boldsymbol{e}\boldsymbol{\kappa} - \boldsymbol{Q}$ (resp. $(\boldsymbol{e} - \boldsymbol{\beta})\boldsymbol{\kappa} - \boldsymbol{C} - \boldsymbol{D}$) is singular for any invariant probability vector $\boldsymbol{\kappa}$ of $\boldsymbol{Q}$, by letting $\boldsymbol{\alpha} = \boldsymbol{\kappa}$, $\boldsymbol{v} = \boldsymbol{e}$ (resp. $\boldsymbol{v} = \boldsymbol{e} - \boldsymbol{\beta}$), and $\boldsymbol{Y} = \boldsymbol{Q}$ (resp. $\boldsymbol{Y} = \boldsymbol{C} + \boldsymbol{D}$) in Lemma 3. The formulae (14) and (18) in Section 3 is thus not applicable to reducible $\boldsymbol{C} + \boldsymbol{D}$ with more than one irreducible classes of states.

Remark 4. *If $\boldsymbol{C} + \boldsymbol{D}$ has transient states and only one irreducible class of states, i.e., $H = 1$, $\boldsymbol{Q}$ has the unique invariant probability vector $\boldsymbol{\kappa}$ even though it is reducible. In this case, we can prove that both of $\boldsymbol{e}\boldsymbol{\kappa} - \boldsymbol{Q}$ and $(\mathbf{e} - \boldsymbol{\beta})\boldsymbol{\kappa} - \boldsymbol{C} - \boldsymbol{D}$ are non-singular.*

Remark 5. *Although analytical results for the M/G/1-type Markov chain corresponding to the case of reducible $\boldsymbol{C} + \boldsymbol{D}$ is obtained in [5, section 3.5], it considers only the case of $H = 1$ with transient states. As shown for the continuous version, however, the case of $H \geq 2$ is essentially different from that of $H = 1$.*

The rest of this section consists of three subsections. In Section 4.1, we consider the LST of the stationary distribution $\boldsymbol{u}^*(s)$ ($\mathrm{Re}(s) > 0$), and derive a formula applicable to reducible $\boldsymbol{C} + \boldsymbol{D}$. In Section 4.2, we provide an efficient computational procedure of reducible $\boldsymbol{Q}$ with the block structure (25). Finally, in Section 4.3, we consider the moments of the stationary distribution. We show that some modification from the irreducible case is necessary to obtain the moments.

4.1. LST of Stationary Distribution

In this subsection, we derive a formula for the LST of the stationary distribution $\boldsymbol{u}^*(s)$ ($\mathrm{Re}(s) > 0$) applicable to reducible $\boldsymbol{C} + \boldsymbol{D}$. Note that (5) and (22) are still valid for reducible $\boldsymbol{C} + \boldsymbol{D}$. The difference from the irreducible case is that $\acute{\boldsymbol{u}}(0)$ cannot be obtained from Lemma 2 because (14) and (18) does not hold for reducible $\boldsymbol{C} + \boldsymbol{D}$ with $H \geq 2$ as shown above.

Therefore, we first derive a formula for the mean first passage time $\boldsymbol{f}(x)$ $(x \geq 0)$ applicable to reducible $\boldsymbol{C}+\boldsymbol{D}$ with the general structure (23). Let $\boldsymbol{\kappa}_h$ $(h \in \mathcal{H})$ denote the invariant probability vector of $\boldsymbol{Q}_h$ (see (25)), which is uniquely determined by

$$\boldsymbol{\kappa}_h \boldsymbol{Q}_h = \boldsymbol{0}, \quad \boldsymbol{\kappa}_h \boldsymbol{e}_h = 1. \tag{27}$$

We then define $M \times M$ matrix $\check{\boldsymbol{Q}}$ as

$$\check{\boldsymbol{Q}} = \begin{pmatrix} \boldsymbol{O} & \check{\boldsymbol{Q}}_{\mathrm{T},1} & \check{\boldsymbol{Q}}_{\mathrm{T},2} & \cdots & \check{\boldsymbol{Q}}_{\mathrm{T},H} \\ \boldsymbol{O} & \boldsymbol{e}_1\boldsymbol{\kappa}_1 & \boldsymbol{O} & \cdots & \boldsymbol{O} \\ \boldsymbol{O} & \boldsymbol{O} & \boldsymbol{e}_2\boldsymbol{\kappa}_2 & \cdots & \boldsymbol{O} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \boldsymbol{O} & \boldsymbol{O} & \boldsymbol{O} & \cdots & \boldsymbol{e}_H\boldsymbol{\kappa}_H \end{pmatrix}, \tag{28}$$

where

$$\check{\boldsymbol{Q}}_{\mathrm{T},h} = (-\boldsymbol{Q}_{\mathrm{T}})^{-1}\boldsymbol{Q}_{\mathrm{T},h}\boldsymbol{e}_h\boldsymbol{\kappa}_h, \quad h \in \mathcal{H}.$$

Lemma 4. *$\boldsymbol{f}(x)$ $(x \geq 0)$ is given by*

$$\boldsymbol{f}(x) = [\boldsymbol{I} - \exp(\boldsymbol{Q}x) + x\check{\boldsymbol{Q}}](\boldsymbol{\Delta} - \boldsymbol{C} - \boldsymbol{D})^{-1}\boldsymbol{e}, \tag{29}$$

where $\boldsymbol{\Delta}$ is defined as

$$\boldsymbol{\Delta} = \check{\boldsymbol{Q}} - \int_0^\infty x d\boldsymbol{D}(x)\check{\boldsymbol{Q}} = \begin{pmatrix} \boldsymbol{O} & \boldsymbol{\Delta}_{\mathrm{T},1} & \boldsymbol{\Delta}_{\mathrm{T},2} & \cdots & \boldsymbol{\Delta}_{\mathrm{T},H} \\ \boldsymbol{O} & (\boldsymbol{e}_1 - \boldsymbol{\beta}_1)\boldsymbol{\kappa}_1 & \boldsymbol{O} & \cdots & \boldsymbol{O} \\ \boldsymbol{O} & \boldsymbol{O} & (\boldsymbol{e}_2 - \boldsymbol{\beta}_2)\boldsymbol{\kappa}_2 & \cdots & \boldsymbol{O} \\ \vdots & \vdots & \vdots & \ddots & \boldsymbol{O} \\ \boldsymbol{O} & \boldsymbol{O} & \boldsymbol{O} & \cdots & (\boldsymbol{e}_H - \boldsymbol{\beta}_H)\boldsymbol{\kappa}_H \end{pmatrix},$$

$$\boldsymbol{\Delta}_{\mathrm{T},h} = \check{\boldsymbol{Q}}_{\mathrm{T},h} - \int_0^\infty x d\boldsymbol{D}_{\mathrm{T}}(x)\check{\boldsymbol{Q}}_{\mathrm{T},h} - \int_0^\infty x d\boldsymbol{D}_{\mathrm{T},h}(x)\boldsymbol{e}_h\boldsymbol{\kappa}_h, \quad h \in \mathcal{H}.$$

Remark 6. $(\boldsymbol{\Delta} - \boldsymbol{C} - \boldsymbol{D})^{-1}$ *is given by*

$$(\boldsymbol{\Delta} - \boldsymbol{C} - \boldsymbol{D})^{-1} = \begin{pmatrix} [-(\boldsymbol{C}_{\mathrm{T}} + \boldsymbol{D}_{\mathrm{T}})]^{-1} & \boldsymbol{J}_{\mathrm{T},1} & \boldsymbol{J}_{\mathrm{T},2} & \cdots & \boldsymbol{J}_{\mathrm{T},H} \\ \boldsymbol{O} & \widehat{\boldsymbol{\Delta}}_1^{-1} & \boldsymbol{O} & \cdots & \boldsymbol{O} \\ \boldsymbol{O} & \boldsymbol{O} & \widehat{\boldsymbol{\Delta}}_2^{-1} & \cdots & \boldsymbol{O} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \boldsymbol{O} & \boldsymbol{O} & \boldsymbol{O} & \cdots & \widehat{\boldsymbol{\Delta}}_H^{-1} \end{pmatrix}, \tag{30}$$

where

$$\widehat{\boldsymbol{\Delta}}_h = (\boldsymbol{e}_h - \boldsymbol{\beta}_h)\boldsymbol{\kappa}_h - \boldsymbol{C}_h - \boldsymbol{D}_h, \quad h \in \mathcal{H},$$
$$\boldsymbol{J}_{\mathrm{T},h} = (-1)\cdot[-(\boldsymbol{C}_{\mathrm{T}} + \boldsymbol{D}_{\mathrm{T}})]^{-1}(\boldsymbol{\Delta}_{\mathrm{T},h} - \boldsymbol{C}_{\mathrm{T},h} - \boldsymbol{D}_{\mathrm{T},h})\widehat{\boldsymbol{\Delta}}_h^{-1}, \quad h \in \mathcal{H}.$$

Proof. Because we can prove Lemma 4 in almost the same way as the irreducible case in [11], we provide only an outline of the proof. By definition of $\check{\boldsymbol{Q}}$, it follows that

$$\boldsymbol{Q}\check{\boldsymbol{Q}} = \boldsymbol{O},$$

from which we obtain a similar result to (15).

$$\sum_{n=1}^{\infty} \frac{x^n}{n!} \boldsymbol{Q}^{n-1} = \left[\boldsymbol{I} - \exp(\boldsymbol{Q}x) + x\check{\boldsymbol{Q}}\right](\check{\boldsymbol{Q}} - \boldsymbol{Q})^{-1}.$$

Note here that the block upper-triangular matrix $\check{\boldsymbol{Q}} - \boldsymbol{Q}$ is non-singular because its diagonal block matrices $-\boldsymbol{Q}_{\mathrm{T}}$ and $\boldsymbol{e}_h\boldsymbol{\kappa}_h - \boldsymbol{Q}_h$ ($h \in \mathcal{H}$) are non-singular. Noting that $\boldsymbol{\Delta} - \boldsymbol{C} - \boldsymbol{D}$ is non-singular by the same reasoning as the non-singularity of $\check{\boldsymbol{Q}} - \boldsymbol{Q}$, we also obtain

$$(-1) \cdot \lim_{s \to 0+} \frac{d}{ds} \boldsymbol{Q}^*(s)\boldsymbol{e} = (\check{\boldsymbol{Q}} - \boldsymbol{Q})(\boldsymbol{\Delta} - \boldsymbol{C} - \boldsymbol{D})^{-1}\boldsymbol{e},$$

which corresponds to (16). We then obtain (29) from (13). □

We then obtain $\boldsymbol{u}^*(s)$ ($\mathrm{Re}(s) > 0$) for reducible $\boldsymbol{C} + \boldsymbol{D}$ using (5), (17), and (22).

Theorem 2. *$\boldsymbol{u}^*(s)$ ($\mathrm{Re}(s) > 0$) satisfies*

$$\boldsymbol{u}^*(s)[s\boldsymbol{I} + \boldsymbol{C} + \boldsymbol{D}^*(s)] = c\boldsymbol{\eta}^{\mathrm{E}}[\boldsymbol{I} - \overline{\boldsymbol{B}}^*(s)], \qquad \mathrm{Re}(s) > 0, \tag{31}$$

and it is given by

$$\boldsymbol{u}^*(s) = c\boldsymbol{\eta}^{\mathrm{E}}\overline{\boldsymbol{R}}^*(s)[\boldsymbol{I} - \boldsymbol{R}^*(s)]^{-1}, \qquad \mathrm{Re}(s) > 0,$$

where $\boldsymbol{\eta}^{\mathrm{E}}$ denotes a $1 \times M$ probability vector which is uniquely determined by

$$\boldsymbol{\eta}^{\mathrm{E}} = \boldsymbol{\eta}^{\mathrm{E}} \int_0^{\infty} d\overline{\boldsymbol{B}}(x) \exp(\boldsymbol{Q}x), \quad \boldsymbol{\eta}^{\mathrm{E}}\boldsymbol{e} = 1, \tag{32}$$

and c is given by

$$c = \frac{1}{\boldsymbol{\eta}^{\mathrm{E}}(\overline{\boldsymbol{\Delta}} + \overline{\boldsymbol{B}} - \boldsymbol{I})(\boldsymbol{\Delta} - \boldsymbol{C} - \boldsymbol{D})^{-1}\boldsymbol{e}}, \tag{33}$$
$$\overline{\boldsymbol{\Delta}} = \int_0^{\infty} x d\overline{\boldsymbol{B}}(x)\check{\boldsymbol{Q}}.$$

Remark 7. *Let $\boldsymbol{\Phi}$ denote an $M \times M$ matrix given by*

$$\boldsymbol{\Phi} = \int_0^{\infty} d\overline{\boldsymbol{B}}(x) \exp(\boldsymbol{Q}x).$$

Since $\{(U(t), S(t)); t \geq 0\}$ is irreducible and positive recurrent, $\boldsymbol{\Phi}$ represents an irreducible transition probability matrix. Therefore, $\boldsymbol{\Phi}$ has its invariant probability vector, so that $\boldsymbol{\eta}^{\mathrm{E}}$ is uniquely determined by (32)

Remark 8. *When we apply Theorem 2 to the case that $\boldsymbol{C} + \boldsymbol{D}$ has no transient states, it is necessary to rewrite (28) and (30) as*

$$\check{\boldsymbol{Q}} = \begin{pmatrix} \boldsymbol{e}_1\boldsymbol{\kappa}_1 & \boldsymbol{O} & \cdots & \boldsymbol{O} \\ \boldsymbol{O} & \boldsymbol{e}_2\boldsymbol{\kappa}_2 & \cdots & \boldsymbol{O} \\ \vdots & \vdots & \ddots & \vdots \\ \boldsymbol{O} & \boldsymbol{O} & \cdots & \boldsymbol{e}_H\boldsymbol{\kappa}_H \end{pmatrix}, \quad (\boldsymbol{\Delta} - \boldsymbol{C} - \boldsymbol{D})^{-1} = \begin{pmatrix} \widehat{\boldsymbol{\Delta}}_1^{-1} & \boldsymbol{O} & \cdots & \boldsymbol{O} \\ \boldsymbol{O} & \widehat{\boldsymbol{\Delta}}_2^{-1} & \cdots & \boldsymbol{O} \\ \vdots & \vdots & \ddots & \vdots \\ \boldsymbol{O} & \boldsymbol{O} & \cdots & \widehat{\boldsymbol{\Delta}}_H^{-1} \end{pmatrix}.$$

4.2. Computation of Reducible $\boldsymbol{Q}$

In this subsection, we consider the computation of $\boldsymbol{Q}$ for reducible $\boldsymbol{C}+\boldsymbol{D}$. As mentioned in Remark 2, $\boldsymbol{Q}$ can be computed based on the recursion (11). However, a straightforward implementation of the computational procedure given in Remark 2 is not efficient for reducible $\boldsymbol{C}+\boldsymbol{D}$ because $\boldsymbol{Q}^{(n)}$ is a sparse block matrix and the number of iterations required is determined by the most slowly convergent sequence among non-zero blocks. Therefore, we can avoid unnecessary calculations by computing $\boldsymbol{Q}$ blockwise as follows.

It is readily to see from (11) that $\boldsymbol{Q}_h$ $(h \in \mathcal{H})$ is given by the limit $\lim_{n\to\infty} \boldsymbol{Q}_h^{(n)}$ of the elementwise increasing sequence of matrices $\boldsymbol{Q}_h^{(n)}$ $(n = 0, 1, \ldots)$ defined as

$$\boldsymbol{Q}_h^{(0)} = \boldsymbol{C}_h, \qquad \boldsymbol{Q}_h^{(n)} = \boldsymbol{C}_h + \int_0^\infty d\boldsymbol{D}_h(y) \exp\big(\boldsymbol{Q}_h^{(n-1)} y\big), \quad n = 1, 2, \ldots. \tag{34}$$

Because $\boldsymbol{Q}_h$ $(h \in \mathcal{H})$ represents an infinitesimal generator and $\boldsymbol{Q}_h \boldsymbol{e}_h = \boldsymbol{0}$ holds, it can be computed individually with an adequate stopping criterion in the same way as the computation of $\boldsymbol{Q}$ with (11).

Similarly, $\boldsymbol{Q}_\mathrm{T}$ is given by the limit $\lim_{n\to\infty} \boldsymbol{Q}_\mathrm{T}^{(n)}$ of the elementwise increasing sequence of matrices $\boldsymbol{Q}_\mathrm{T}^{(n)}$ $(n = 0, 1, \ldots)$ defined as

$$\boldsymbol{Q}_\mathrm{T}^{(0)} = \boldsymbol{C}_\mathrm{T}, \qquad \boldsymbol{Q}_\mathrm{T}^{(n)} = \boldsymbol{C}_\mathrm{T} + \int_0^\infty d\boldsymbol{D}_\mathrm{T}(y) \exp\big(\boldsymbol{Q}_\mathrm{T}^{(n-1)} y\big), \quad n = 1, 2, \ldots.$$

However, because $\boldsymbol{Q}_\mathrm{T}$ represents a defective infinitesimal generator, the stopping criterion of $\boldsymbol{Q}_\mathrm{T}^{(n)}$ is not clear. We thus need to compute $\boldsymbol{Q}_\mathrm{T}$ along with $\boldsymbol{Q}_{\mathrm{T},h}$ $(h \in \mathcal{H})$. Let $\widehat{\boldsymbol{Q}}_{\mathrm{T},h}^{(n)}$ $(n = 0, 1, \ldots)$ denote a sequence of matrices defined as

$$\begin{aligned}
\widehat{\boldsymbol{Q}}_{\mathrm{T},h}^{(0)} &= \boldsymbol{C}_{\mathrm{T},h}, \\
\widehat{\boldsymbol{Q}}_{\mathrm{T},h}^{(n)} &= \boldsymbol{C}_{\mathrm{T},h} + \int_0^\infty d\boldsymbol{D}_{\mathrm{T},h}(y) \exp\big(\boldsymbol{Q}_h y\big) \\
&\quad + \int_0^\infty d\boldsymbol{D}_\mathrm{T}(y) \int_0^y \exp\big(\boldsymbol{Q}_\mathrm{T}^{(n-1)} t\big) \widehat{\boldsymbol{Q}}_{\mathrm{T},h}^{(n-1)} \exp\big(\boldsymbol{Q}_h (y-t)\big) dt, \quad n = 1, 2, \ldots.
\end{aligned} \tag{35}$$

According to the probabilistic interpretation of $\boldsymbol{Q}_\mathrm{T}^{(n)}$ [3], it can be verified that $\lim_{n\to\infty} \widehat{\boldsymbol{Q}}_{\mathrm{T},h}^{(n)} = \boldsymbol{Q}_{\mathrm{T},h}$ for $h \in \mathcal{H}$. Therefore, we first compute $\boldsymbol{Q}_h$ $(h \in \mathcal{H})$ with (34), and then we compute $\boldsymbol{Q}_\mathrm{T}$ and $\boldsymbol{Q}_{\mathrm{T},h}$ with an adequate stopping criterion. More specifically, for a given allowable error $\epsilon > 0$, we may stop the iteration at n^* satisfying

$$\max_{i \in \mathcal{M}_\mathrm{T}} \Big| \big[\boldsymbol{Q}_\mathrm{T}^{(n^*)} \boldsymbol{e}_\mathrm{T} + \sum_{h \in \mathcal{H}} \widehat{\boldsymbol{Q}}_{\mathrm{T},h}^{(n^*)} \boldsymbol{e}_h\big]_i \Big| < \epsilon.$$

Remark 9. *The second integral on the right-hand side of (35) can be computed with uniformization as follows. Let θ denote the maximum absolute value of the diagonal elements of the matrix $\boldsymbol{C}$. We then have*

$$\begin{aligned}
&\int_0^\infty d\boldsymbol{D}_\mathrm{T}(y) \int_0^y \exp(\boldsymbol{Q}_\mathrm{T}^{(n)} t) \widehat{\boldsymbol{Q}}_{\mathrm{T},h}^{(n)} \exp\big(\boldsymbol{Q}_h (y-t)\big) dt \\
&\qquad = \sum_{m=0}^\infty \boldsymbol{D}_\mathrm{T}^{(m+1)}(\theta) \sum_{j=0}^m [\boldsymbol{I}_\mathrm{T} + \theta^{-1} \boldsymbol{Q}_\mathrm{T}^{(n)}]^{m-j} \theta^{-1} \widehat{\boldsymbol{Q}}_{\mathrm{T},h}^{(n)} [\boldsymbol{I}_h + \theta^{-1} \boldsymbol{Q}_h]^j,
\end{aligned} \tag{36}$$

where

$$\boldsymbol{D}_{\mathrm{T}}^{(m)}(\theta) = \int_0^\infty \exp(-\theta x) \frac{(\theta x)^m}{m!} d\boldsymbol{D}_{\mathrm{T}}(x).$$

The derivation of (36) is given in Appendix C.

Remark 10. *If $\boldsymbol{C} + \boldsymbol{D}$ has no transient states, $\boldsymbol{Q}$ is given by*

$$\begin{pmatrix} \boldsymbol{Q}_1 & \boldsymbol{O} & \cdots & \boldsymbol{O} \\ \boldsymbol{O} & \boldsymbol{Q}_2 & \cdots & \boldsymbol{O} \\ \vdots & \vdots & \ddots & \vdots \\ \boldsymbol{O} & \boldsymbol{O} & \cdots & \boldsymbol{Q}_H \end{pmatrix},$$

so that we only need to compute $\boldsymbol{Q}_h$ ($h \in \mathcal{H}$) based on (34).

4.3. Moments of the Stationary Distribution

In this subsection, we derive a recursive formula for the moments of the stationary distribution. For this purpose, we introduce some notations. We first rewrite (31) to be

$$\boldsymbol{u}^*(s)[s\boldsymbol{I} + \boldsymbol{C} + \boldsymbol{D}^*(s)] = -\boldsymbol{\eta}^*(s) \qquad \mathrm{Re}(s) > 0, \tag{37}$$

where $\boldsymbol{\eta}^*(s)$ is given by

$$\boldsymbol{\eta}^*(s) = c\boldsymbol{\eta}^{\mathrm{E}}[\overline{\boldsymbol{B}}^*(s) - \boldsymbol{I}].$$

Let $\boldsymbol{u}^{(m)}$ ($m = 0, 1, \ldots$) and $\boldsymbol{\eta}^{(m)}$ ($m = 0, 1, \ldots$) denote $1 \times M$ vectors given by

$$\boldsymbol{u}^{(0)} = \lim_{s \to 0+} \boldsymbol{u}^*(s), \qquad \boldsymbol{u}^{(m)} = \lim_{s \to 0+} \frac{(-1)^m}{m!} \cdot \frac{d^m}{ds^m}\big[\boldsymbol{u}^*(s)\big], \quad m = 1, 2, \ldots,$$
$$\boldsymbol{\eta}^{(0)} = \lim_{s \to 0+} \boldsymbol{\eta}^*(s), \qquad \boldsymbol{\eta}^{(m)} = \lim_{s \to 0+} \frac{(-1)^m}{m!} \cdot \frac{d^m}{ds^m}\big[\boldsymbol{\eta}^*(s)\big], \quad m = 1, 2, \ldots.$$

Note that the jth ($j \in \mathcal{M}$) element of $\boldsymbol{u}^{(0)}$ represents the stationary probability that the phase is equal to j.

We develop a recursion to compute $\boldsymbol{u}^{(m)}$ ($m = 0, 1, \ldots$) utilizing the fact that $\boldsymbol{C}$ and $\boldsymbol{D}^*(s)$ are sparse block matrices. We thus partition $\boldsymbol{u}^*(s)$, $\boldsymbol{\eta}^*(s)$, $\boldsymbol{u}^{(m)}$ ($m = 0, 1, \ldots$), and $\boldsymbol{\eta}^{(m)}$ ($m = 0, 1, \ldots$) as follows.

$$\boldsymbol{u}^*(s) = (\boldsymbol{u}_{\mathrm{T}}^*(s), \boldsymbol{u}_1^*(s), \boldsymbol{u}_2^*(s), \ldots, \boldsymbol{u}_H^*(s)), \qquad \boldsymbol{\eta}^*(s) = (\boldsymbol{\eta}_{\mathrm{T}}^*(s), \boldsymbol{\eta}_1^*(s), \boldsymbol{\eta}_2^*(s), \ldots, \boldsymbol{\eta}_H^*(s)),$$
$$\boldsymbol{u}^{(m)} = (\boldsymbol{u}_{\mathrm{T}}^{(m)}, \boldsymbol{u}_1^{(m)}, \boldsymbol{u}_2^{(m)}, \ldots, \boldsymbol{u}_H^{(m)}), \qquad \boldsymbol{\eta}^{(m)} = (\boldsymbol{\eta}_{\mathrm{T}}^{(m)}, \boldsymbol{\eta}_1^{(m)}, \boldsymbol{\eta}_2^{(m)}, \ldots, \boldsymbol{\eta}_H^{(m)}),$$

where $\boldsymbol{u}_{\mathrm{T}}^*(s)$, $\boldsymbol{\eta}_{\mathrm{T}}^*(s)$, $\boldsymbol{u}_{\mathrm{T}}^{(m)}$, and $\boldsymbol{\eta}_{\mathrm{T}}^{(m)}$ denote $1 \times M_{\mathrm{T}}$ vectors and $\boldsymbol{u}_h^*(s)$, $\boldsymbol{\eta}_h^*(s)$, $\boldsymbol{u}_h^{(m)}$, and $\boldsymbol{\eta}_h^{(m)}$ ($h \in \mathcal{H}$) denote $1 \times M_h$ vectors.

Note that (37) is equivalent to

$$\boldsymbol{u}_{\mathrm{T}}^*(s)[s\boldsymbol{I}_{\mathrm{T}} + \boldsymbol{C}_{\mathrm{T}} + \boldsymbol{D}_{\mathrm{T}}^*(s)] = -\boldsymbol{\eta}_{\mathrm{T}}^*(s), \tag{38}$$
$$\boldsymbol{u}_h^*(s)[s\boldsymbol{I}_h + \boldsymbol{C}_h + \boldsymbol{D}_h^*(s)] = -\boldsymbol{\phi}_h^*(s), \quad h \in \mathcal{H}, \tag{39}$$

where

$$\boldsymbol{\phi}_h^*(s) = \boldsymbol{\eta}_h^*(s) + \boldsymbol{u}_{\mathrm{T}}^*(s)[\boldsymbol{C}_{\mathrm{T},h} + \boldsymbol{D}_{\mathrm{T},h}^*(s)], \quad h \in \mathcal{H}.$$

We define $\boldsymbol{\phi}_h^{(m)}$ ($h \in \mathcal{H}$, $m = 0, 1, \ldots$) as

$$\boldsymbol{\phi}_h^{(0)} = \lim_{s \to 0+} \boldsymbol{\phi}_h^*(s) = \boldsymbol{\eta}_h^{(0)} + \boldsymbol{u}_{\mathrm{T}}^{(0)}(\boldsymbol{C}_{\mathrm{T},h} + \boldsymbol{D}_{\mathrm{T},h}),$$

$$\boldsymbol{\phi}_h^{(m)} = \lim_{s\to 0+} \frac{(-1)^m}{m!} \cdot \frac{d^m}{ds^m}\left[\boldsymbol{\phi}_h^*(s)\right] = \boldsymbol{\eta}_h^{(m)} + \boldsymbol{u}_{\mathrm{T}}^{(m)}\boldsymbol{C}_{\mathrm{T},h} + \sum_{l=0}^{m} \boldsymbol{u}_{\mathrm{T}}^{(l)}\boldsymbol{D}_{\mathrm{T},h}^{(m-l)}, \quad m = 1,2,\ldots,$$

where for $h \in \mathcal{H}$,

$$\boldsymbol{D}_{\mathrm{T},h}^{(0)} = \boldsymbol{D}_{\mathrm{T},h}, \qquad \boldsymbol{D}_{\mathrm{T},h}^{(m)} = \lim_{s\to 0+} \frac{(-1)^m}{m!} \cdot \frac{d^m}{ds^m}\left[\boldsymbol{D}_{\mathrm{T},h}^*(s)\right], \quad m = 1,2,\ldots.$$

We also define $\boldsymbol{D}_{\mathrm{T}}^{(m)}$ $(m = 0,1,\ldots)$ and $\boldsymbol{D}_h^{(m)}$ $(h \in \mathcal{H},\, m = 0,1,\ldots)$ as

$$\boldsymbol{D}_{\mathrm{T}}^{(0)} = \boldsymbol{D}_{\mathrm{T}}, \qquad \boldsymbol{D}_{\mathrm{T}}^{(m)} = \lim_{s\to 0+} \frac{(-1)^m}{m!} \cdot \frac{d^m}{ds^m}\left[\boldsymbol{D}_{\mathrm{T}}^*(s)\right], \quad m = 1,2,\ldots,$$

$$\boldsymbol{D}_h^{(0)} = \boldsymbol{D}_h, \qquad \boldsymbol{D}_h^{(m)} = \lim_{s\to 0+} \frac{(-1)^m}{m!} \cdot \frac{d^m}{ds^m}\left[\boldsymbol{D}_h^*(s)\right], \quad m = 1,2,\ldots.$$

Theorem 3. $\boldsymbol{u}^{(m)} = (\boldsymbol{u}_{\mathrm{T}}^{(m)}, \boldsymbol{u}_1^{(m)}, \boldsymbol{u}_2^{(m)}, \ldots, \boldsymbol{u}_H^{(m)})$ *$(m = 0,1,\ldots)$ is given recursively by*

$$\boldsymbol{u}_{\mathrm{T}}^{(0)} = \boldsymbol{\eta}_{\mathrm{T}}^{(0)}[-(\boldsymbol{C}_{\mathrm{T}} + \boldsymbol{D}_{\mathrm{T}})]^{-1},$$

$$\boldsymbol{u}_{\mathrm{T}}^{(m)} = \left[\boldsymbol{\eta}_{\mathrm{T}}^{(m)} - \boldsymbol{u}_{\mathrm{T}}^{(m-1)} + \sum_{l=0}^{m-1} \boldsymbol{u}_{\mathrm{T}}^{(l)}\boldsymbol{D}_{\mathrm{T}}^{(m-l)}\right][-(\boldsymbol{C}_{\mathrm{T}} + \boldsymbol{D}_{\mathrm{T}})]^{-1}, \quad m = 1,2,\ldots,$$

and for $h \in \mathcal{H}$,

$$\boldsymbol{u}_h^{(0)}\boldsymbol{e}_h = \frac{1}{1 - \boldsymbol{\pi}_h\boldsymbol{\beta}_h}\left[\boldsymbol{\phi}_h^{(1)}\boldsymbol{e}_h + \boldsymbol{\phi}_h^{(0)}(\boldsymbol{e}_h\boldsymbol{\pi}_h - \boldsymbol{C}_h - \boldsymbol{D}_h)^{-1}\boldsymbol{\beta}_h\right], \tag{40}$$

$$\boldsymbol{u}_h^{(0)} = \boldsymbol{u}_h^{(0)}\boldsymbol{e}_h\boldsymbol{\pi}_h + \boldsymbol{\phi}_h^{(0)}(\boldsymbol{e}_h\boldsymbol{\pi}_h - \boldsymbol{C}_h - \boldsymbol{D}_h)^{-1}, \tag{41}$$

$$\boldsymbol{\psi}_h^{(m)} = \left(\sum_{l=0}^{m-1} \boldsymbol{u}_h^{(l)}\boldsymbol{D}_h^{(m-l)} - \boldsymbol{u}_h^{(m-1)} + \boldsymbol{\phi}_h^{(m)}\right)(\boldsymbol{e}_h\boldsymbol{\pi}_h - \boldsymbol{C}_h - \boldsymbol{D}_h)^{-1} \quad m = 1,2,\ldots, \tag{42}$$

$$\boldsymbol{u}_h^{(m)}\boldsymbol{e}_h = \frac{1}{1 - \boldsymbol{\pi}_h\boldsymbol{\beta}_h}\left[\sum_{l=0}^{m-1} \boldsymbol{u}_h^{(l)}\boldsymbol{D}_h^{(m+1-l)}\boldsymbol{e}_h + \boldsymbol{\phi}_h^{(m+1)}\boldsymbol{e}_h + \boldsymbol{\psi}_h^{(m)}\boldsymbol{\beta}_h\right], \quad m = 1,2,\ldots, \tag{43}$$

$$\boldsymbol{u}_h^{(m)} = \boldsymbol{u}_h^{(m)}\boldsymbol{e}_h\boldsymbol{\pi}_h + \boldsymbol{\psi}_h^{(m)}, \quad m = 1,2,\ldots. \tag{44}$$

Proof. We first consider $\boldsymbol{u}_{\mathrm{T}}^{(m)}$ $(m = 0,1,\ldots)$. It follows from (38) that

$$\boldsymbol{u}_T^{(0)}(\boldsymbol{C}_{\mathrm{T}} + \boldsymbol{D}_{\mathrm{T}}) = -\boldsymbol{\eta}_T^{(0)},$$

$$\boldsymbol{u}_T^{(m)}(\boldsymbol{C}_{\mathrm{T}} + \boldsymbol{D}_{\mathrm{T}}) - \boldsymbol{u}_T^{(m-1)} + \sum_{l=0}^{m-1} \boldsymbol{u}_T^{(l)}\boldsymbol{D}_{\mathrm{T}}^{(m-l)} = -\boldsymbol{\eta}_T^{(m)}, \quad m = 1,2,\ldots.$$

Since $\boldsymbol{C}_{\mathrm{T}} + \boldsymbol{D}_{\mathrm{T}}$ is a defective infinitesimal generator, it is non-singular. We thus obtain the recursion for $\boldsymbol{u}_{\mathrm{T}}^{(m)}$ $(m = 0,1,\ldots)$ from the above equations.

Next we consider $\boldsymbol{u}_h^{(m)}$ $(h \in \mathcal{H},\, m = 0,1,\ldots)$ based on (39). Since $\boldsymbol{C}_h + \boldsymbol{D}_h$ $(h \in \mathcal{H})$ is an irreducible infinitesimal generator, the recursion for $\boldsymbol{u}_h^{(m)}$ $(h \in \mathcal{H},\, m = 0,1,\ldots)$ can be obtained by standard manipulations in matrix-analytic methods (e.g., [11]), and therefore we omit the proof. □

Remark 11. *By definition of $\boldsymbol{\phi}_h$ $(h \in \mathcal{H})$, we need to compute $\boldsymbol{u}_{\mathrm{T}}^{(m+1)}$ before computing $\boldsymbol{u}_h^{(m)}$ by (40)–(44). When $\boldsymbol{C} + \boldsymbol{D}$ has no transient states, on the other hand, $\boldsymbol{u}_h^{(m)}$ $(h \in \mathcal{H})$ can be immediately obtained from (40)–(44) noting*

$$\boldsymbol{\phi}_h^{(m)} = \boldsymbol{\eta}_h^{(m)}, \qquad m = 0,1,\ldots.$$

5. Conclusion

We extended the matrix analytic methods for bivariate Markov process $\{(U(t), S(t));\, t \ge 0\}$ introduced in [7] to the case of reducible $\boldsymbol{C}+\boldsymbol{D}$. We first proved Lemma 3, which implies that some known results for the boundary vector $\acute{\boldsymbol{u}}(0)$ of the stationary distribution is not valid for reducible $\boldsymbol{C}+\boldsymbol{D}$, when there exist more than one irreducible classes of states. We then derived a formula for the LST of the stationary distribution applicable to the reducible $\boldsymbol{C}+\boldsymbol{D}$ in Section 4.1. Furthermore, we provided an efficient computational procedure of the fundamental matrix $\boldsymbol{Q}$ and the moments of the stationary distribution.

Recall that the Markov process considered in this paper corresponds to the (censored) workload processes in MAP/G/1 queues with various features (see Examples 1-A, 1-B, 2-A, and 2-B in Section 2). Based on the results in this paper, we can obtain performance measures of the corresponding queueing model such as the waiting time distribution and the queue length distribution in a straightforward manner by following the discussion in [7] and [8] for an ordinary MAP/G/1 queue.

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A. Non-Singularity of $\boldsymbol{I}-\boldsymbol{R}^*(s)$ for Reducible $\boldsymbol{C}+\boldsymbol{D}$

In this appendix we provide a brief proof that $\boldsymbol{I}-\boldsymbol{R}^*(s)$ ($\mathrm{Re}(s)>0$) is non-singular for reducible $\boldsymbol{C}+\boldsymbol{D}$, as is the case of irreducible $\boldsymbol{C}+\boldsymbol{D}$. Note first that $\boldsymbol{R}^*(s)$ takes the form

$$\boldsymbol{R}^*(s) = \begin{pmatrix} \boldsymbol{R}^*_{\mathrm{T}}(s) & \boldsymbol{R}^*_{\mathrm{T},1}(s) & \boldsymbol{R}^*_{\mathrm{T},2}(s) & \cdots & \boldsymbol{R}^*_{\mathrm{T},H}(s) \\ \boldsymbol{O} & \boldsymbol{R}^*_1(s) & \boldsymbol{O} & \cdots & \boldsymbol{O} \\ \boldsymbol{O} & \boldsymbol{O} & \boldsymbol{R}^*_2(s) & \cdots & \boldsymbol{O} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \boldsymbol{O} & \boldsymbol{O} & \boldsymbol{O} & \cdots & \boldsymbol{R}^*_H(s) \end{pmatrix},$$

and its diagonal block matrices are given by

$$\boldsymbol{R}^*_{\mathrm{T}}(s) = \int_0^\infty \exp(-sx) \int_x^\infty d\boldsymbol{D}_{\mathrm{T}}(y) \exp(\boldsymbol{Q}_{\mathrm{T}}(y-x)),$$
$$\boldsymbol{R}^*_h(s) = \int_0^\infty \exp(-sx) \int_x^\infty d\boldsymbol{D}_h(y) \exp(\boldsymbol{Q}_h(y-x)), \qquad h \in \mathcal{H}.$$

Since $\boldsymbol{C}_h+\boldsymbol{D}_h$ ($h \in \mathcal{H}$) is irreducible, we can verify that $\boldsymbol{I}_h-\boldsymbol{R}^*_h(s)$ ($\mathrm{Re}(s)>0$) is non-singular in the same way as in [9]. Furthermore, noting that $\boldsymbol{C}_{\mathrm{T}}+\boldsymbol{D}_{\mathrm{T}}$ denotes a defective infinitesimal generator, we can prove that $\boldsymbol{I}_{\mathrm{T}}-\boldsymbol{R}^*_{\mathrm{T}}(s)$ ($\mathrm{Re}(s)>0$) is non-singular in the same way as in [3, Appendix I]. Therefore, $\boldsymbol{I}-\boldsymbol{R}^*(s)$ ($\mathrm{Re}(s)>0$) is non-singular because its diagonal block matrices are all non-singular. □

B. Proof of Lemma 3

We first consider the case that $\boldsymbol{Y}$ has two irreducible classes of states and no transient states, i.e.,

$$\boldsymbol{Y} = \begin{pmatrix} \boldsymbol{Y}_1 & \boldsymbol{O} \\ \boldsymbol{O} & \boldsymbol{Y}_2 \end{pmatrix},$$

where $\boldsymbol{Y}_1$ and $\boldsymbol{Y}_2$ denote irreducible infinitesimal generators. (26) is then rewritten to be

$$\boldsymbol{v} = \begin{pmatrix} \boldsymbol{v}_1 \\ \boldsymbol{v}_2 \end{pmatrix}, \qquad \boldsymbol{\gamma}_h \boldsymbol{v}_h \neq 0 \quad \text{for some } h \in \{1,2\}.$$

Without loss of generality, we assume that $\boldsymbol{\gamma}_2 \boldsymbol{v}_2 \neq \boldsymbol{0}$. We then define a $1 \times M$ vector $\boldsymbol{y}$ as

$$\boldsymbol{y} = \left(\boldsymbol{\gamma}_1,\ \frac{-\boldsymbol{\gamma}_1 \boldsymbol{v}_1}{\boldsymbol{\gamma}_2 \boldsymbol{v}_2} \cdot \boldsymbol{\gamma}_2 \right)$$

It then follows that

$$\boldsymbol{y}\boldsymbol{v} = \boldsymbol{\gamma}_1 \boldsymbol{v}_1 - \frac{\boldsymbol{\gamma}_1 \boldsymbol{v}_1}{\boldsymbol{\gamma}_2 \boldsymbol{v}_2} \cdot \boldsymbol{\gamma}_2 \boldsymbol{v}_2 = 0.$$

Therefore we have

$$\boldsymbol{y}(\boldsymbol{v}\boldsymbol{\alpha} - \boldsymbol{Y}) = \boldsymbol{0}.$$

Since $\boldsymbol{y} \neq \boldsymbol{0}$, $\boldsymbol{v}\boldsymbol{\alpha} - \boldsymbol{Y}$ is singular.

In the exactly same way, we can easily verify that $\boldsymbol{v}\boldsymbol{\alpha} - \boldsymbol{Y}$ is still singular for the general case that $\boldsymbol{Y}$ has transient states and more than two irreducible classes. □

C. Derivation of (36)

With uniformization at rate θ, we have

$$\begin{aligned}
&\int_0^\infty d\boldsymbol{D}_{\mathrm{T}}(y) \int_0^y \exp(\boldsymbol{Q}_{\mathrm{T}}^{(n)} t) \boldsymbol{Q}_{\mathrm{T},h}^{(n)} \exp\big(\boldsymbol{Q}_h (y-t)\big) dt \\
&\quad = \int_0^\infty d\boldsymbol{D}_{\mathrm{T}}(y) \int_0^y \exp(-\theta t) \exp\big(\theta(\boldsymbol{I}_{\mathrm{T}} + \theta^{-1}\boldsymbol{Q}_{\mathrm{T}}^{(n)})t\big) \boldsymbol{Q}_{\mathrm{T},h}^{(n)} \\
&\qquad\qquad \cdot \exp(-\theta(y-t)) \exp\big(\theta(\boldsymbol{I}_h + \theta^{-1}\boldsymbol{Q}_h)(y-t)\big) dt \\
&\quad = \int_0^\infty \exp(-\theta y) d\boldsymbol{D}_{\mathrm{T}}(y) \int_0^y \sum_{i=0}^\infty \frac{\theta^i t^i}{i!} [\boldsymbol{I}_{\mathrm{T}} + \theta^{-1}\boldsymbol{Q}_{\mathrm{T}}^{(n)}]^i \boldsymbol{Q}_{\mathrm{T},h}^{(n)} \cdot \sum_{j=0}^\infty \frac{\theta^j (y-t)^j}{j!} [\boldsymbol{I}_h + \theta^{-1}\boldsymbol{Q}_h]^j dt \\
&\quad = \sum_{m=0}^\infty \sum_{j=0}^m \int_0^\infty \exp(-\theta y) d\boldsymbol{D}_{\mathrm{T}}(y) \int_0^y \frac{\theta^{m-j} t^{m-j}}{(m-j)!} \cdot \frac{\theta^j (y-t)^j}{j!} dt \\
&\qquad\qquad \cdot [\boldsymbol{I}_{\mathrm{T}} + \theta^{-1}\boldsymbol{Q}_{\mathrm{T}}^{(n)}]^{m-j} \boldsymbol{Q}_{\mathrm{T},h}^{(n)} [\boldsymbol{I}_h + \theta^{-1}\boldsymbol{Q}_h]^j,
\end{aligned}$$

where $m = i + j$. Furthermore, calculating the integral with respect to t using

$$\int_0^y t^{m-j} (y-t)^j dt = \frac{j!(m-j)!}{(m+1)!} \cdot y^{m+1},$$

we obtain

$$\begin{aligned}
&\int_0^\infty d\boldsymbol{D}_{\mathrm{T}}(y) \int_0^y \exp(\boldsymbol{Q}_{\mathrm{T}}^{(n)} t) \boldsymbol{Q}_{\mathrm{T},h}^{(n)} \exp\big(\boldsymbol{Q}_h (y-t)\big) dt \\
&\quad = \sum_{m=0}^\infty \sum_{j=0}^m \int_0^\infty \exp(-\theta y) \frac{(\theta y)^{m+1}}{(m+1)!} d\boldsymbol{D}_{\mathrm{T}}(y) \cdot [\boldsymbol{I}_{\mathrm{T}} + \theta^{-1}\boldsymbol{Q}_{\mathrm{T}}^{(n)}]^{m-j} \theta^{-1} \boldsymbol{Q}_{\mathrm{T},h}^{(n)} [\boldsymbol{I}_h + \theta^{-1}\boldsymbol{Q}_h]^j.
\end{aligned}$$

(36) now follows immediately.

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Yoshiaki Inoue
Graduate School of Engineering, Osaka University
2-1 Yamadaoka, Suita, Osaka 565-0871, Japan
E-mail: `y-inoue@post.comm.eng.osaka-u.ac.jp`