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SEARCH GAMES: LITERATURE AND SURVEY

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Abstract The purpose of this paper is to review literature published to date on search games, almost all of which have originated from *search theory*. Search theory itself is a research field of *operations research* that arose from realistic research efforts into air defense operations in England during World War II while the concept of search theory originated in anti-submarine warfare operations conducted by the U.S. Navy against German U-boats during the same war. The search game is an application of game theory to search problems in search theory. Search games have two main players: searchers and targets.

There are some models in the search game: binary search game, linear search game, hide-search game, hide-allocation game, evasion-search game, princess-monster game, ambush game, search allocation game, path-constrained search game and search-search game. We fully survey literature on these models. We also outline other games which are closely related to the search game. Search theory has been evolving to be not method-oriented but problem-oriented for practical applications. Therefore we focus on the description of the discovery of problems and their modeling in this paper while we seldom mention the methodologies the authors devised for their search problems.

Keywords: Search, game theory, survey

1. Introduction

The purpose of this paper is to review literature published to date on search games, almost all of which are known to have originated from the field of *search theory*. Search theory itself is a research field of *operations research* (OR) that arose from realistic research efforts into air defense operations in England during World War II, while the concept of search theory originated in anti-submarine warfare (ASW) operations conducted by the U.S. Navy against German U-boats during the same war. The search game is an application of game theory, which was developed by Von Neumann (1944) [233], to problems in search theory. Here, it should be noted that while OR and game theory became popular around the world after WWII, we can deduce that researchers and practitioners exchanged information and knowledge during wartime from the fact that Morse and Kimball applied game theory to the analysis on the passage control of submarines by ASW aircraft in restricted waters in their well-known book *Methods of Operations Research* (1951) [178], which is the most famous OR textbook.

From the history mentioned above, it can be seen that search theory focuses on realistic problems encountered by searchers attempting to locate targets. In fact, search theory can be defined as the study of finding effective ways to search for targets in the search activities that take place between searchers and targets. In this sense, it may be helpful to think of the search problems that look for optimal solutions in optimization problems as examples of search theory, but these will be excluded from our review because the problems are not directly descended from search theory in a historical context. Classically, search problems have two main players: searchers and targets. Therefore, before starting our review of search game literature, we will begin with an overview of the historical developments of search theory.

1.1. Search theory development and search game classifications

Koopman first showed us the OR methodologies developed for ASW operations undertaken by the U.S. Navy in *Search and Screening* (1946) [156]. This book marks the beginning of academic studies into search theory. *Screening* in the title refers to using naval ships and aircraft to protect convoys against submarine attacks. The details of these academic discussions on search theory are found in various papers [158–160] published in *Operations Research*, the journal of the Operations Research Society of America, then. More specifically, he discussed the kinematic search problems of searchers and targets in maritime environments in Koopman (1956) [158], the basics of sensor-based target detection in (1956) [159], and optimal distributions of search efforts, such as search times and forces, in (1957) [160]. While these remain the primary themes in search theory, the third theme has attracted the most attention in papers published in OR-related journals.

Now, let us review the variational problem Koopman proposed in the third reference. Let X be a random variable indicating the real-valued position of a target on a one-dimensional line $\boldsymbol{R}$ and $p(x)$ be the probability density function of X. If a searcher distributes his or her search effort based on density $\varphi(x)$ at position $x \in \boldsymbol{R}$, he or she can detect the target with a probability $1 - \exp(-\varphi(x))$. Thus, what is the most effective distribution of a total amount Φ of effort for the detection of the target? An optimal distribution that maximizes the detection probability is given by an optimal solution $\{\varphi^*(x), x \in \boldsymbol{R}\}$ of the following variational problem:

$$
\begin{aligned}
(K_c) \qquad & \max_{\{\varphi(x)\}} \int_{-\infty}^{\infty} p(x)\left\{1-\exp(-\varphi(x))\right\} dx \\
s.t. \qquad & \int_{-\infty}^{\infty} \varphi(x)dx = \Phi,\ \varphi(x) \geq 0,\ x \in \boldsymbol{R}. \qquad (1.1)
\end{aligned}
$$

According to Ibaraki and Katoh (1988) [139], this problem, which is named *Koopman's problem*, is regarded as an original problem in the more general research field of *optimal resource allocation*. If we redefine Koopman's problem in a discrete search space consisting of n cells $\boldsymbol{K} = \{1, \cdots, n\}$, we have the following formulation:

$$
\begin{aligned}
(K_d) \qquad & \max_{\{\varphi_i\}} \sum_{i=1}^{n} p_i\left\{1-\exp(-\varphi_i)\right\} \\
s.t. \qquad & \sum_{i=1}^{n} \varphi_i = \Phi,\ \varphi_i \geq 0,\ i = 1, \cdots, n, \qquad (1.2)
\end{aligned}
$$

where $\{p_i, i \in \boldsymbol{K}\}$ is the probability distribution of the target over the cells and φ_i is the amount of search effort distributed into cell i by the searcher. From the Euler-Lagrange equation for problem (K_c) and the method of Lagrange multipliers for problem (K_d), we first find an optimal multiplier λ^* by solving the respective equations $\int \log(p(x)/\lambda)dx = \Phi$ and $\sum_i \log(p_i/\lambda) = \Phi$, and lastly find an optimal distribution of the search effort by substituting the multiplier into $\varphi^*(x) = \log(p(x)/\lambda^*)$ or $\varphi_i^* = \log(p_i/\lambda^*)$.

As in these problems, during the early history of search theory, researchers discussed the effective distribution of search resources after determining the probability distribution of a stationary target. De Guenin (1961) [68] and Kadane (1968) [146] generalized and extended Koopman's problem in a continuous search space and a discrete space, respectively, while Stone gathered excellent theoretical results for search problems with a stationary target in his book *Theory of Optimal Search* (1969) [224] and won the Lanchester Prize. As an objective function in the stationary-target problem, these researchers adopted the risk criterion of combining target value and search costs, and whereabouts probability defined as a target detection probability plus a true guess probability of the target position following the search operation as well as the detection probability.

In the 1970s, the stationary-target problem was extended to search problems of moving targets, and an additional element, time, was explicitly embedded into search models. In terms of target motion, researchers created various movement types that were appropriate to particular search problems. These include a target path that changes its geographic position as time goes by, a Markov movement in which the next position depends solely on the present position, and a deterministic motion that is a kind of fixed target path produced when randomized parameters are determined before the motion.

Now, let's take the target path as an example. Here, we will denote a finite discrete geographic space by $\boldsymbol{K}$ and a finite discrete time space by $\boldsymbol{T}$. If a target takes path ω from among a finite number of paths Ω, it is assumed to be at cell $\omega(t) \in \boldsymbol{K}$ at time $t \in \boldsymbol{T}$. A searcher has the total amount of arbitrary-divisible resources $\Phi(t)$ available at each time $t \in \boldsymbol{T}$ and makes a distribution plan for his or her search resources while knowing the probability distribution of the target path $\pi = \{\pi(\omega), \omega \in \Omega\}$, where $\pi(\omega)$ is the probability of choosing a path ω. Next, we will denote a searcher's resource distribution by $\varphi = \{\varphi(i,t), i \in \boldsymbol{K}, t \in \boldsymbol{T}\}$, where $\varphi(i,t)$ is the amount of resources distributed in cell i at time t. Assume that for detection of the target at cell i at time t, only the resources $\varphi(i,t)$ distributed just-in-time are effective and the detection probability is given by $1 - \exp(-\alpha_i \varphi(i,t))$, where parameter α_i indicates the effectiveness of unit resource in cell i for detection. Since the target on path ω is at cell $\omega(t)$ at time t, a searcher having a plan φ detects the target on the path ω with probability $1 - \exp(-\alpha_{\omega(t)} \varphi(\omega(t), t))$ at time t. Since the searcher detects the target when the detection occurs at most once at a time, we have $1 - \exp(-\sum_{t \in T} \alpha_{\omega(t)} \varphi(\omega(t), t))$ as the total detection probability. Therefore, we formulate an optimal search resource distribution problem with detection probability as its objective, as follows:

$$
\begin{aligned}
(P) \qquad & \max_{\{\varphi\}} \sum_{\omega \in \Omega} \pi(\omega) \left\{ 1 - \exp(-\sum_{t \in T} \alpha_{\omega(t)} \varphi(\omega(t), t)) \right\} \\
s.t. \qquad & \sum_{i \in \boldsymbol{K}} \varphi(i,t) \leq \Phi(t), \; t \in \boldsymbol{T}, \quad \varphi(i,t) \geq 0, \; i \in \boldsymbol{K}, t \in \boldsymbol{T}.
\end{aligned} \tag{1.3}
$$

Pollock (1970) [196] and Dobbie (1974) [71] discussed optimal search problems for a target moving between two cells in an analytical manner, while Iida (1972) [140], Brown (1980) [52], and Washburn (1983) [237] developed computational methods for the optimal distribution of search resources for a moving target. Washburn's algorithm is called the forward and backward (FAB) algorithm.

From a theoretical point of view, numerous problems related to the optimal distribution of search resources belong to the concave programming problem. In addition to the optimal resource distribution problem, researchers have studied other search problems in order to find optimal search routes, optimal search plans with some constraints on resource distribution and search routes, and other variations. These include the works of Eagle and Yee (1990)

[76] along with Hohzaki and Iida (1997) [127], who made use of branch-and-bound and relaxation methods for discrete optimization.

As discussed above, determining the optimal search policy of a searcher has been the primary focus of research in the early history of search theory. This results from the fact that ASW operational success previously depended primarily on search plans because submarine mobility is generally poor in relation to that of ASW forces. However, because submarine mobility has significantly improved in the current era, it is natural that researchers expand their thema to game-theoretical modeling of search problems or search games in ways that not only discuss the searcher's plan but also take target movements into consideration. This is making game theory increasingly relevant, and in current search games, the target is assumed to be an active competitor against the searcher.

Reviewing what we have discussed thus far, searchers decide how to distribute search resources and can choose search routes, while stationary targets generate an existence distribution in the search space and moving targets choose paths across the search space. Almost all search game studies now treat searchers and targets as players. In Table 1, we classify search games by the combinations of a searcher's strategy (S's strat.) and a target strategy (T's strat.) and each game is identified by the names used most commonly in references, although with some slight modifications. For example, the search game involving both moving targets and searchers is most often called the *search-and-evasion game* or the *search-evasion game*. However, we changed it to the evasion-search game (ESG) for brevity and naming coherency. It should be noted that the classification of Table 1 is not permanently acceptable because model definitions are sometimes extended or slightly modified from different points of view, and because models assigned names by one specialty might need a different nomenclature when it is utilized by researchers from other fields.

Table 1: Classification of search games by player strategies

\S's strat. T's strat.	special	moving	resource distribution
special	Smuggling game		Inspection game
stationary	Binary search game	Linear search game Hide-search game	Hide-allocation game
moving	Path-constrained search game	Evasion-search game Princess-monster game	Ambush game Search allocation game Network interdiction game
resource distribution			Search-search game Blotto game Attack-defense game

1.2. Inspection and smuggling games

The smuggling game (SG), which is shown in Table 1 (both special strategies by target and searcher), is a non-cooperative game in which smugglers work to move contraband in secret, and customs officials work to detect the smuggling and capture the smuggler. Thus, the SG has a story similar to the search game. Historically, SGs are seen as offshoot from the inspection game (IG) listed in Table 1. In this section, we will provide outlines of both games before fully surveying search game literature because we regard them as special cases of the search game. However, for additional information on the IG, please refer to Avenhaus (1986) [24] as a textbook and see Avenhaus (2002) [25] and Hohzaki (2013) [122] for a review and explanation of the game.

In 1962, Dresher (1962) [73] first proposed an IG model for use in analyzing the effectiveness of inspections related to arms reduction treaties. After the development of Phase I of the IG, which included Dresher's work, analysts utilized Phase II of the game from the 1960s to the 1980s in order to make an effective inspection plan of the International Atomic Energy Agency (IAEA) for application to the Non-Proliferation Treaty for Nuclear Weapons (NPT). In Phase III (after the 1980s), the IG was applied to new arm reduction treaties such as the Treaty on Intermediate Nuclear Forces and the Treaty on Conventional Forces in Europe. Important IG references are as follows: Maschler (1966) [171] and Kuhn (1963) [162] in Phase I, Bierlein (1968) [46] and Avenhaus (1986) [24] in Phase II, and Brams and Davis (1987) [51] and Avenhaus and Canty (1996) [26] in Phase III.

During the 1980s through 1990s, when the United States faced serious problem related to drugs smuggled from Central and South American countries; the IG was used to analyze effective ways to shut down the flow of drugs under the name of smuggling game. Thomas and Nisgav (1979) [229], Baston and Bostock (1991) [33] and Garnaev (1994) [101] were studies on the SG.

Now, let us review the model of Dresher's multi-stage IG. In the Dresher's game, there are two players: an inspectee and an inspector. During n stages, the inspectee has at most a chance to violate a treaty in order to obtain a reward while the inspector dispatches an inspection team m times ($m \le n$) to the inspectee's country. At each stage, the inspectee has two options, 'violate' or 'comply', and the inspector can choose to either 'inspect' or 'not inspect'. When the inspectee violates the treaty, an inspection brings the inspector definite exposure of the inspectee's illegal behavior and reward of one, and the inspectee loses the same amount of reward. No inspection brings the inspectee a reward of one and the inspector loses the same amount. Legal behavior on the part of the inspectee brings nothing to either player. This problem is a sequential two-person zero-sum (TPZS) game with parameters n and m. Using the notation $I(n,m)$ as the value of the game, we have the following payoff matrix, in which the inspector is a row-player, the inspectee is a column-player, and both players have two pure strategies each.

Table 2: Payoff matrix of Dresher's inspection game

$$I(n,m) = val \quad \begin{array}{c} \textit{Inspector}\backslash\textit{Inspectee} \\ \textit{inspection} \\ \textit{no inspection} \end{array} \begin{array}{cc} \textit{legal action} & \textit{illegal action} \\ \left(\begin{array}{c} I(n-1,m-1) \\ I(n-1,m) \end{array} \right. & \left. \begin{array}{c} 1 \\ -1 \end{array} \right) \end{array}$$

The val symbol indicates the value of the matrix game following the symbol. Because we can express any equilibrium of the matrix game with two rows and two columns in an analytical manner, we have a recurrence formula for a relationship between $I(n,m)$ and $I(n-1,\cdot)$. From this formula, we have an analytical form of $I(n,m)$.

$$I(n,m) = -\binom{n-1}{m} \Big/ \sum_{i=0}^{m} \binom{n}{i}.$$

Dresher's game has two strategies {*legal action*, *illegal action*} for the inspectee and {*inspection*, *no inspection*} for the inspector. As a substitute for this game, we can consider an equivalent search game that has a special set of strategies: {*hide*, *not hide*} for a target corresponding to the inspectee and {*search*, *not search*} for a searcher corresponding to the inspector.

In the years following its development, the IG became generalized and increasingly complicated due to the need to take into account imperfect detection of illegal actions, increases in the number of inspectees and inspectors, and the acquisition of other information, as can be found in Sakaguchi (1994) [215], Hohzaki et al. (2006) [135] and Hohzaki (2012) [120]. Additionally, while sequential or stochastic multi-stage games have been used to model numerous smuggling problems, each game has a comparatively small number of pure strategies for the smuggler and custom officials. IGs were often modeled as non-cooperative games, with analysis focusing on the inspector's strategy of resource distribution (such as budget and personnel) in order to determine inspection effectiveness, as in Avenhaus and Kilgour (2004) [27] and Hohzaki (2007) [114]. That is why we categorize the SG as a game that combines two special strategies and the IG as a game with a special target strategy and a resource-distribution strategy for the searcher.

As we explained above, even though the IGs and SGs could be regarded as search games for effectively searching for contraband and illegal actions, they are not direct descendants from Koopman's work. Therefore, we will conclude our review of those games at this point.

1.3. Policy explanations and chapter structure of this paper

Herein, we will provide an explanation of the basic policy used in this survey paper. First, we acknowledge that search theory does not have so many inherent methodologies that provides high generality and high applicability to problems in other research fields. Additionally, we know that we must often borrow methodologies from discrete/continuous optimization, mathematical programming, game theory, graph/network theory, geometry, and others fields, and then modify them for application to individual search problem. For addressing practical problems, search theory has been evolving towards being solution-oriented rather than method-oriented. Its interest exists in the discovery of problems and modeling. Therefore, we must apologize because, in this paper, we will seldom mention the methodology the authors devised for their search problems.

In Section 2, we outline some textbooks and survey papers on search theory, published to date that have bountiful contents from basic theory to up-to-date search theory developments. Following that section, we describe the expository writing for each search game model based on its classification in Table 1. These include the binary search game (BSG) in Section 3, the linear search game (LSG), hide-search game (HSG), and hide-allocation game (HAG) in Section 4, the evasion-search game (ESG) in Section 5, the princess-monster game (PMG) in Section 5, the ambush game (AG) in Section 7, the search allocation game (SAG) and path-constrained search game (PCSG) in Section 8, the search-search game (SSG), Blotto game (BG), and attack-defense game (ADG) in Section 9 and miscellaneous other search game models in Section 10.

2. Textbooks and Survey Papers on Search Theory

In this section, we will introduce a number of textbooks and survey papers on search theory. As discussed above, the theory originated from *Search and Screening* by Koopman (1946) [156], in which he covered a wide range of general information on the theory from his examinations of the physical and kinetic characteristics of search operations and the realistic usage of the radar and sonar device invented during WWII. However the book does not specifically discuss search games.

Differential Games by Isaacs (1965) [144] is a textbook that discusses ESGs defined in a continuous geographic and time space. Although some researchers sometimes refer to this type of evasion-search game as *pursuit-evasion game* (PEG), Isaacs' proposed game is named the *princess-monster problem* (PMP). In this game, which is set in a dark circular arena, a princess and a monster start from respective points. The princess wants to avoid being found by the monster while the monster wants to find the princess. Neither player has any visual information on the other, but detection occurs if both players come to within a certain preset distance of each other. The maximum speeds of both players are limited. The game payoff is the time until the first detection occurrence. Because the motions of both players are expressed by differential equations, the PMP is a differential game, which is defined as a game with constraints expressed by differential equations. The use of an optimal control method is often necessary to solve a differential game, as demonstrated in Dockner et al. (2000) [72].

Search Games by Gal (1980) [95] explicates HSGs with stationary targets and ESGs with moving targets using a special search space of lines, trees, or circles, while *Search and Detection* by Washburn (1981) [236] is an introductory textbook that describes the basics of search theory. The 4th edition, which was published in 2002, shows a simple ESG model on a continuous space. In *Geometric Games and their Applications* by Ruckle (1983) [209], we can study a number of AGs defined in various geometric spaces, while *Search Games and Other Applications of Game Theory* by Garnaev (2000) [102] discusses AGs, HSGs, SSGs, BGs, as well as IGs.

The first half part of *The Theory of Search Games and Rendezvous* by Alpern and Gal (2003) [18], is an extended version of Gal's textbook mentioned above, with addition of a discussion about HSG and ESG in geometric search spaces. In the second half, the authors introduce rendezvous search, a special search problem in which multiple players start from randomized points and rendezvous with each other primarily in geometric search space. The search problem often utilizes the criterion of time until the rendezvous. Additionally, since there are some differences between players' recognition about their initial situations and orientations, rendezvous is often more difficult than it would appear at a glance. Although rendezvous search is interesting, it was not chosen as a target for our survey because it has not yet been formulated into a game.

Chapter 8 of *Two-Person Zero-Sum Games* by Washburn (2014) [239] consists of 15 pages where the author provides search game examples such as HSGs, which are regarded as a matrix game. It also explains BGs in brief in Chapter 6. *Search Theory* by Iida and Hohzaki (2007) [141] is a Japanese textbook that devotes numerous pages to the discussion of one-sided search problems. In Chapter 10, the focus changes to search games, especially SAGs.

Next, we will introduce a number of survey or explanatory papers related to search theory. Enslow (1966) [78] cites and outlines 13 papers related to search games, while Dobbie (1968) [70] introduces 10 papers on search games in a section of 'two-sided search'. Additionally, Moore (1970) [177] refers to four papers on the HSG, focusing on the problem of finding ways to look for targets sequentially in possible regions. This author also mentions SAGs while considering the density of distributed search resources in a continuous search space. Nakai (1989) [180] authored a survey paper on the general contents of search problems in which he cites about 134 references in total, while Benkoski (1991) [43] sets two categories (one-sided and two-sided search) and two subcategories (stationary and moving target) in each category in order to classify past research. This author uses 62 references to cite ambush search games and tactical games involving military affairs.

Moreover, Hohzaki (2008) [117] mainly explicates optimization methods of search theory and search games in Section 4, while Gal (2011) [96] explains the so-called infiltration game, which is a variation of the search game, from the standpoint of infiltrators or invaders. It also surveys PMPs, along with ESGs and HSGs according to the classification by the shape of the search space, e.g., line, tree, star, and plane. Alpern (2011) [8] primarily explains rendezvous searches using a few exceptional examples of the HSGs and rendezvous-evasion games, in which two players attempt to meet as quickly as possible without being detected by their opponent (searcher). Washburn (2011) [238] is an explanatory paper for BGs, which he defines in a general way while describing TPZS BG models. Evans and Bishop (2013) [79] is an ordinary paper rather than a survey paper, and was written to explain how the authors solved a static spatial search game (SSSG) for a special HSG model in which a target is hiding in place and a searcher of looking into that place to detect the target. The authors contribute Section 3 to the plethora of search game surveys: HSGs, ESGs, SAGs, AGs and rendezvous search. Hohzaki (2013) [123] explains and surveys SAGs, as the manuscript title indicates.

3. Binary Search Game

In his early research, Johnson (1964) [145] dealt with a BSG, which has other names of a dichotomy search game and a high-low search game. Blue chooses an integer k from a set of integers $1, \cdots, n$ and Red guesses the integer. Red is told whether the guessed integer is higher or lower than the chosen integer (true number). Blue and Red are players, and the game payoff is the number of guesses until the chosen and the guessed integer match by coincidence. Blue (maximizer) takes a mixed strategy with p_k as the probability of choosing an integer k, while Red (minimizer) determines a pure strategy for an ordered set of n integers as well as a mixed strategy for the probability of choosing it. Each number of an ordered set tells us which integer to call after learning whether the previous guessed number is higher or lower. Table 3 shows an example of an ordered set, say set i, for a game with $n = 7$. S_{ij} indicates the order that integer j is called in the set i. In this example, integer 3 is called in the first place. In the second place, call 6 if 3 is lower and call 1 if higher. If the second call 6 is lower, call 7 and the game ends. If 6 is higher, call 4 in the third place. If the second call 1 is not true, call 2 and end. If the third call 4 is not true, it is lower than the true integer and the game ends with call 5. Red determines the probability of taking this set of calls i, t_i, as a mixed strategy.

Table 3: Example of an ordered set

j	1	2	3	4	5	6	7
S_{ij}	2	3	1	3	4	2	3

Thus, we derive an equilibrium to solve the game with the payoff given by

$$V = \min_{\{t_i\}} \max_{\{p_j\}} \sum_i \sum_j t_i S_{ij} p_j = \max_{\{p_j\}} \min_{\{t_i, S_{ij}\}} \sum_i \sum_j t_i S_{ij} p_j$$

using p_j and t_i.

Johnson analyzed the general properties of optimal strategies and showed us equilibrium points in the case of $n \leq 11$. Gal (1974) [91] extended Johnson's results for the same problem.

In his book, Ulam (1976) [231] asked a question that relates to the binary search problem. His question, which was called *Ulam's problem*, became a target of binary search researches. In Ulam's problem, Blue chooses an integer from 1 to $1,000,000$. Red is allowed to ask up to 20 questions, to each of which Blue is supposed to answer either yes or no. If we suppose that Blue is allowed to lie once or twice to Red's questions, how many questions does Red need to get the right answer? This is Ulam's question. Ulam's problem is a BSG that considers lying. Gal (1978) [93] proposed a BSG, where Blue chooses a real number from a continuous interval $[a, b]$ and Red guesses it. The probability of a lie in each of Blue's yes or no answers is given. If the final interval reduced by Red's guesses contains the true number after n guesses of Red, Blue obtains a payoff based on the length of the final interval.

As explained above, in many models, the BSG defined on a discrete number of intervals is played until the true and guessed numbers coincide. However, in games defined in a continuous interval, the accuracy of the finally guessed number or the accumulation of each difference between each guessed and true number is adopted as the payoff. Other versions of the BSG were studied by Rivest et al. (1980) [201], Ravikumar and Lakshmanan (1984) [199], Baston and Bostock (1985) [30], Alpern (1985) [4], Pelc (1986, 1989) [191, 193] and Niven (1988) [185]. Spencer (1984) [222], Pelc (1987) [192] and Czyzowicz and Pelc (1988) [62] researched Ulam' problem. In the last paper, the authors determined that at least 29 calls are needed to reach the right answer.

4. Linear Search, Hide-Search, and Hide-Allocation Games

In the HSG model, the searcher moves in a search space to find a stationary target while the target hides in an attempt to avoid detection.

The simplest search space within which a target can hide is a line. Beck and Newman (1970) [41] discussed an LSG. A hider chooses one point on a line to hide, but the distance between the origin and hiding point, called the hiding distance, is constrained in a probabilistic fashion. The searcher moves from the origin to find the target in a continuous motion with speed 1. Perfect detection, which means that detection is certain, occurs when the searcher passes the hiding point. The payoff of the game is the time until target detection. The authors found a minimax searcher strategy. Gal (1972) [90] is a generalization of the Beck and Newman's model. Gal (1974) [92] and Gal and Chazan (1976) [97] also dealt with Beck and Newman's game with the modified detection time payoff divided by the hiding distance and derived a minimax searcher strategy. They then tried to extend their models to a two-dimensional (2D) search space. Fristedt (1977) [88] discussed Gal's model and Fristedt and Heath (1974) [89] adopted various payoffs depending on the hiding distance and detection time in their LSG. We have numerous references on the HSG of a hiding target versus a searcher because we can find examples of such search games in our real life and society.

Now, let us model a simple HSG on a discrete search space consisting of n cells. A target (hider) hides in a cell and a searcher looks into a cell just once to detect the hider under the assumption that perfect detection occurs if the searcher looks into the hiding cell. The detection at cell i brings the searcher reward α_i. When the game is in equilibrium, the hider and the searcher have the same mixed strategy in which the hiding or looking probability of cell i is given by $(1/\alpha_i)/\sum_{k=1}^{n}(1/\alpha_k)$ and the value of the game is $1/\sum_{k=1}^{n}(1/\alpha_k)$.

Von Neumann (1953) [232] considered the above game to be a preliminary model and discussed a TPZS HSG based on the search space of a matrix. A hider hides in an element of the matrix, say (i, j), and the searcher chooses a row or a column. If the element is in the chosen row i or the chosen column j, the searcher detects the hider and obtains reward q_{ij}. Norris (1962) [186] proposed an HSG basic model in which a hider chooses a cell to hide in and a searcher sequentially looks into the cells. However, in this game, even if the searcher looks into the hider's cell, there is a probability that the hider will be overlooked (imperfect detection). The game payoff is defined by the number of looks required to detect the hider. Norris obtained an analytical form of equilibrium for the game defined on a two-cell space. Bram (1963) [50] extended Norris's game to a general discrete search space with multiple cells. Neuts (1963) [184] added the acquisition of the hider's value by its detection and the search cost expenditure to the Norris model, and considered a discount factor for the reward at each looking stage of the game. He dealt with the following two models: a multi-stage game model in which the hider hides once and the searcher repeats looking until the hider is detected, and the repeated game model in which the players repeat a subgame that the hider hides once and the searcher looks once. Efron (1964) [77] considered a repeated HSG with the number of looks of the searcher as payoff by adding constraints that prevent the hider from reusing the past hiding cell and the searcher from reexamining a cell that had been examined previously. Roberts and Gittins (1978) [203] and Gittins and Roberts (1979) [104] extended the two-cell model and n-cell model, respectively. Flood (1972) [81] derived an equilibrium of Von Neumann's game in a more general way by using the methodologies for optimal allocation problems. Sakaguchi (1973) [214] complemented the results of Neuts (1963) and analyzed the modified game with the acquisition of the hider's value and the discount factor. Subelman (1981) [227] modified the multi-looking game so that the searcher is restricted to a limited number of looks and the payoff is given by the total detection probability of the hider. For this game, a pure searcher strategy is not represented by the order in which cells are examined, but by a simpler form of how many times the searcher looks into each cell. Berry (1986) [44] proposed some heuristics for an optimal search and compared the detection probabilities that resulted from the proposed heuristics. The HSG by Ruckle (1990) [210] has a special model, in which a hider puts gold and a landmine in different cells and a searcher is rewarded only if he finds the gold before encountering the landmine.

After the early research mentioned above, the HSG was discussed under some special geometric space and network-structured space settings. Bostock (1984) [48] studied a search game on a network with two nodes and three arcs between the nodes. In the Kikuta model (1990, 1991) [149, 150], a searcher continues searching for a hider in a cell of a multi-cell discrete space while expending traveling costs and search costs to move between and search in cells. The game payoff is the total traveling cost. Anderson (1990) [21] modeled a TPZS HSG, in which a searcher uses a maximum speed 1 to search for a hider hidden at a point on a continuous geographic and time space. The payoff is the time that the searcher expends to detect the hider. Reijnierse and Potters (1993) [200] derived a relationship between an optimal search path and Euler paths for a search game defined on a cyclic graph. Kikuta and Ruckle (1994) [152] and Alpern (2011) [9] investigated a so-called *find-fetch game*, which ends when a searcher finds a hider and returns to the starting point. The former research considered the find-fetch game with the search cost as payoff while the latter game used the return time of the searcher. The model created by Alpern and Reyniers (1994) [20] is a find-fetch game where two searchers meet at a preplanned point on a line after finding a hider. This game is a combination of the rendezvous search and the search game. In addition to the games mentioned so far, Cao (1995) [55], Alpern (2008, 2010) [6, 7] and Dagan and Gal (2008) [63] studied HSGs defined on trees, and Pavlovic (1995) [190], Kikuta (2004) [151], Alpern et al. (2008) [13] and Alpern and Baston (2009) [12] discussed HSGs on networks.

Another type of HSG is called an *accumulation game*. In this game the hider conceals several treasures simultaneously or sequentially in different places and the searcher picks them up while searching for them. If the searcher cannot gather a given number of treasures during a given time period, the hider wins the game and obtains some reward. There are several versions of the accumulation games depending on the properties of hiding places, discreteness or continuity of the treasure, information about previously searched places, and the capacity of treasure location. These are investigated by Kikuta and Ruckle (1997, 2002)

[153, 154], Ruckle and Kikuta (2000) [212], Zoroa et al. (2004) [249], Alpern et al. (2010) [14].

The game between a submarine versus an ASW airplane has provided a good theme for search games. Here, the submarine moves on a chosen fixed course at a chosen fixed speed on a 2D sea area and the airplane searches for the submarine using its detection sensor without knowing the submarine's course and speed. The course and speed selection by the submarine is equivalent to a choice of a point in a circle that represents a set of speed vectors. The airplane's search can be visualized by sequentially placing a small disk that represents the detection range of its sensor in the speed circle. If any part of the small disk overlaps a speed vector chosen by the submarine, the airplane is assumed to have detected the submarine. Danskin (1968, 1990) [66, 67] solved a geometric submarine vs. airplane game in the speed circle, taking the detection probability as payoff. We could categorize the game as an ESG because the submarine moves in practice, or as a hide-allocation game (HAG) if we regard the sequential placement of detection disks as search resource allocations.

As an HAG example, in which a searcher looks for a stationary target by distributing search resources, we look at Nakai (1988) [183] and Iida et al. (1994) [143]. Nakai considered a variety of HAGs with continuously divisible search resources, even though he also discussed one-sided search problems.

5. Evasion-Search Game

If the HSG has a moving target (evader), we call it *the evasion-search game* (ESG). For elaborate target motion, a simpler search space is more convenient for ESG modeling. In the early phase of the research, ESGs were studied on a line.

In the model created by Meinardi (1964) [173], the evader moves to reach a point on a line while knowing the searcher's position. This author did not explicitly derive any equilibrium for his multi-stage ESG, but instead focused on improving the evader motion in order to make its distribution more uniform on the line. Lee (1983, 1985) [166, 167] investigated a model, in which an evader starts from a cell n of cells $\{0, 1, \cdots, n\}$ aligned on a line bound for a destination cell 0 that would provide a shelter. On the way to the shelter, the evader can move to a few neighboring cells adjacent to his or her current cell. A searcher knows the current position of the evader and decides the next cell to move to and look into. The game payoff is the number of detections until the evader reaches his or her destination under the assumption of perfect detection. Like Lee's model, there were numerous ESG models with start and destination points for the evader. Nakai (1983, 1986) [179, 181] and Lalley (1988) [164] studied a game in which a searcher overlooks an evader and the payoff is defined as the non-detection probability until a final time, or the accumulated local reward obtained from each non-detection. For an ESG defined on a discrete cell space and a discrete time, the dynamic programming method is usually powerful enough to derive its equilibrium by manipulating some recursive equations.

An ESG with an evader goal is often simulated as a passage control problem of a submarine in a strait, and is often called an *infiltration game*. Many researchers have used a submarine as an attractive player. In Dubins' ESG model (1957) [74], a ship (an evader) has two options for his or her next position and a bomber (searcher) drops a bomb on the ship once at the most. The bomber knows the ship's current position, but there is a significant time lag between dropping the bomb and its explosion. The game payoff is the probability of the bomb's hitting the ship. For the ESG emulation of an ASW airplane equipped with depth charges versus a submarine, there are several versions that can be produced by changing the number of depth charges, the effective range or power of the depth charge, time lag of the depth charge explosion, velocities of players and others. These are reviewed in Charnes and Schroeder (1967) [56], Langford (1973) [163], Baston and Bostock (1989) [32], and Garnaev (1993) [100]. As a more complicated ASW game, Brown et al. (2011) [54] investigated an ESG game with a task force consisting of an aircraft carrier and destroyers vs. a submarine that receives information about the position of the task force and is attempting to infiltrate the task force in order to destroy the aircraft carrier.

Some researchers set the search space to arbitrary cell spaces and networks. Washburn (1980) [235] investigated a multi-stage ESG with a traveling cost as payoff, in which the evader and searcher choose their next cells to move to, but only the evader knows the searcher's present position. The play continues until the evader is detected or a final time under the perfect detection assumption. Thomas and Washburn (1991) [230] also studied a multi-stage ESG that used the evader's recognition of the searcher's position and a limit on the searcher's speed. Eagle and Washburn (1991) [75] considered a multi-stage ESG with information about the players' positions, in which the payoff is the total reward given by accumulating reward determined by the players' positions at each stage. He named the game the *cumulative search-evasion game*. The model of Baston and Garnaev (1996) [35] is a TPZS ESG defined on a network with two nodes and n arcs connecting the nodes, in which there are limits on the evader's maximum speed and the number of looks by a searcher.

The Ruckle game (1981) [207] has a payoff of the time until the detection of an evader in a search situation where neither the searcher nor the evader have any information about their opponents and where the players move in a probabilistic way on a cyclic network. In the Corwin model (1981) [59], an evader takes a probabilistic move (Markov motion) but the searcher takes action involving a sequence of ordered looks. If the searcher does not detect the evader during the limited number of looks, the searcher anticipates the last position of the evader. The game has the whereabouts probability as payoff, explained in Section 1.1. In the Auger TPZS game (1991) [23], a searcher makes an effort to intercept an evader moving toward a goal node on a network. Alpern (1992) [5] generalized Auger's results. Anderson and Aramendia (1992) [2] also studied an ESG with the payoff of detection time on a network, but they formulated the game into an infinite-dimensional linear programming problem, which had not been previously seen in ordinary solution methods. The Kierstead and DelBalzo (2003) [148] solution method was distinctive because they found sub-optimal strategies for an ESG under complicated circumstances using a genetic algorithm.

At this point, let us take the time to list some special ESG models: Owen and McCormick (2008) [188] and Alpern et al. (2011) [16]. In the former model, a fugitive evades authorities that are trying to find him in a continuous search space, while taking account of the increasing risk that someone may inform the authorities about him if he stays at one position for a long time. The latter model is a biological ESG between a predator and prey. By means of the ESG and the HSG, the authors worked to answer the following question: which is the best hunting method for the predator, cruising (positive movement to seek prey) or ambushing (hiding in ambush to wait for prey)?

6. Princess-Monster and Differential Games

An ESG defined on a continuous search space, known as the *princess-monster game* (PMG), was first proposed by Isaacs (1965) [144]. In this ESG type, it is necessary to derive an equilibrium from some differential equations that represent player motions and an integrated function that represents the payoff. That is why the game is called a *differential game* (DG).

The PMG is a differential game specialized to the evasion-search of two players. For the DG, we often construct a basic equation called a Hamilton-Jacobi-Bellman (HJB) equation, which comes from optimal control theory, and then derive an equilibrium of optimal player trajectories satisfying the HJB equation.

The PMG is also known as the *lion-and-man problem* (LMP) and *robot-and-rabbit problem* (RRP), which are slightly different models. The PMG was defined in a dark circular arena, as explained in Section 2. There are several PMG variations depending on the search space, player velocities, acquisition of player information, discreteness or continuity of search space, and so on.

As for the PMGs defined on a circle, which would be the easiest to solve, Zelikin (1972) [244] and Alpern (1974) [3] assumed a uniform distribution for a monster's initial position on the circle, and Wilson (1972) [242] and Forema (1974) [86] took general distribution functions for that distribution in their PMGs and assumed that the princess knows the initial position of the monster.

Croft (1964) [60] defined an LMP in a circular arena, which is the same as the Isaacs' problem except that both players, the lion and the man, are always visible to each other. They proved that the lion would not catch the man if a polygonal trajectory was used during a finite time period. Flynn (1973, 1974, 1974) [82–84] proved the existence of a trajectory that brings the lion to the closest distance d^* to the man in a finite time as well as the existence of a trajectory for the man that maintains a distance more than $d^* - \varepsilon$ ($\varepsilon > 0$) from the lion. Lewin (1986) [169] gave an analytical expression for the optimized distance d^* by the relative ratio of lion and man velocities.

Halpern (1969) [106] considered a Stackelberg game resembling the RRP. Here, a robot with a higher velocity is programmed to get close to a rabbit with a lower velocity, but the rabbit wants to stay away from the robot and has the advantage of knowing how the robot's program works. The author solved a maximin optimization in order to obtain the players' optimal movement.

Alpern et al. (2008) [15] solved a PMG that is defined in an interval $[-1, 1]$ on a line and has princess and monster players that move at the same speed 1. The researchers proved that the game value V, defined as the time when both players meet for the first time, is $15/11 \leq V \leq 13/9$. A PMG on a circle and a plane was discussed by Foreman (1977) [87]. The PMG on a network has the following general model. For a PMG on a network, arcs of which have unit length in total, the value of the game is the time needed for a searcher traveling at unit speed to find the princess moving on the network. This is called the search value of the relevant network. Alpern and Asic (1985) [10] showed that the search value is not larger than 15/16 for the network with one node and two loops of length 1/2 each on the node. Alpern and Asic (1986) [11] dealt with a PMG on a network consisting of two nodes and three arcs of length 1/3 each between the two nodes. The search space Garnaev (1991) [98] handled for his PMG is a continuous square. If two searchers start from a same point on a circle and move in reverse directions, they are certain to find an evader moving on the circle prior to their rendezvous. Like this problem, Parsons (1976) [189] asked how many searchers are needed to be certain of finding an evader on various networks. In the PGM devised by Dendris et al. (1997) [69], the searcher does not have any information but an intelligent evader knows the searcher's position. Since the evader is only allowed to move in the region that has not yet been searched, the available free region becomes smaller as time goes by. When there is no free region for the evader, the game ends.

Garnaev (1992) [99] handled networks with two nodes and a convex region as the domain space of his PMG, and assumed imperfect detection in which a searcher detects an evader with probability p when they are at the same place and with lower probability q when they are on two neighboring arcs. In a 2D convex region Ω, Gal (1979) [94] and Fitzgerald (1979) [80] considered PMGs with the first time of detection as payoff when a searcher enters detectable range ε from an evader. They showed an equation $\lim_{\varepsilon \to 0} 2\varepsilon V(\varepsilon) = |\Omega|$, where $V(\varepsilon)$ is the value of the game and $|\Omega|$ is the area of Ω. Lalley and Robbins (1988) [165] proposed an algorithm for generating a sequence of trajectories that approach the minimax searcher's strategy regardless of ε.

Now, let us cite other PMG research efforts. Melikyan (1973) [174] took account of information acquisition, and Olsder and Pourtallier (1995) [187] counted information as a cost in their PMG. Chkhartishvile and Shikin (1995) [57] found the relation between detectable range and player velocities that make detection occur in their PMG when an evader senses the position of a higher-speed searcher. Worsham (1974) [234] handled a PMG defined on a discrete time space, in which players have a restricted memory of players' past positions, and produce sub-optimal trajectories. Hagedorn and Breakwell (1976) [105] considered a PMG with multiple searchers and Stipanovic et al. (2010) [225] proposed a new method that substitutes a Liaponov function for the HJB equation. Their method can be easily applied to PMGs with multiple players. Bhattacharya et al. (2009) [45] considered a PMG where players play the game while avoiding various obstacles put in the search space.

As a PMG textbook, we refer readers to Dockner et al. (2000) [72], from which the basic theorems of general differential games can be learned. These theorems have a number of applications, primarily in economics. An ESG defined in a continuous geographic and time space is sometimes called a *pursuit-evasion game* (PEG). In explanatory papers on PEGs, Krasovskii et al. (1987) [161] introduces Isaacs' theory and describes the dynamics of playing the games, while Melikyan (1998) [175] focuses on the applications of PEGs with differentiable functions.

7. Ambush Game

As in the submarine passage control problem by Morse and Kimball (1951) [178] described in Section 1, searchers lie in ambush for evaders in *ambush game* (AG). Ruckle first developed the AG model. AGs often need geometric knowledge to be handled correctly. Ruckle (1983) [209] specializes in geometry-related AGs.

In an AG, a searcher literally detects or captures an evader from ambush. The evader adopts a movement strategy and the searcher determines an ambush position or a distribution plan of search resource as a barrier against the evader's movement. Here, we will provide a rough overview of the original model devised by Ruckle et al. (1976) [211]. First, Player I tries to pass through a square region with length and width of 1 from the left to right side along a continuous curve. Player II emplaces a length of barriers in the region. If the evader's path touches the barriers, a capture or detection event occurs. The game payoff might be either the time to detection or the distance the evader has traveled. Player I is a maximizer and Player II is a minimizer. Through a series of papers of Ruckle et al. (1976) [211], Ruckle (1981) [208], Ruckle and Reay (1981) [213] and Ruckle (1979, 1979) [205, 206], Ruckle proposed several AG versions where two players choose several intervals from a fixed interval $[0, 1]$ and the payoff is determined based on whether two intervals has any intersection or the length of the intersection.

Baston and Bostock (1987) [31], Lee (1990) [168], and Zoroa and Fernandez-Saez (1999) [246] performed an in-depth consideration of a TPZS AG on a lattice region. In these works, an evader chooses a point from $[0, 1]$ as a crossing point and a searcher emplaces two barriers of length a and b ($a \leq b$) in an attempt at interception. Ruckle (1983) [209] solved AGs of (i) $a = b$, (ii) $a = 1/3, b = 1/2$ and (iii) $a = 1/5, b = 1/2$. Baston and Bostock (1987) showed a perfect form of equilibrium in the case of $b \geq 1/2$ and some cases of $b < 1/2$, Lee (1990) in the case of $b \leq 1/3$, Zoroa and Fernandez-Saez (1999) in the case of $a < b < 1/2$. We have examined other research papers that discuss AGs on a lattice, such as Zoroa and Zoroa (1993) [251] and Zoroa and Fernandez-Saez (2001) [247]. Zoroa et al. (2009) [252] considered a game with the payoff weighted by invasion points.

AGs involve a choice of numbers from intervals or sets as the player's strategy. Baston et al. (1989) [34] studied an AG called a *number search game*, where each player chooses a set of sequential integers and the payoff is the number of shared integers. Some researchers considered a cyclic set of numbers $\{1, 2, \cdots, n\}$ where cell n is thought to be next to cell 1. The boundary around a country can be considered a cyclic set. Researchers such as Zoroa and Fernandez-Saez (2003) [248], Alpern et al. (2011) [19] and Zoroa et al. (2012) [250] analyzed rational player choices from a cyclic set using AGs. Zoroa and Fernandez-Saez (2003) proposed a general solution method to produce equilibrium. Alpern et al. (2011) considered a TPZS AG, in which an infiltrator intrudes into a borderland represented by m cyclic-numbered cells during n time points and a patrol team moves in the same land in an attempt to catch the infiltrator. This model could be also regarded as an ESG defined on a borderland of lattice shape. Zoroa et al. (2012) also considered an AG on a cyclic-numbered lattice, where a searcher emplaces some stationary obstacles to intercept an infiltrator.

In the model of Alpern et al. (2013) [17], the searcher chooses one from two search options: a search in a finite region and an ambush during time periods. If an evader stays in the region selected for the search or if he moves during the ambush time periods, the searcher detects the evader. The payoff is the detection time. Hohzaki and Iida (2001) [131] studied an AG with a distribution strategy of search resource on a network. Here, an evader chooses a path to travel along and a searcher distributes a limited amount of search resources on an arc. The detection probability of the evader on an arc depends on the property of the arc and the amount of resources distributed there. The total detection probability is the payoff.

Arnold (1962) [22] and Baston and Kikuta (2004, 2009) [37, 38] are studies on submarines interceptions in confined waterways. Arnold (1962) discussed optimal invasion points of submarine and optimal points to install sound sensor while taking into consideration sonar detection sensitivities. Baston and Kikuta (2004, 2009) analyzed the same AG with additional information acquisition in strait barrier operations.

As surveyed so far, AG researchers tend to be theoretical and handle comparatively simple search spaces in their research efforts. It would take more time to extend their results to general search spaces.

8. Search Allocation and Path-Constrained Search Games

In a *search allocation game* (SAG), the searcher distributes search resources in an attempt to detect a moving target. Considering the way used to name search games so far, we should probably name such a game an evasion-allocation game, but that name has never been used in the past. For information on basic modeling and a short survey of SAGs, please refer to Hohzaki (2013) [123].

One of the early SAG studies was Stewart (1981) [223], in which an SAG was modeled on a two-cell space and a discrete time space. Here, a target starts from cell 1 and goes to a goal cell 2. The target strategy is to decide when to go to cell 2 and when to move to the goal. The searcher strategy is to make a search resource distribution plan in two cells. The author obtained equilibrium for the special SAG model with the detection probability as payoff.

We can easily review the formulation of SAG on a discrete cell space $\boldsymbol{K}$ and a discrete time space $\boldsymbol{T}$ by recalling the objective described in Problem (P) in Section 1.1. To begin, let us denote a mixed strategy of the target by $\pi = \{\pi(\omega), \omega \in \Omega\}$, where Ω is a set of target paths and $\pi(\omega)$ is the probability of taking path ω. We denote a pure searcher strategy by $\varphi = \{\varphi(i,t), i \in \boldsymbol{K}, t \in \boldsymbol{T}\}$, where $\varphi(i,t)$ is the amount of resources to be distributed in cell i at time t. The expected detection probability is given by

$$R(\varphi, \pi) = \sum_{\omega \in \Omega} \pi(\omega) \left\{ 1 - \exp(-\sum_{t \in T} \alpha_{\omega(t)} \varphi(\omega(t), t)) \right\}. \tag{8.1}$$

If the searcher gains reward 1 when the target is detected but not otherwise, while the target losses the same amount of reward if detected, the SAG is a TPZS game with the payoff of $R(\varphi, \pi)$. Iida et al. (1996) [142] proposed a basic SAG, after which Hohzaki and Iida (1998) [128] discussed basically the same model except that the payoff includes a certain reward gain on detection and losses of search cost. Hohzaki and Iida (1999) [129] generalized the results by producing a general payoff function.

After these basic model propositions, researchers added practical conditions to the SAG model. For example, in an effort to achieve high applicability, Hohzaki et al. (2002) [132] and Hohzaki (2006) [112] considered some practical constraints on target movement and energy possession. In addition, in Hohzaki (2006) [112], the author showed an especially notable correspondence between a discrete SAG defined in a discrete search space and a continuous SAG defined in a continuous space. In this study, the continuous SAG equilibrium is given as a convergence point of equilibria of discrete SAGs from a computational point of view, even though the continuous SAG is formulated into a variational problem and is difficult to solve. In addition to practical conditions on target motion, various features of search resource were incorporated into the SAG model, such as those mentioned by Dambreville and Le Cadre (2002, 2002) [64, 65] and Hohzaki (2008, 2012) [116, 119]. As in actual search operations, the target's mobility, which is mainly a function of his/her energy, affects operational results. When considering this situation, Hohzaki and Ikeda (2009) [133] discussed a SAG in which a target factors in an energy supply strategy as well as a movement strategy. The event that occurs when a searcher overlooks a target is a kind of Type I error in statistics. Type II errors, in which the searcher gains false contact signals implying target detection in an impossible detection situation, often occur in actual search operations. However, there have been few studies taking the false contact into account. Hohzaki (2004, 2007) [111, 113] and Kekka and Hohzaki (2013) [147] analyzed the effects of false contacts on SAGs, assuming that the resource distribution of the searcher increases the frequency of false contacts.

Hohzaki and Joo (2015) [134] proposed a new SAG model with incomplete information about an initial target position and its equilibrium methodology that uses convex programming formulation.

While almost all SAGs have been modeled into one-shot games, Hohzaki (2007) [115] first discussed a multi-stage SAG in which both players can change their predetermined strategies after obtaining new information in the middle of the game. Specifically, the target learns the residual amount of search resources of the searcher and the searcher observes the residual energy and current position of the target. Additionally, Washburn and Hohzaki (2001) [240] and Hohzaki and Washburn (2003) [138] modeled a SAG named a *datum search game*, which starts from observation of a target position (datum point) by the searcher, on a continuous plane and a continuous time space. They derived an approximation of equilibrium of the target speed/course control and searcher resource distribution. Later, Hohzaki and Ida (2009) [126] performed an experimental simulation of the datum search game involving human operators and verified a good correspondence between the theoretical equilibrium and the experimental results.

There are a number of other SAG models as well. For example Hohzaki (2013) [121] considered a three-person non-zero-sum SAG with a target and two searchers, in which the target desires to minimize its detection probability, and each searcher acts in a selfish manner to maximize his or her expected payoff based on the assumption that the searchers gain different rewards depending on whether detection is accomplished by a single searcher or by simultaneous detection involving two searchers. For this model, the author calculated an equilibrium using a proposed computational algorithm. Another model, described by Hohzaki (2009) [118], is a cooperative SAG model in which multiple searchers are motivated to cooperate with each other. More specifically, when several searchers offer to combine their resources, a joint search operation for the target becomes possible. The target mission is to evade detection by the searchers, and the game is non-cooperative between the searchers and the target, but cooperative for the searchers. To facilitate this, based on the concept of core of cooperative game, the author proposed a persuadable imputation among all searchers if the searchers gain the target value upon detection.

The last SAG model we will introduce is the *path-constrained search game* (PCSG), which inherits the name from one-sided search problems for an optimal distribution of search resources (Eagle and Yee (1990) [76], and Hohzaki and Iida (1997) [127]). The name refers to the fact that the searcher strategy, such as his or her search path, is constrained. Some constraints on searcher strategy have a discretization effect that makes the problem non-deterministic polynomial-time hard (NP-hard), and thus difficult to solve. A relaxation of those constraints makes the problem an optimal distribution problem of search resource. That is why we categorize the PCSG into the SAG. Many such games are as difficult to solve as the one-sided problems used to be and there are not very many studies on this type of game. One such study is Hohzaki and Iida (2000) [130], in which the searcher is given a path and he decides to look or not look in his position each time.

9. Search-Search, Blotto, and Attack-Defense Games

For multi-player search games, in which all players have distribution strategies of their resources, Nakai (1986) [182] applied the name of *search-search game* (SSG). However, a similar such game used to be called a *Blotto game* (BG) even though that name is not used solely by search games. Many OR researchers have recently focused on anti-terror operations because of the increasing risk of terrorism around the world. This interest can be seen in BGs where an attacker and a defender are assumed to be competitors for effective resources. To clarify, we will refer to this here as an *attack-defense game* (ADG) and provide an outline of it as well as the SSG. Washburn (2011) [238] provides an introductory explanation about BG and Garnaev (2000) [102] is helpful in the study of SSGs. As explained in Section 1.1, since the one-sided problem of optimal search resource distribution is connected with general resource allocation problems, convex programming [204] and nonlinear programming [40] are very useful when working to solve SSGs and BGs.

The original BG is a simple noncooperative TPZS game with two resource distributors. The fictional story behind the game that Borel (1921) [49] (translated into English by Savage

(1953) [218]) first proposed and McDonald (1949) [172] introduced, involves Colonel Blotto who commands four platoons and his enemy, who commands three platoons. Both sides will dispatch their platoons to capture four forts. If Blotto allocates more platoons to a fort than the enemy, he wins the fort. Otherwise, the enemy occupies the fort. For example, if Blotto allocates a vector of dispatch $(2, 1, 1, 0)$ to four forts and the enemy does a vector $(1, 0, 1, 1)$, Blotto wins the first and the second fort. Both players want to win more forts than the other. Because the allocation of discrete resources has a finite number of allocation vectors, the game is easily solved as a matrix game in theory.

Now, let's review the BG model proposed by Blackett (1958) [47] in a general way. Two players have the total amount of a and b of resources, respectively, and assign them to n targets. If the first player has an allocation plan $\boldsymbol{x} = (x_1, x_2, \cdots, x_n)$ ($\sum_k x_k \leq a$) and the second a plan of $\boldsymbol{y} = (y_1, y_2, \cdots, y_n)$ ($\sum_k y_k \leq b$), player j gets reward $R_k^j(x_k, y_k)$ from target k and the total reward $R^j(\boldsymbol{x}, \boldsymbol{y}) = \sum_{k=1}^n R_k^j(x_k, y_k)$. Each player wants to maximize his or her reward $R^j(\boldsymbol{x}, \boldsymbol{y})$ during the game. As explained above, the reward $R_k^j(x_k, y_k)$ from each target k is usually assumed to depend solely on the resources allocated to the target. In the case of $R^2(\boldsymbol{x}, \boldsymbol{y}) = -R^1(\boldsymbol{x}, \boldsymbol{y})$, the game is a TPZS game. The resource allocation by the players could be discrete and continuous. Macdonell and Mastronardi (2012) [170] gave a perfect form of equilibrium for an original BG with two forts.

We have examined the following studies on SSGs from the context of search game. Nakai (1986) [182] dealt with a noncooperative TPZS game that involved two searchers, each of which has an independent estimation on the hider's distribution probability. The hider conceals himself in n cells, and the each searcher works to maximize his or her hider detection probability before it is detected by the competing searcher. Garnaev (2007) [103] also considered a similar game, in which two searchers have common information about the distribution probability of the hider, but with the difference that the both searchers gain some reward in the case of either simultaneous detection by both searchers or a single searcher detection.

Croucher (1975) [61] investigated a search game with a searcher and a protector in which a target lies in cell $i = 1, \cdots, n$ with probability p_i, which is known to the players. If the searcher allocates x_i resources to cell i and the target is there, the searcher detects the target with probability $1 - \exp(-\lambda_i x_i)$. However, the probability is reduced by a factor of $\exp(-\mu_i y_i)$ if the protector provides a y_i resource allocation. Croucher's game is a TPZS SSG where the searcher has a resource, the protector has b resource and the target detection probability is the payoff. Based on the above assumption, the expected payoff can be given by the following expression:

$$R(\boldsymbol{x}, \boldsymbol{y}) = \sum_{i=1}^{n} p_i \left\{1 - \exp(-\lambda_i x_i)\right\} \exp(-\mu_i y_i). \tag{9.1}$$

The function is separable for variable vector $\boldsymbol{x}$ or $\boldsymbol{y}$ on targets, while it is concave in $\boldsymbol{x}$ and convex in $\boldsymbol{y}$. Therefore we can easily apply the theory of convex programming to the game and produce analytical expressions for optimal distribution of $\boldsymbol{x}$ and $\boldsymbol{y}$ using two optimal Lagrange multipliers. To find optimal multipliers and the game equilibrium, we can propose a simple numerical algorithm. The model of Baston and Garnaev (2000) [36] is an extension of Croucher's model, but is a non-zero-sum search game caused by different distribution costs of search resource.

Shubik and Weber (1981) [221] modified a BG to be applicable to military applications such as strategic missile deployment and anti-ballistic missile (ABM) defense. To generate a payoff function, they created a strategic value determination from surviving targets after two competitors distribute resources to destroy or defend valuable targets. Roberson (2006) [202] also considered a complex payoff similar to the Shubik and Weber model. As in that study, Robertson more often used the BG model to analyze attack-defense problems than search-search problems. When we roughly survey ADGs, we note that many of them could be interpreted as SSGs.

As an early research on the ADG, we can include Cohen (1966) [58] who assumed a convex payoff function and considered a model similar to that of Shubik and Weber. An attempt to apply a BG model to anti-terror operations can be found in Powell (2007) [197]. Here, the author analyzed the optimal allocation of anti-terror resources needed to defend official sites based on the following assumptions: (1) a direct allocation of anti-terror resources to one site has no effect on any other site, (2) an allocation of investment resources to intelligence, border defense, and counter-terrorism operations will have effects on the defensive posture of all sites, (3) there are two type of threats, strategic terror threats and non-strategic disaster/infection threats, and (4) the defender does not have access to information about target sites terrorist aims at. Powell (2009) [198] extended his previous results using a novel non-zero-sum sequential game model and derived its subgame perfect equilibrium. Hohzaki and Nagashima (2009) [137] discussed a Stackelberg ADG model in which an attacker budgets for the construction of various types of missiles that will be used to attack containers or silos that the defender must protect. The defender observes the missile construction plan, divides his or her strategic materials appropriately and transports the divided portions to containers or silos. The game payoff is the amount of destroyed material. Hausken (2010, 2011) [107, 109] investigated a non-zero-sum ADG with a valuable system consisting of various elements aligned in series along with parallel and combined composition.

Zhuang et al. (2010) [245] modeled an attack-defense problem using a multi-stage incomplete-information game that includes secrecy and deception. At each stage of play, the defender must decide whether to make a direct resource allocation that provides temporary effects or an indirect resource investment with potential long-term effects. Shan and Zhuang (2013) [220] discussed a Stackelberg game in which a leader/defender distributes his or her resources to defend targets that two types of attacker/followers will attempt to destroy.

In this section, we have reviewed SSG variations. We also examine BG and ADG studies that have come before the footlights recently again due to historical requests, primarily because those game models resemble the SSG.

10. Miscellaneous Search Games

In this section, we will discuss the *network interdiction game* (NIG) listed in Table 1 in Section 1.1 as well as other miscellaneous games. As was done in the previous section, we provide an outline of the NIG while paying attention to related search games.

In modern society, we utilize a variety of networks for various purposes such as information transmission/reception, various forms of travel, power transmission/usage, and so on. Mathematical models used to analyze the interdiction of entities on such networks are referred to as *network interdiction models* (NIMs). A game version of NIM is a NIG. The SG we explained previously is the historical root of problem-oriented NIGs designed to intercept contraband flows on smuggling networks. Using graph and network theory, this game provides advanced insights that could be used in the foundations of methodology-oriented NIGs.

In today's fight against terrorism, problem-oriented NIGs are increasingly utilized to help create effective defenses for infrastructure such as information and communication networks, electricity/gas/water networks, road/railway traffic networks and others. Among those others, the NIG provides good tools for efficient guard patrol plans in various facilities, effective ways to control infectious diseases along infection routes in order to prevent worldwide pandemics, and (as always) application to military problems. Methodology-oriented NIMs have been studied for many years by researchers such as Ford and Fulkerson (1962) [85]. They discussed a good way to separate a start node from a goal node on a network as an application of a minimum cut problem. Herein, we will outline several NIG models, but for more in depth information, please refer to Hohzaki (2015) [124], which is an introductory paper on NIMs from the safe life standpoint.

As published papers on the interdiction of smuggling, we have examined Washburn and Wood (1995) [241], Salmeron (2012) [216] and Bakir (2011) [28]. In the Washburn and Wood (1995) model, a smuggler wants to pass through a smuggling network undetected by customs, while customs works to find the smuggler by setting up inspection sites on the network. The authors formulated the game into a mixed integer-programming problem and solved it by means of graph theory and maximum flow theory. Salmeron (2012) used mathematical programming formulation to create a sensor placement plan that would offer the optimum chance of detecting an intruder traveling through a network based on the criterion of the intruder detection probability. Bakir (2011) used a Stackelberg model to analyze the allocation of security resources aimed at intercepting the flow of illegal weapons via imported or exported containers, primarily through harbors. The former two models are regarded as SAGs because customs detects smuggling or intruders via the distribution of discrete resources.

As for infrastructure network defense, we have the following references. In the first, Kodialam and Lakshman (2003) [155] developed a sampling strategy for packet flows travelling through a communication network within a given budget in order to effectively detect malicious intrusions. In the second, Salmeron et al. (2004) [217] proposed a good way to mitigate damage inflicted by terrorist attacks on electric power grids using a Stackelberg NIG. Later, Bell et al. (2008) [42] evaluated the vulnerability of road networks in London against terrorist attacks in order to secure a minimum traffic flow, while Perea and Puerto (2013) [194] considered the design of railway networks that are resilient against intentional attacks. These research efforts can be regarded as being related to the SAG because terrorist efforts are aimed at inflicting the maximum possible damage by distributing attacking resources on infra-nets.

Scaparra and Churce (2008) [219] adopted a Stackelberg game model to analyze an effective defense policy in a situation where a defender first fortifies facilities and terrorists attack those facilities after observing the fortification effort, while Basilico et al. (2012) [29] investigated an effective patrol policy for finding damage inflicted by terrorist's attacks. Baykal-Gursoy et al. (2014) [39] considered a mitigation policy involving stationary node investigations and active network patrols after an attacker has already damaged nodes. By replacing an attacker with a hider and the defender with a searcher, these NIGs could be used for hide-allocation games.

In another study into facility patrol problems, Pita et al. (2009) [195] and Tambe (2012) [228] designed the Assistant for Randomized Monitoring over Routes (ARMOR) security system of the Los Angeles International Airport, which exploits an equilibrium of a Bayesian Stackelberg game as a rational randomized patrol plan, while Morita et al. (2011) [176] and Hohzaki et al. (2013) [136] evaluated the vulnerability of guard patrol routes against intruder routes in facilities and used a TPZS game to design effective patrol routes from the viewpoint of facility security automation involving robots and intelligent sensors. As can be seen from the above review, numerous patrol problems are modeled using Stackelberg games because general information on security systems is often available to potential intruders. If we substitute an evader for an intruder and searchers for guards in patrol games, we can regard these patrol games as evasion-search games.

As an NIG application to military problems or an ADG, we reviewed a study on the ABM problem by Brown et al. (2005) [53], in which the authors used a Stackelberg game to discuss effective deployments of defensive ABMs when defending various targets under threat from adversary theater ballistic missiles. Using a model in which defenders are waiting on arcs to intercept attackers advancing on a network, and by employing Lanchester's attrition model to cover force attrition caused by conflict between defenders and attackers, Hohzaki and Chiba (2015) [125] clarified the relationship between effective defender deployment and available information on attackers. This model is regarded as an SAG where defenders are responsible for allocating defensive resources against moving attackers on a network.

Finally, let us check some advanced NIG studies on graph/network theory. Wood (1993) [243] considered a Stackelberg game with arc deletions followed by maximum flows on a surviving network, while Akgun et al. (2011) [1] also used a Stackelberg game model to analyze the interception problem of multi-commodity maximum flows, which they called the multi-terminal maximum-flow network-interdiction problem.

11. Conclusions

Herein, we surveyed literature on search games. As can be seen in Table 1 in Section 1.1, our targets of search game models are the smuggling game (SG), inspection game (IG), binary search game (BSG), linear search game (LSG), hide-search game (HSG), hide-allocation game (HAG), evasion-search game (ESG), princess-monster game (PMG), ambush game (AG), search allocation game (SAG), path-constrained search game (PCSG), search-search game (SSG), Blotto game (BG), attack-defense game (ADG), and network interdiction game (NIG). Due to their lack of historical relation to search theory, we did not carry out a full survey for the SG, IG, BG, ADG and NIG models. However, we did provide outlines for them while paying attention to their search problems.

Search games are applications of game theory to search problems. While authors naturally borrow solution concepts and models from game theory, they must devise concrete methodologies to derive equilibrium points in almost all cases because game theory cannot be stretched sufficiently to supply solution methods. However, since we did not have sufficient space to show the methodologies of their solutions to readers, it was necessary to focus on the characteristics and differences of search game models, and several basic formulations.

Numerous search game researchers have aimed at solving realistic search problems on a case-by-case basis. In such cases, game theory cannot always be directly applied to the search models, even though its evolution affects search games. However, the application of various advanced concepts of game theory has provided welcome enrichments to the applicability and analytical power of search games. A keyword is 'information', which is now necessary in advanced search games. Information plays an important role in games, as Harsanyi (1967) [108] pointed out. Additionally, as mentioned before, the presence of falsehoods in the 'yes' and 'no' answers is integral to some BSG models. Information about the history of past searched location characterizes some HSG and ESG models. Similarly, information about princess and monster's position, target position and target mobility are important in PMG and SAG models while secrecy and deception might be involved in information for some ADG models. Since we want to make search game models more realistic, efforts should be made to encourage adoption of the following models: a multi-stage model with information acquisition as a stage breakpoint, an incomplete-information game model where players face uncertainty about rules and game payoff, and a Stackelberg game model with asymmetric acquisition of player information. Information-related issues can be expected to become significantly more important in future studies on search games.

New paradigms of search game evolution should be tried in search spaces, especially in PMG and AG scenarios, where researchers have developed theories based on comparatively simple search spaces. For more realistic applications, a wider variety of general spaces should be included, which will require the development of computational methods other than theoretical expressions for equilibria.

We will conclude this paper with some remarks on cooperative search game modeling. Since its formation, search theory has been used in actual search and rescue (SAR) operations from a theoretical point of view. It would seem natural that SAR activities would require cooperative game modeling where searchers and targets behave in a cooperative manner. However, we found just one or two studies on the cooperative search games to review in Section 8. It is also interesting that the expectable elements of cooperative-game modeling and information utility could be incorporated into the rendezvous search problems we mentioned, but did not survey, in Section 2. In an ordinary rendezvous search, several searchers may want to meet each other, but rendezvous is difficult because of the lack of consistency or connection between the various searcher viewpoints in search space, present positions, orientation, or other factors. However, we can expect that various versions of cooperative search games will branch from rendezvous search in the near future.

Acknowledgements: The author is grateful to editors Shinji Mizuno and Katunori Ano for inviting me to write this survey paper and anonymous reviewers for their helpful comments. Even though search games are a primary theme of my published papers, I hesitated to undertake this project due to the difficulty of exhaustively looking over the vast amount of literature published over the last half century, and due to the differences of writing style between survey papers and research papers on a specific theme. At that time, we had a plan to transfer the huge amount of paper manuscripts that my predecessor Koji Iida and pre-predecessor Takashi Kisi had accumulated in our laboratory at the National Defense Academy onto hard disks as electronic files. This plan was accomplished with the help of my colleagues Emiko Fukuda and Yutaka Sakuma, and graduate students Taihei Matsuo, Takuya Adachi, Ikki Sakai and temporary worker Kazuko Yamagishi. I knew that completion of this task would help me to survey the early literature (I actually cited 25 references from the saved files) and that I needed a form of pressure to attempt the difficult chore of writing a survey paper. Finally, based on the lesson in the old saying "jumping down from the stage at Kiyomizu temple", which we use when we are confronted with a difficulty and wish to make a positive decision, I decided to take up the task. In Section 2, I introduced numerous textbooks and explanatory papers related to search theory, the contents of which are reasonably biased based on the interest of their authors, and explanatory papers having limited numbers of pages. In this sense, no full-spectrum survey on search theory has been written since the publication of Benkoski et al. (1991) [43] about a quarter of a century ago, which cited 62 papers related to search games and 239 references in total. This provided me with an additional reason to write this paper. I also extend my appreciation to Teruhisa Nakai, whose memorandum with 160 or more references on search theory was very helpful.

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